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Fourier Analysis – Exercise sheet 5
(to be discussed on June 11)

Ex 5.1: *(Pointwise convergence of Dirchlet means for differentiable functions)*
Let $f \in L^1(\mathbb{T})$ be differentiable at $t_0 \in \mathbb{T}$. Then the partial sums $D_n * f$ of the Fourier series of f converge to $f(t_0)$ at t_0.
Hint: Use Ex. 4.1.

Ex 5.2: *(The maximum principle for entire functions)*
(*This exercise may be well-known for those who familiar with basic complex analysis*)
A function $f : \mathbb{C} \to \mathbb{C}$ is called *entire* if f is a complex power series with radius of convergence equal to ∞, i.e. there exists $(a_n)_{n\in\mathbb{N}}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \forall z \in \mathbb{C}.$$

(a) Show that for any entire f,

$$\max_{z\in\overline{\mathbb{D}}} |f(z)| = \max_{|z|=1} |f(z)|.$$

Hint: Follow the steps

(i) Reduce the claim to complex polynomials f of degree larger than 1.

(ii) Let $n > 1$ and $z \in \mathbb{D}$. Consider $U \in \mathbb{C}^{(n+1)\times(n+1)}$ defined by

$$U = \begin{pmatrix} z & 0 & \dots & 0 & \sqrt{1-|z|^2} \\ \sqrt{1-|z|^2} & 0 & \dots & 0 & \bar{z} \\ 0 & & & & 0 \\ \vdots & & I_{n-1} & & \vdots \\ 0 & & & & 0 \end{pmatrix}$$

where I_{n-1} denotes the identity matrix of dimension $(n-1)\times(n-1)$.
Show that U is unitary, i.e. $U^*U = UU^* = I_{n+1}$ [1] and that for polynomials f with $deg(f) = n$,

$$f(z) = P_1 f(U) P_1^T$$

where $P_1 = (1, 0, \dots, 0) \in \mathbb{C}^{1\times(n+1)}$. Conclude that $|f(z)| \le \|f(U)\|_{2\to2}$ where the operator norm is induced by the Euclidean norm.

(iii) Conclude the assertion by arguing why $\|f(U)\|_{2\to2} \le \max_{|z|=1} |f(z)|$ (use that U is unitary and the spectral theorem from linear algebra).

(b) Show that in (a) the set $\mathbb{D}$ can be replaced by any bounded, open, connected set Ω in $\mathbb{C}$, i.e.

$$\max_{z\in\overline{\Omega}} |f(z)| = \max_{\partial\Omega} |f(z)|,$$

where $\partial\Omega$ denotes the boundary of the open set Ω.
Hint: Assume that there exists $z \in \Omega$ such that $|f(z)| \ge \max_{\tilde{z}\in\overline{\Omega}} |f(\tilde{z})|$.

(c) (for people familiar with basic complex analysis) Show above statements for functions f that are analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ (or Ω and $\overline{\Omega}$ respectively).

[1] where $T^* = (\overline{t_{j,i}})_{i,j}$ denotes the hermitian transpose of the matrix $T = (t_{i,j})_{i,j}$.

Ex. 5.3: (Isoperimetric inequality in 2D) The goal of this exercise is to show the statement

For any closed, regular, nonself-intersecting, positively orientated C^1-curve Γ in $\mathbb{R}^2$ [2] *of length L and with enclosing area A the inequality*

$$(*) \qquad 4\pi A \le L^2$$

holds with equality if and only if the curve is a circle. Here, **regular** *means that $\gamma'(t) \neq 0$ for all $t \in \mathbb{T}$ for any C^1-parametrization $\gamma : \mathbb{T} \to \mathbb{R}^2$ of Γ.*

For that consider the following steps, where $\gamma : [0, 2\pi] \to \mathbb{R}^2$, with components γ_1 and γ_2, denotes a C^1-parametrization of Γ, see [1].

(a) Show that the area A enclosed by Γ equals

$$A = \frac{1}{2} \int_0^{2\pi} \gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s)\, ds.$$

Hint: Use Green's theorem / Stoke's theorem)

(b) *(Poincaré–Wirtinger inequality in 1D)*
Show that for $f \in C^1(\mathbb{T})$ (or more generally, for f being absolutely continuous with $f' \in L^2$) it holds that

$$\|f - \hat{f}(0)\|_{L^2(\mathbb{T})} \le \|f'\|_{L^2(\mathbb{T})}.$$

(c) Show $(*)$ in the case that $\|\gamma'(t)\|_2 = (\gamma_1(t)^2 + \gamma_2(t)^2)^{\frac{1}{2}} = 1$ for all $t \in \mathbb{T}$.
(Hint: Consider $f(s) = \gamma_1(s) + i\gamma_2(s)$, show that $A = \frac{1}{2} \operatorname{Im} \int_{\mathbb{T}} f'(s)\overline{f(s)}\, ds$ and note that $\int_{\mathbb{T}} f'(s)\, ds = 0$)

(d) Show why the assumption in (c) on γ can always be made by reparametrizing.
(Hint: $\gamma \rightsquigarrow \gamma \circ h^{-1}$ where $h(t) = \frac{1}{L} \int_0^t \|\gamma'(s)\|_2\, ds$.)

(e) Show the statement on the equality by investigating when equality holds in (b) and the inequalities in the proof of (c).

Ex. 5.4: (Young's inequality for convolutions) Prove Theorem 2.3 from the lecture for $\Omega = \mathbb{T}$.

Let $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ for $p, q, r \in [1, \infty]$ and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then $f * g \in L^r(\Omega)$ and

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Hint: Use Riesz–Thorin's theorem (and also Minkowski's inequality, Ex. 2.4).
Conclude why the statement also holds for $\Omega = \mathbb{R}$.

[2]here we mean that there exists a continuously differentiable $\gamma : [0, 2\pi] \to \mathbb{R}^2$ such that γ is injective on $[0, 2\pi)$, $\gamma'(t) \neq 0$ for all $t \in \mathbb{T}$, $\gamma(0) = \gamma(2\pi)$ and $\Gamma = \gamma(\mathbb{T})$. The length (or perimeter) L of Γ can be expressed as $L = \int_0^{2\pi} \|\gamma'(t)\|_2\, ds$.