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Fourier Analysis — Solutions and Remarks to Exercise sheet 2

In Lectures 2 and 3 we discussed an abstract way to approximate functions in certain functions spaces — by means of *approximate identities* and most prominently Theorem 3.7. In the following we start investigating the connection to Fourier series and also highlight how this approach provides direct proofs for previously seen results (see Ex. 2.2).

> Take-Home-Message: *Theorem 3.7 (Approximation in homogeneous Banach spaces) gives a universal tool to approximate f's by $f * k_n$ where (k_n) is an approximate identity. The Fejér kernel $(k_n)_n = (F_n)_n$ is an important example, in particular since it is the arithmetic means of the Dirichlet kernel (D_n). Recall that $D_n * f$ equals the n-th partial sum of the Fourier series of f. In contrast to (F_n), the Dirichlet kernel is not an approximate identity, which indicates the complexity of the question whether the partial sums of a Fourier series converge.*

Ex 2.0:

Let $(a_n)_{n\in\mathbb{N}}$ be a converging sequence in a Banach space. Show that the sequence of arithmetic means $(b_n)_{n\in\mathbb{N}}$, $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k$, converges to the same limit. Also show that the converse fails in general.

Ex 2.1: *(Fejér kernel and Dirichlet kernel)*

Recall the definition of the *Dirichlet kernel* $(D_n)_{n\in\mathbb{N}_0} \subset C(\mathbb{T})$, $D_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n} \mathrm{e}^{ikt}$. Further define the *Fejér kernel* $(F_n)_{n\in\mathbb{N}_0}$ by

$$F_n = \frac{1}{n+1}\sum_{k=0}^{n} D_k.$$

(a) Show that $D_n(t) = \frac{1}{2\pi}\frac{\sin\left((n+\frac{1}{2})t\right)}{\sin t/2}$ for all $t \in \mathbb{T}$ and $n \in \mathbb{N}_0$.

(b) Show that $F_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n}(1-\frac{|k|}{n+1})\mathrm{e}^{ikt} = \frac{1}{2\pi(n+1)}\left(\frac{\sin\left((n+1)\frac{t}{2}\right)}{\sin t/2}\right)^2$

(c) Use (b) to conclude that $(F_n)_n$ is an approximate identity with the additional properties

$$F_n(t) \geq 0, \qquad F_n(t) = F_n(-t), \qquad \lim_{n\to\infty}\sup_{s\in[\delta,2\pi-\delta]}|F_n(s)| = 0$$

for $n \in \mathbb{N}_0$, $t \in \mathbb{T}$, $\delta \in (0,\pi)$.

(d) Let $f \in L^p(\mathbb{T})$, $p \in [1,\infty)$. Show that $\sum_{k=-n}^{n}\hat{f}(k)(1-\frac{|k|}{n+1})\mathrm{e}^{ik\cdot}$ converges to f in $L^p(\mathbb{T})$.

Ex 2.2:

Reprove the following results which we have already encountered in the lecture.

(a) Prove Weierstrass's Theorem — the set of trigonometric polynomials $\mathrm{Trig}(\mathbb{T})$ lies dense in $C(\mathbb{T})$ — by a constructive argument (given $f \in C(\mathbb{T})$ explicitly construct a sequence in $\mathrm{Trig}(\mathbb{T})$ that converge to f) *Hint: approximate identity.*

(b) Prove that $C(\mathbb{T})$ lies dense in $L^1(\mathbb{T})$ with the methods of Section 3
⚠ *Be aware of a circular argument*

(c) Prove that the Fourier coefficients of an $L^1(\mathbb{T})$ function are unique, i.e. $T : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is injective by using the Fejér kernel from Exercise 2.1.

Ex 2.3:

(a) *There was a TYPO in the original formulation of the exercise in Ex. 2.3: On the right-hand-side h should have been replaced by $R(h) = h(-\cdot)$.* We will discuss (a) again on Exercise sheet 3.

Let $f, g \in L^1(\mathbb{T})$ and $h \in L^\infty(\mathbb{T})$. Show that $\int_{\mathbb{T}} (f * g)(s) h(s)\, ds = \int_{\mathbb{T}} f(s)(g * R(h))(s)\, ds$.

Note that this identity can be linked to the "dual operator (also called "conjugate operator") of

$$M_g : L^1(\mathbb{T}) \to L^1(\mathbb{T}), f \mapsto f * g.$$

Recall that the dual operator T' of a bounded linear operator $T : X \to Y$ (X, Y Banach spaces) is defined through

$$\langle Tx, x' \rangle_{X,X'} = \langle x, T'x' \rangle_{X,X'} \qquad \forall x \in X, x' \in X'.$$

(b) Use (a) and Ex. 2.1 to show that for any $h \in L^\infty(\mathbb{T})$ there exists a sequence $(h_n)_{n\in\mathbb{N}} \subset C(\mathbb{T})$ which converges to h in weak* sense, i.e.

$$\int_{\mathbb{T}} f(s)\overline{h_n(s)}\, ds \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{T}} f(s)\overline{h(s)}\, ds \qquad \forall f \in L^1(\mathbb{T}).$$

Ex 2.4: *(Minkowski's inequality)*

(a) Prove *Minkowski's inequality* for homogeneous Banach spaces X on $\mathbb{T}$:

For $f \in X$ and $g \in L^1(\mathbb{T})$ it follows that $f * g \in X$ and $\|f * g\|_X \leq \|g\|_{L^1(\mathbb{T})}\|f\|_X$. [1]

This can be rephrased as

$$\mathcal{M}_g : \begin{cases} f & \mapsto & g * f \\ X & \to & X \end{cases}$$

is a bounded linear operator from X to X with norm less or equal $\|g\|_{L^1(\mathbb{T})}$.

(b) Show Minkowski's inequality for $X = L^\infty(\mathbb{T})$.

(c) Show that Minkowski's inequality is sharp[2] for the cases

$$X = L^1(\mathbb{T}), X = L^\infty(\mathbb{T}), X = C(\mathbb{T})$$

by showing that $\|\mathcal{M}_g\|_{X\to X} = \|g\|_{L^1(\mathbb{T})}$.

(d) For $X = L^2(\mathbb{T})$ show that $\|\mathcal{M}_g\|_{X\to X} = 2\pi\|\hat{g}\|_{\ell^\infty(\mathbb{Z})}$ and again conclude that Minkowski's inequality is sharp.

Hints: (a): Consider first $g \in C(\mathbb{T})$ and inspect (the proof) of Theorem 3.7.
(c): To show the case $X = L^1$ "imagine" that there exists $e \in X \setminus \{0\}$ such $e * g = g$. For $X = L^\infty(\mathbb{T})$ find a suitable f and for $X = C(\mathbb{T})$ use Ex. 2.1.

<u>Solution:</u> *In the proof of Theorem 3.7 ("Approximation on homogeneous Banach spaces") we have proved the following "lemma"*

For $k \in C(\mathbb{T})$ we have shown that $B_1 : X \to X : f \mapsto \int_{\mathbb{T}} k(s) f(\cdot - s)\, ds$ is a well-defined bounded operator with, for $f \in X$, $B_1 f = B_2 f := k * f$ and

$$(\star) \qquad \|B_2 f\|_X = \|B_1 f\|_X \leq \|k\|_{L^1(\mathbb{T})} \max_{s\in\mathbb{T}} \|f(\cdot - s)\|_X = \|k\|_{L^1(\mathbb{T})}\|f\|_X,^{3}$$

where the last equality follows from properties of X.

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[1] Note that this proves the special case $p = r$ of *Young's inequality for convolutions*,

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \qquad f \in L^p(\mathbb{T}), g \in L^q(\mathbb{T}), \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r}.$$

[2] Here "sharp" means that the inequality in the above statement (for fixed X) can not be replaced by $\|f * g\|_X \leq c\|g\|_{L^1}\|f\|_X$ for any $c < 1$ (with c not depending on f and g!)

[3] the inequality simply follows from $\|\int\| \leq \int \|\|$.

In the following we write g instead of k and note that $B_2 = \mathcal{M}_g$. Then the above shows that for $g \in C(\mathbb{T})$ we have that $\mathcal{M}_g$ is bounded from X to X with norm $\|\mathcal{M}_g\|_{X\to X} \le \|g\|_{L^1(\mathbb{T})}$. It remains to consider general $g \in L^1(\mathbb{T})$. Let $(g_n)_{n\in\mathbb{N}} \subset C(\mathbb{T})$ approximate g in the $L^1(\mathbb{T})$ norm, e.g. set $g_n = g * F_n$ and use Ex. 2.1(d). For $f \in X$, we observe the following

- *As $f, g \in L^1(\mathbb{T})$, $\mathcal{M}_g f = f * g \in L^1(\mathbb{T})$ and $\mathcal{M}_{g_n} f \to \mathcal{M}_g f$ in $L^1(\mathbb{T})$ by basic properties of the convolution (Thm. 2.2).*
- *$\mathcal{M}_{g_n} f \in X$ for all $n \in \mathbb{N}$, by the first step in the proof since $g_n \in C(\mathbb{T})$.*
- *The sequence $(\mathcal{M}_{g_n} f)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. This follows from $(\star)$ and the fact that $(g_n)_{n\in\mathbb{N}}$ is Cauchy in $L^1(\mathbb{T})$. Hence, $(\mathcal{M}_{g_n} f)_{n\in\mathbb{N}}$ has a limit in X.*
- *Since X is continuously embedded in $L^1(\mathbb{T})$ ("the $\|\cdot\|_X$ is stronger than $\|\cdot\|_{L^1(\mathbb{T})}$"), the limits of $(\mathcal{M}_{g_n} f)_{n\in\mathbb{N}}$ in $L^1(\mathbb{T})$ and X coincide.*

Thus we have shown that $\mathcal{M}_g f$ lies in X and hence $\mathcal{M}_g$ is well-defined from X to X. That $\|\mathcal{M}_g\|_{X\to X} \le \|g\|_{L^1(\mathbb{T})}$ now follows from the assertion for continuous g, i.e.

$$\|\mathcal{M}_g f\|_X = \lim_{n\to\infty} \|\mathcal{M}_{g_n} f\|_X \le \lim_{n\to\infty} \|g_n\|_{L^1(\mathbb{T})} \|f\|_X = \|g\|_{L^1} \|f\|_X,$$

where we used again that $\mathcal{M}_{g_n} f \to \mathcal{M}_g f$ in X for $n \to \infty$, as seen above.

(b): first note that $X = L^\infty(\mathbb{T})$ is not a homogeneous Banach space and hence (a) does not apply. However, the assertion follows easily by the definition of the convolution and triangle inequality.

(c): $X = L^1(\mathbb{T})$. Obviously we would be done if there existed an $e \in X$ such that $e * g = g$ and $\|e\|_X = 1$. As such e does not exists (see Ex. 2.6), we consider an approximate identity $(k_n)_n$ with $\|k_n\|_{L^1(\mathbb{T})} = 1$ instead. By Theorem 3.7, $k_n * g \to g$ in $L^1(\mathbb{T})$. Thus $\|\mathcal{M}_g k_n\| \to \|g\|_{L^1(\mathbb{T})}$ which shows that $\|\mathcal{M}_g\|_{L^1\to L^1} = \|g\|_{L^1(\mathbb{T})}$. For $X = L^\infty(\mathbb{T})$ one can explicitly find $f \in X$ such that $\|f\|_X = 1$ and $\|\mathcal{M}_g f\|_X = \|g\|_{L^1(\mathbb{T})}$. Alternatively, one may argue by duality (and using Ex. 2.3). For $X = C(\mathbb{T})$, one can approximate the special $f \in L^\infty(\mathbb{T})$ from the case $X = L^\infty(\mathbb{T})$ suitably in $C(\mathbb{T})$, i.e. consider $f_n = f * F_n$ where $(F_n)_n$ denotes the Fejer kernel. Then $f_n \in C(\mathbb{T})$, thus $\mathcal{M}_g f_n \in C(\mathbb{T})$ and, by (b) and since $g * F_n \to g$ in L^1,

$$\|\mathcal{M}_g f_n\|_{C(\mathbb{T})} = \|(g * F_n) * f\|_{L^\infty(\mathbb{T})} = \|\mathcal{M}_{g*F_n} f\|_{L^\infty(\mathbb{T})} \to \|\mathcal{M}_g f\|_{L^\infty} = \|g\|_{L^1(\mathbb{T})}.$$

(d) Use that $\widehat{f * g} = 2\pi \hat{f}\hat{g}$ and the fact that $L^2(\mathbb{T})$ is isometric isomorphic to $\ell^2(\mathbb{Z})$ via the Fourier coefficients. That $\|\mathcal{M}_g\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})}$ then follows from basic properties of multiplication operators on ℓ^2 (check!). For the sharpness of Minkowski's inequality find $g \in L^1(\mathbb{T})$ such that $2\pi\|\hat{g}\|_{\ell^\infty(\mathbb{Z})} = \|g\|_{L^1(\mathbb{T})}$.

Ex 2.5: (*Convergence of Fourier series in $L^1(\mathbb{T})$, $C(\mathbb{T})$ and pointwise*)

(a) Show that the following statement is wrong for general functions $f \in L^1(\mathbb{T})$.

> *The partial sums*
> $$S_N f = \sum_{k=-N}^{N} \hat{f}(k) e^{ikt}$$
> *of the Fourier series converge to f in the $L^1(\mathbb{T})$-norm (as $N \to \infty$).*

Can we find a "large" (say "dense") subspace D of $L^1(\mathbb{T})$ such that the above statement holds indeed true for all $f \in D$?

(b) Show that the following statement is wrong for general functions $f \in C(\mathbb{T})$.

> *The partial sums of the Fourier series converge to f pointwise.*

(c) Conclude that the Dirichlet kernel $(D_n)_{n\in\mathbb{N}_0}$ is not an approximate identity.

Hint: (a): you may freely use that $\sup_{n\in\mathbb{N}} \|D_n\|_1 = \infty$ and Exercise 2.4(c).
(b): similar as in (a), also note that $\mathcal{M}_g \tau_s = \tau_s \mathcal{M}_g$, where $\tau_s f = f(\cdot - s)$.

Solution: (a) and the statement we would get if "$L^1(\mathbb{T})$" is replaced by "$C(\mathbb{T})$" in (a) follow from the following general theorem, we will prove on Monday May 7:

Let X be a homogeneous Banach space. Then the partial sums of the Fourier series, $D_n * f$, converge in X for every $f \in X$ if and only if the operator norm of

$$\mathcal{M}_{D_n} : X \to X$$

is uniformly bounded in $n \in \mathbb{N}$ (where we use the notation from Ex. 2.4).

Note that this is a direct consequence of the uniform boundedness principle. Here, we only need the direction "$\Rightarrow$". Hence, it suffices to show that $\sup_{n\in\mathbb{N}} \|\mathcal{M}_{D_n}\|_{X\to X} = \infty$ for $X = L^1(\mathbb{T})$ and $X = C(\mathbb{T})$. By Ex. 2.4(c) we have that in both cases $\|\mathcal{M}_{D_n}\|_{X\to X} = \|D_n\|_{L^1(\mathbb{T})}$ which is unbounded in n (it behaves like $\log(n)$ as we have seen in the lecture and Ex. 1.3.
The second question in (a) can be answered affirmatively by setting $D = L^2(\mathbb{T})$ (recall why!).
(b): The assertion is stronger than what we have just shown for $X = C(\mathbb{T})$. Assume that the statement was true and denote by $E_t : C(\mathbb{T}) \to \mathbb{C}, f \mapsto f(t)$ the point-evaluation operator for $t \in \mathbb{T}$. Therefore for any $f \in C(\mathbb{T})$ and $t \in \mathbb{T}$, the sequence $(E_t\mathcal{M}_{D_n} f)_{n\in\mathbb{N}}$ would be bounded. Hence, by the uniform boundedness principle, for fixed $t \in \mathbb{T}$, the operators $E_t\mathcal{M}_{D_n}$ are uniformly bounded in n, i.e.

$$(\star\star) \qquad \sup_{n\in\mathbb{N}} \|E_t\mathcal{M}_{D_n}\|_{C(\mathbb{T})\to\mathbb{C}} = \sup_{n\in\mathbb{N}} \sup_{\|f\|_\infty=1} |E_t\mathcal{M}_{D_n} f| < \infty.$$

In (a) we have already observed that $\sup_{n\in\mathbb{N}} \|\mathcal{M}_{D_n}\|_{C(\mathbb{T})\to C(\mathbb{T})} = \infty$. Therefore, there exists a sequence $f_n \in C(\mathbb{T})$ such that $\|f_n\|_{C(\mathbb{T})} = 1$ and $\|\mathcal{M}_{D_n} f_n\|_{C(\mathbb{T})} \to \infty$ as $n \to \infty$. Since the maximum norm of a continuous function is attained at some point in $\mathbb{T}$, we also find a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{T}$ such that

$$|E_{t_n}\mathcal{M}_{D_n} f_n| = \|\mathcal{M}_{D_n} f_n\|_{C(\mathbb{T})} \to \infty.$$

Finally note that $E_t f = E_0\tau_{-t} f$ where $\tau_{-t} f = f(\cdot + t)$ and that $\mathcal{M}_g\tau_{-t} = \tau_{-t}\mathcal{M}_g$ (check!) and thus,

$$E_{t_n}\mathcal{M}_{D_n} f_n = E_0\mathcal{M}_{D_n}\tau_{-t_n} f_n.$$

Defining $h_n = \tau_{-t_n} f_n$ now gives a sequence in $C(\mathbb{T})$ with $\|h_n\|_{C(\mathbb{T})} = 1$ for all $n \in \mathbb{N}$ and $\|E_0\mathcal{M}_g h_n\|_{C(\mathbb{T})} \to \infty$. This contradicts $(\star\star)$.

Ex 2.6*: *(Approximate identity)*
Show that there exists no element e in $L^1(\mathbb{T})$ such that $e * f = f$ for all $f \in L^1(\mathbb{T})$.