Universität Hamburg
Fachbereich Mathematik
Dr. Felix Schwenninger
felix.schwenninger[ät]uni-hamburg.de

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Fourier Analysis – Exercise sheet 7
(discussed on July 9) — including sketches of the solutions

Important information:
This last exercise class will be held in a slightly different format than what has been the case so far: During the exercise class the participants should work (alone or in groups) on some given exercises and I will assist with problems and difficulties. Some exercises on the topics that are currently discussed in the lecture are already included below. Other exercises — dealing with topics that were already covered by previous exercise sheets — will be provided on Monday.

Ex 7.1: *Discuss whether the following objects are tempered distributions:*

(a) *the functionals*

$$L_f : \phi \mapsto \int_{\mathbb{R}} \phi(s) f(s)\, ds$$

for the choices of functions $f(x) = c$ for all $x \in \mathbb{R}$ or $f(x) = e^{x^2}$.

(b) L_f *for* $f \in L^p(\mathbb{R})$, $p \in [1, \infty]$.

(c) L_f *for* $f \in L^1_{loc}(\mathbb{R})$

(d) *the functional* δ_t *given by* $\phi \mapsto \phi(t)$ *for fixed* $t \in \mathbb{R}$

(e) L_μ *for any finite Borel measure* μ, *where* L_μ *is defined by*

$$L_\mu \phi = \int_{\mathbb{R}} \phi(s)\, d\mu(s)$$

(f) $L_{\log|\cdot|}$

(g) *the functional given by*

$$\phi \mapsto \lim_{\varepsilon \to 0^+} \int_{|s| \geq \varepsilon} \frac{\phi(s)}{s}\, ds$$

Solution: Yes, the functional, call it u is a tempered distribution. To see this, first observe that for $\phi \in S(\mathbb{R})$,

$$u(\phi) = \lim_{\varepsilon \to 0^+} \int_{|s| \geq \varepsilon} \frac{\phi(s)}{s}\, ds$$

is well-defined as the limit exists by the following argument: Since

$$\left| \int_{|s| \geq \varepsilon} \frac{\phi(s)}{s}\, ds \right| = \left| \int_{\varepsilon \leq |s| \leq 1} \frac{\phi(s)}{s}\, ds + \int_{|s| \geq 1} \frac{\phi(s)}{s}\, ds \right| = \left| \int_{\varepsilon \leq |s| \leq 1} \frac{\phi(s) - \phi(0)}{s}\, ds + \int_{|s| \geq 1} \frac{\phi(s)}{s}\, ds \right|$$

and $|\frac{\phi(s)-\phi(0)}{s}| \leq \|\phi'\|_{L^\infty(\mathbb{R})}$ for all $s \in (0,1)$ by Rolle's theorem, we conclude by dominated convergence that

$$\left| \int_{|s| \geq \varepsilon} \frac{\phi(s)}{s}\, ds \right| \leq 2\|\phi'\|_{L^\infty(\mathbb{R})} + \sup_{x \in \mathbb{R}} |x\phi(x)| \int_{|s| \geq 1} \frac{ds}{s^2} = 2\rho_{0,1}(\phi) + 4\rho_{1,0}(\phi)$$

(for the first term observe that the factor 2 comes from the integration), and where $\rho_{\alpha,\beta}(\phi) = \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta \phi(x)|$ are the seminorms that define the convergence on $S(\mathbb{R})$. Then it is also clear that if $\phi_n \overset{S}{\to} \phi$ as $n \to \infty$ — which means that $\rho_{\alpha,\beta}(\phi_n) \to \rho_{\alpha,\beta}(\phi)$ for all indices $\alpha, \beta \in \mathbb{N}_0$ — we have that $u(\phi_n) \to u(\phi)$. Thus, $u \in S'(\mathbb{R})$.

Ex 7.2:_(Derivative and Fourier transform of tempered distribution)_
Let $n \in \mathbb{N}$. We define the n-th derivative $\partial^n u \in S'(\mathbb{R})$ of a tempered distribution $u \in S'(\mathbb{R})$ by

$$\langle \partial^n u, \phi \rangle = \langle u, (-1)^n \partial^n \phi \rangle$$

(recall that by definition $\langle u, \phi \rangle = u(\phi)$). Also, define the Fourier transform $\mathcal{F}u$ by

$$\langle \mathcal{F}u, \phi \rangle = \langle u, \mathcal{F}\phi \rangle$$

and similarly the inverse Fourier transform by

$$\langle \mathcal{F}^\star u, \phi \rangle = \langle u, \mathcal{F}^\star \phi \rangle$$

Show the following

(a) _This definition of the Fourier transform is consistent with the definition we have seen for functions u in L^p for $p = 1$ and $p = 2$ (note what we have shown in Ex. 7.1)._
Sketch of solution: We have shown in the lecture that up to the identification of elements in $f \in L^1(\mathbb{R})$ as elements in $S'(\mathbb{R})$ via the functional L_f, see Ex.7.1., that

$$L_{\mathcal{F}f}(\psi) = \int_{\mathbb{R}} \mathcal{F}(f)(s)\psi(s)\,ds = \int_{\mathbb{R}} f(s)\mathcal{F}(\psi)(s)\,ds = \langle L_f, \mathcal{F}\psi \rangle$$

Thus, $\langle L_{\mathcal{F}f}, \psi \rangle = \langle L_f, \mathcal{F}\psi \rangle$ for all $\psi \in S(\mathbb{R})$ which shows that the distributional Fourier transform coincides with the definition on $L^1(\mathbb{R})$ and particularly on $S(\mathbb{R})$. For $f \in L^2$, we defined $\mathcal{F}$ by the unique bounded extension of the operator

$$\mathcal{F}|_{S(\mathbb{R})\to S(\mathbb{R})} : S(\mathbb{R}) \to S(\mathbb{R}), f \mapsto \mathcal{F}(f),$$

where boundedness refers to the inequality $\|\mathcal{F}(f)\|_{L^2(\mathbb{R})} \leq (2\pi)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})}$ for all $f \in S(\mathbb{R})$ and hence all $f \in L^2$. Therefore, let (ϕ_n) be a sequence of functions in $S(\mathbb{R})$ which converge (in L^2) to a given $f \in L^2$. Then by the first part,

$$\int_{\mathbb{R}} \mathcal{F}(\phi_n)(s)\psi(s)\,ds \overset{Def}{=} \langle L_{\mathcal{F}(\phi_n)}, \psi \rangle = \langle L_{\phi_n}, \mathcal{F}\psi \rangle \overset{Def}{=} \int_{\mathbb{R}} \phi_n(s)\mathcal{F}(\psi)(s)\,ds \qquad \forall n \in \mathbb{N}.$$

By Cauchy-Schwarz and the fact that $\phi_n \overset{L^2}{\to} f$ as well as $\mathcal{F}\phi_n \overset{L^2}{\to} \mathcal{F}f$, we conclude that $\langle L_{\mathcal{F}f}, \psi \rangle = \langle L_f, \mathcal{F}\psi \rangle$ for all $\psi \in S(\mathbb{R})$ which shows the assertion.

(b) _Compute the Fourier transform of $\partial\delta_0$ where δ_0 is defined as in Ex 7.2_
Solution: By definition and basics on the Fourier transform for $L^1(\mathbb{R})$-functions, we have for $\phi \in S(\mathbb{R})$ that

$$\langle \mathcal{F}\partial\delta_0, \phi \rangle = -\langle \delta_0, \partial\mathcal{F}(\phi) \rangle = -\langle \delta_0, \mathcal{F}(m_{-i\mathbf{t}}\phi) \rangle = -\mathcal{F}(m_{-i\mathbf{t}}\phi)(0) = \int_{\mathbb{R}} (it)\phi(t)\,dt = \langle L_{i\mathbf{t}}, \phi \rangle,$$

where we have used the notation $\mathbf{t} = t \mapsto t$ and $m_g f = t \mapsto g(t)f(t)$. Thus, the Fourier transform of $\partial\delta_0$ is given by the function $t \mapsto it$ (up to identification with a tempered distribution).

(c) _Compute the derivative of the function step function $f(s) = \begin{cases} 1 & |s| \leq 1 \\ 0 & |s| > 1 \end{cases}$_
Solution: By definition $\langle \partial f, \phi \rangle = -\langle f, \partial\phi \rangle$an since $f \in L^1$, this can be rewritten as

$$\langle f, \partial\phi \rangle = \int_{\mathbb{R}} f(s)\partial\phi(s)\,ds = \int_{-1}^{1} \partial\phi(s)\,ds = \phi(1) - \phi(-1) \overset{Def}{=} \langle \delta_1 - \delta_{-1}, \phi \rangle$$

for all $\phi \in S(\mathbb{R})$. Thus, $\partial f = \delta_{-1} - \delta_1$.

(d) _Compute the Fourier transform of the distributions defined by the functions_ sin _and_ cos.
Use that $\sin(t) = \frac{1}{2i}(\mathrm{e}^{it} - \mathrm{e}^{-it})$ and show first that $\mathcal{F}(\mathrm{e}^{ik\mathbf{t}}) = \delta_k$ for all $k \in \mathbb{R}$. Then, by linearity if follows that $\mathcal{F}(\sin) = \frac{1}{2i}(\delta_1 - \delta_{-1})$. For cos _one can proceed analogously or use the following argument. Since $\cos(t) = \sin(t + \frac{\pi}{2}) \overset{Def}{=} \tau_{\frac{\pi}{2}} \sin$ we have by basics of the Fourier transform (of $L^1(\mathbb{R})$-functions) that_

$$\langle \mathcal{F}L_{\cos}, \phi \rangle = \langle L_{\cos}, \mathcal{F}\phi \rangle = \langle L_{\tau_{\frac{\pi}{2}}\sin}, \mathcal{F}\phi \rangle \overset{*}{=} \langle L_{\sin}, \tau_{-\frac{\pi}{2}}\mathcal{F}\phi \rangle = \langle L_{\sin}, \mathcal{F}(m_{\mathrm{e}^{i\frac{\pi}{2}\mathbf{t}}}\phi) \rangle$$

where $(*)$ *follows from* $\int_{\mathbb{R}} f(t)[\tau_s g](t)\,dt = \int_{\mathbb{R}}[\tau_{-s}f](t)g(t)\,dt$*. By what we have shown for* $\sin$*, we get*

$$\langle L_{\sin}, \mathcal{F}(m_{e^{i\frac{\pi}{2}t}}\phi)\rangle = \frac{1}{2i}(e^{i\frac{\pi}{2}}\phi(1) - e^{-i\frac{\pi}{2}}\phi(-1)) = \frac{1}{2}(\phi(1)+\phi(-1)) = \langle\frac{1}{2}(\delta_1+\delta_{-1}),\phi\rangle.$$

Hence, $\mathcal{F}(\cos) = \frac{1}{2}(\delta_1+\delta_{-1})$*.*

Solutions to the additional exercises discussed in the Exercise class

Ex 7.3: *Prove that* $\|f\|^2_{L^\infty(\mathbb{R})} \le 2\|f\|_{L^p(\mathbb{R})}\|f'\|_{L^q(\mathbb{R})}$ *for all* $f \in S(\mathbb{R})$*,* $p,q\in[1,\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=1$*.*
Hint: Use the identity $f(t) = \int_{-\infty}^t \frac{\partial}{\partial s}(f(s)^2)\,ds$*, and apply the chain rule, as well Holder inequality.*

Ex 7.4:*(Show that the Fourier transform is not surjective as mapping from* $L^1(\mathbb{R})$ *to* $C_0(\mathbb{R}) = \{f:\mathbb{R}\to\mathbb{C}\mid f$ *continuous and* $\lim_{t\to\pm\infty} f(t)=0\}$*)*
To do so prove the following steps

(1) For all $0<\epsilon<T<\infty$*,* $\left|\int_\epsilon^T \frac{\sin t}{t}\,dt\right| \le 4$

Solution: Since $\sin t \le t$ for all $t\ge 0$, we have that $0 \le \int_0^\pi \frac{\sin t}{t}\,dt \le \pi$. It is also (geometrically) clear that $-2 \le \int_\pi^T \frac{\sin t}{t}\,dt \le 0$ for all $T>\pi$. This directly gives the assertion as $\pi\le 4$.

(2) For all $0<\epsilon<T<\infty$ *and* $f\in L^1(\mathbb{R})$ *with* $f(t)=-f(-t)$ *for a.e.* $t\in\mathbb{R}$*, it holds that* $\left|\int_\epsilon^T \frac{\mathcal{F}(f)(t)}{t}\,dt\right| \le 4\|f\|_{L^1(\mathbb{R})}$*.*

Solution: By definition of the Fourier transform and the property that f is odd, it follows that $\mathcal{F}(f)(t) = 2\int_0^\infty \sin(ts)f(s)\,ds$. Inserting this in $\left|\int_\epsilon^T \frac{\mathcal{F}(t)}{t}\,dt\right|$ and applying Fubini, as well as noting that $\sin(t)\frac{dt}{t}$ is invariant under scaling $t \rightsquigarrow \alpha t$, readily leads to the assertion (also note that $2\int_0^\infty |f(s)|ds = \|f\|_{L^1(\mathbb{R})}$ since f is odd).

(3) Conclude that there exists no function $f\in L^1(\mathbb{R})$ *such that* $\mathcal{F}(f)(s)=g(s)$ *for all* $s\in\mathbb{R}$ *where* g *is a continuous, odd function such that* $g(s)=\frac{1}{\log(s)}$ *for all* $s\ge 2$*.*

Solution: This follows by contraction. If such f exists, then consider the odd function $\tilde{f}$ defined by $\tilde{f}(t)=\frac{1}{2}(f(t)-f(-t))$. Since $R\mathcal{F}=\mathcal{F}R$, where R denotes the reflection operator $Rh = h(-\cdot)$, we have by linearity of $\mathcal{F}$ that $\mathcal{F}(\tilde{f})(s) = \frac{1}{2}(g(s)-g(-s)) = g(s)$ since g was assumed to be odd. Now apply part (2) for $\epsilon=2$ and conclude that for all $T>2$,

$$\left|\int_2^T \frac{g(t)}{t}\,dt\right| = \left|\int_2^T \frac{\mathcal{F}(f)(t)}{t}\,dt\right| \le 4\|f\|_{L^1(\mathbb{R})}$$

But, $g(t)=\frac{1}{\log(t)}$ for $t\ge 2$ by assumption which implies that $\lim_{T\to\infty}\int_2^T \frac{dt}{t\log(t)} = \infty$ (the latter follows for instance by the fact that $t\mapsto t\log(t)$ is strictly increasing on $(2,\infty)$ and hence $\int_2^T \frac{dt}{t\log(t)} \ge \sum_{n=2}\frac{1}{n\log(n)} = \infty$ where the last identity holds by Cauchy's condensation test)

Ex 7.5: *Show that the sequence* $(e^{in\cdot})_{n\in\mathbb{N}}$ *converges to* 0 *in* $S'(\mathbb{R})$*.*
Solution: We have to show that for any $\phi\in S(\mathbb{R})$, $\langle L_{e^{int}},\phi\rangle = \int_{\mathbb{R}} e^{int}\phi(t)\,dt$ converge to 0 as $n\to\infty$. This, however, follows since $\int_{\mathbb{R}} e^{int}\phi(t)\,dt = \mathcal{F}(\phi)(-n)$ and $\mathcal{F}(\phi)\in C_0(\mathbb{R})$ — the latter being a basic on the Fourier transform (in fact, it even holds that $\mathcal{F}(\phi)\in S(\mathbb{R})$ since $\phi\in S(\mathbb{R})$).
Note that the sequence does not converge with respect to any L^p-norm and hence also not in the topology of $S(\mathbb{R})$.

Ex 7.6:*(Uncertainty principle) Let* $f\in S(\mathbb{R})$*. Show that the following inequality holds*

$$\|f\|^2_{L^2(\mathbb{R})} \le C\inf_{x\in\mathbb{R}}\|(\cdot-x)f(\cdot)\|_{L^2(\mathbb{R})}\cdot\inf_{y\in\mathbb{R}}\|(\cdot-y)\mathcal{F}(f)(\cdot)\|_{L^2(\mathbb{R})}$$

where C *is an absolute constant.*

Solution: Fix $x \in \mathbb{R}$ and write $|f(t)|^2 = f(t)\overline{f(t)}\partial_t(t-x)$ and use integration by parts to obtain

$$\|f\|_{L^2}^2 = -\int_{\mathbb{R}} 2\Re[f(t)\partial_t \overline{f(t)}](t-x)\, dt$$

Estimating the real part by the modulus and using Cauchy-Schwarz gives

$$\|f\|_{L^2}^2 \leq 2\|(\cdot - x)f(\cdot)\|_{L^2}\|\partial_t f\|_{L^2(\mathbb{R})}.$$

By Parseval's identity, Lem. II.2.9, we have that

$$\|\partial_t f\|_{L^2(\mathbb{R})} = \sqrt{\frac{1}{2\pi}}\|\mathcal{F}(\partial_t f)\|_{L^2} = \sqrt{\frac{1}{2\pi}}\|s \mapsto is\mathcal{F}(f)(s)\|_{L^2} = \sqrt{\frac{1}{2\pi}}\|s \mapsto s\mathcal{F}(f)(s)\|_{L^2}$$

where the latter identity follows by basics of the Fourier transform. Altogether this gives

$$\|f\|_{L^2(\mathbb{R})}^2 \leq 2\sqrt{\frac{1}{2\pi}}\|(\cdot - x)f(\cdot)\|_{L^2}\|s \mapsto s\mathcal{F}(f)(s)\|_{L^2}.$$

Now let $y \in \mathbb{R}$ and apply this inequality to the function $t \mapsto f(t)\mathrm{e}^{-iyt}$ instead of f. This only chances the last term on the right-hand side: By basics of the Fourier transform we have that $\mathcal{F}(\mathrm{e}^{-iy\cdot} f)(s) = \mathcal{F}(f)(s+y)$ and hence

$$\|s \mapsto s\mathcal{F}(\mathrm{e}^{iy\cdot} f)(s)\|_{L^2} = \|(\cdot - y)\mathcal{F}(f)(\cdot)\|_{L^2},$$

which yields the assertion.