

A COARSE HALIN GRID THEOREM WITH APPLICATIONS TO QUASI-TRANSITIVE, LOCALLY FINITE GRAPHS

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ABSTRACT. We prove a coarse version of Halin’s Grid Theorem: Every one-ended, locally finite graph that contains the disjoint union of infinitely many rays as an asymptotic minor also contains the half-grid as an asymptotic minor. More generally, we show that the same holds for arbitrary (not necessarily one-ended or locally finite) graphs under additional, necessary assumptions on the minor-models of the infinite rays. This resolves a conjecture of Georgakopoulos and Papasoglu.

As an application, we show that every one-ended, quasi-transitive, locally finite graph contains the half-grid as an asymptotic minor and as a diverging minor. This in particular includes all locally finite Cayley graphs of one-ended finitely generated groups and solves a problem of Georgakopoulos and Papasoglu.

1. INTRODUCTION

An *end* of a graph G is an equivalence class of rays in G where two rays are equivalent if there are infinitely many pairwise disjoint paths between them in G . An end is *thick* if it contains infinitely many pairwise disjoint rays. The *half-grid* (i.e. the grid¹ on $\mathbb{N} \times \mathbb{N}$), is an obvious example: it has infinitely many disjoint rays, and they all belong to the same end, which hence must be thick.

One of the cornerstones of infinite graph theory is the following result of Halin [Hal65, Satz 4’], which asserts that this observation has a converse, and thus the half-grid can be thought of as a prototype for thick ends:

Halin’s Grid Theorem. *Every graph with a thick end contains the half-grid as a minor.*

However, the half-grid minor which Halin’s Grid Theorem provides can have arbitrary distortion compared to the geometry of the ambient graph. Consider for example the infinite clique $K_{\aleph_0}$. Its unique end is thick, and it clearly contains the half-grid as a minor. However, $K_{\aleph_0}$ is quasi-isometric² to a point, so $K_{\aleph_0}$ inherits none of the geometric properties of the half-grid. In contexts where the geometry matters it would be desirable to have a version of Halin’s Grid Theorem which preserves some of the geometry of the half-grid.

Let us first consider Cayley graphs, a class of graphs for which this problem is particularly interesting. In geometric group theory, where one is interested in the large-scale geometry of (the Cayley graph of) a group Γ , one wants the analysis of Γ to be stable under quasi-isometries and, in particular, independent

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¹In a grid two vertices (m, n) and (m', n') are adjacent if and only if $|m - m'| + |n - n'| = 1$

²Two graphs G, H are *quasi-isometric* if there exists a function $f: V(H) \rightarrow V(G)$ and $M \geq 1$ and $A \geq 0$ such that $M^{-1} \cdot d_H(u, v) - A \leq d_G(f(u), f(v)) \leq M \cdot d_H(u, v) + A$ for all $u, v \in V(H)$ and $d_G(f(V(H)), w) \leq A$ for all $w \in V(G)$.

of the choice of a finite generating system. Since having a thick end is a quasi-isometric invariant for Cayley graphs, it is natural to ask whether Cayley graphs with a thick end contain a minor-model of the half-grid that survives under quasi-isometries.

For this, we consider ‘fat’ and ‘asymptotic’ minors, a special kind of minors that play an important role in the new area of ‘coarse graph theory’ (see [GP25] for an overview). A graph X is a *K -fat* minor of a graph G , for some $K \in \mathbb{N}$, if there is a minor-model of X in G whose branch sets and branch paths are pairwise at least K apart, except that we do not require this for incident branch set-path pairs (cf. Section 2.3). The graph X is an *asymptotic minor* of G if X is a K -fat minor of G for every $K \in \mathbb{N}$.

In contrast to the usual minors, asymptotic minors are preserved under quasi-isometries [GP25]. In particular, if a graph contains the half-grid as an asymptotic minor, then its large-scale geometry will still resemble that of a graph with a thick end. Moreover, it does not depend on the choice of a finite generating set whether a locally finite Cayley graph of a finitely generated group contains a fixed graph as an asymptotic minor.

A natural question that arises is whether we can improve Halin’s Grid Theorem for Cayley graphs so that it yields an asymptotic half-grid minor. This question was already posed by Georgakopoulos and Papasoglu, who asked this for Cayley graphs of one-ended, finitely generated groups. For this, note that by a result of Thomassen [Tho92, Proposition 5.6], the unique end of a one-ended, locally finite Cayley graph must be thick.

Problem 1.1 ([GP25, Problem 7.3]). *Let G be a locally finite Cayley graph of a one-ended, finitely generated group. Must the half-grid be an asymptotic minor of G ?*

Our main result resolves Problem 1.1. In fact, instead of constructing a K -fat minor-model of the half-grid for every $K \in \mathbb{N}$, we find a single minor-model of the half-grid which is *ultra-fat*: for every $K \in \mathbb{N}$ its ‘submodel’ corresponding to the subgraph of the half-grid on the vertex set $\mathbb{N}^2 \setminus [K]^2$ is K -fat.

Theorem 1. *Every one-ended, quasi-transitive, locally finite graph G contains the half-grid as an ultra-fat minor. In particular, the half-grid is an asymptotic minor and a diverging minor of G .*

A graph G is *quasi-transitive* if the automorphism group of G acts on $V(G)$ with only finitely many orbits. Note that every Cayley graph of a finitely generated group is quasi-transitive.

Theorem 1 is also related to problems posed by Georgakopoulos and the second author [GH24], who asked for diverging minors of grids. A minor-model of a graph X in a graph G *diverges*, if for every two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of vertices and/or edges of X such that $d_X(x_n, y_n) \rightarrow \infty$, the distances in G between their corresponding branch sets or paths tend to infinity. It is easy to see that every ultra-fat minor-model of the half-grid diverges (see Proposition 2.2).

We remark that Theorem 1 lies at the frontier of what is still true: We cannot ask for coarse embeddings³ of the (hexagonal⁴) half-grid as e.g. the Cayley-graph of the lamplighter group, which is one-ended, does not contain a coarsely embedded (hexagonal) half-grid (cf. [GH24, Discussion preceding Problem 4.2]).

What about quasi-transitive, locally finite graphs that have more than one end? A quasi-transitive graph is *accessible* if there exists $n \in \mathbb{N}$ such that every two of its ends can be ‘separated’ by at most n vertices. By a standard argument about accessible, quasi-transitive graphs, we obtain that we may relax

³See Section 5.1 for the definition.

⁴See Section 3 for the definition.

the condition on G in [Theorem 1](#) from ‘one-ended’ to ‘accessible with a thick end’ (see [Corollary 6.2](#)). Since every connected, quasi-transitive graph without thick ends is quasi-isometric to a tree by Krön and Möller [[KM08](#), Theorem 5.5], this implies the following corollary:

Corollary 2. *Every accessible, connected, quasi-transitive, locally finite graph is quasi-isometric to a tree if and only if it does not contain the half-grid as an ultra-fat minor.*

[Theorem 1](#) has another application. Two rays S, R in a graph G *diverge* if for every $K \in \mathbb{N}$ there exist tails S', R' of S, R , respectively, such that $d_G(S', R') \geq K$. As the half-grid contains infinitely many pairwise disjoint rays, [Theorem 1](#) implies the following:

Corollary 3. *Every quasi-transitive, locally finite graph that contains infinitely many pairwise disjoint rays also contains infinitely many pairwise diverging rays.*

This solves [[GH24](#), Conjecture 4.3] for quasi-transitive graphs.

Let us now leave the realm of Cayley graphs and quasi-transitive graphs and consider arbitrary infinite graphs with thick ends. As mentioned earlier, not every graph with a thick end contains the half-grid as an asymptotic minor, and hence we can in general not hope to replace ‘minor’ with ‘asymptotic minor’ in [Halin’s Grid Theorem](#). Indeed, the already mentioned $K_{\aleph_0}$ is an obvious example, but there are also locally finite graphs that have a thick end but that do not contain the half-grid as an asymptotic minor.⁵ However, there is another formulation of [Halin’s Grid Theorem](#), which we will prove to have a coarse counterpart. For this, consider the following equivalent way of stating [Halin’s Grid Theorem](#) for one-ended graphs: *Let G be a one-ended graph. If $\omega \cdot R$ is a minor of G , then the half-grid is a minor of G .* Here, $\omega \cdot R$ denotes the disjoint union of countably many rays.

Since having $\omega \cdot R$ as an asymptotic minor is a necessary condition for having the half-grid as an asymptotic minor, but graphs with thick ends need in general not have $\omega \cdot R$ as an asymptotic minor, a more natural way to transfer [Halin’s Grid Theorem](#) into the coarse world is the following, as proposed by Georgakopoulos and Papasoglu:

Problem 1.2 ([[GP25](#), Problem 7.2]). *Let G be a one-ended graph. If $\omega \cdot R$ is an asymptotic minor of G , then the half-grid is an asymptotic minor of G .*

The problem is false for general graphs, as we will show by presenting a counterexample in [Section 4](#). However, we prove that [Problem 1.2](#) is true for locally finite graphs.

Theorem 4. *Let G be a one-ended, locally finite graph. If $\omega \cdot R$ is an asymptotic minor of G , then the half-grid is an asymptotic minor of G .*

We remark that we also prove a second version of [Theorem 4](#) (see [Theorem 3.1 \(2\)](#)) where we impose a stronger condition on the minor-model of $\omega \cdot R$, and in return obtain the half-grid as an ultra-fat minor. This result will be an important tool for our proof of [Theorem 1](#).

In fact, we will prove a more general version of [Theorem 4](#) for arbitrary infinite (not necessarily one-ended or locally finite) graphs (see [Theorems 3.2](#) and [3.3](#)), where we impose two further conditions on the

⁵Let G be the graph obtained from the ray $r_1 r_2 \dots$ by replacing each vertex r_i with a clique K_i of size i and each edge $r_i r_{i+1}$ by a complete bipartite graph between K_i and K_{i+1} . Then G is locally finite by definition, and it is easy to check that G has a thick end, but the half-grid is not even a 2-fat minor of G .

rays in the fat minor-models of $\omega \cdot R$. To cover graphs with more than one end, we assume that the rays in the fat minor-models of $\omega \cdot R$ are equivalent. Further, to cover graphs with vertices of infinite degree, we assume that no two distinct rays of the fat minor-model of $\omega \cdot R$ can be separated by removing finitely many balls of finite radius. While the first condition is clearly necessary, we will show in [Example 4.1](#) that also the second condition is necessary.

1.1. More related work. Our work complements existing work about coarse graph theory on infinite (quasi-transitive) graphs [[AH24](#); [EGG25](#); [GP25](#); [LMR23](#); [Leh25](#); [Mac25](#)]. It is also related to existing work on characterising quasi-transitive, locally finite graphs with thick ends, e.g. [[AH24](#); [Ant11](#); [GH24](#); [KM08](#)].

In [[AH24](#)] we showed that every quasi-transitive, locally finite graph with a thick end whose cycle space is generated by cycles of bounded length⁶ contains the full-grid (i.e. the grid on $\mathbb{Z} \times \mathbb{Z}$) as an asymptotic minor and as a diverging minor. Since every such graph is accessible, [Theorem 1](#) (more precisely [Corollary 2](#)) covers a larger class of graphs, but only yields an asymptotic and diverging minor of the half-grid. We remark that our proof techniques in this paper are mostly independent of [[AH24](#)].

Lee, Martínez-Pedroza and Rodríguez-Quinche [[LMR23](#)] introduced a coarse version of the classical cops and robbers game. Using the above mentioned result in [[AH24](#)], Esperet, Gahlawat and Giocanti [[EGG25](#)] proved for locally finite Cayley graphs of finitely presented groups that if one cop cannot catch the robber in the coarse game, then infinitely many cops are needed. Replacing in their proof the result in [[AH24](#)] by [Theorem 1](#), this yields the same for the larger class of accessible, finitely generated groups, a result that has directly been obtained by Lehner [[Leh25](#)].

Our proof of [Theorem 4](#) relies on a strengthening of [Halin's Grid Theorem](#), which was proved by Kurkofka, Melcher and Pitz in [[KMP22](#)].

1.2. How this paper is organised. This paper is structured as follows. In [Section 2](#) we recall some important definitions. We prove [Theorem 4](#) in [Section 3](#) and discuss some examples regarding coarse versions of [Halin's Grid Theorem](#) in [Section 4](#). In [Section 5](#) we deduce [Theorem 1](#) from [Theorem 3.1](#), our second version of [Theorem 4](#), and in [Section 6](#) we prove [Corollaries 2](#) and [3](#).

2. PRELIMINARIES

Our notions mainly follow [[Die24](#)]; in particular, $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$. In what follows, we recap some definitions which we need later. We denote the half-grid by HG .

2.1. Paths, rays and ends. For two sets A, B of vertices of a graph G , an A – B *path* meets A precisely in its first vertex and B precisely in its last vertex.

A *ray* is a one-way infinite path, and a *double ray* is a two-way infinite path. A *tail* of a (double) ray R is any ray $S \subseteq R$. If $R = r_0 r_1 \dots$ is a ray, then we denote by $r_i R r_j$ for $i, j \in \mathbb{N}$ the subpath $r_i \dots r_j$ of R , and by $r_i R$ or $R_{\geq i}$ the tail $r_i r_{i+1} \dots$ of R . Further, we denote by $R r_i$ or $R_{\leq i}$ the subpath $r_0 \dots r_i$ of R . We use these notions analogously for double rays; in particular, if $R = \dots r_{-1} r_0 r_1 \dots$ is a double ray, then $R r_i$ and $R_{\leq i}$ denote the tail $r_i r_{i-1} \dots$ of R .

A (possibly infinite) set $U \subseteq V(G)$ *separates* two rays R, S in G if U separates $V(R)$ and $V(S)$ in G , i.e. if $G - U$ does not contain an R – S path.

⁶See [[AH24](#), Section 2.3] or [[Die24](#), Section 1.9] for the definition. In particular, Cayley graphs of finitely presented groups have this property.

An *end* ε of a graph G is an equivalence class of rays in G where two rays are equivalent if they are joined by infinitely many disjoint paths in G or, equivalently, if for every finite set $U \subseteq V(G)$ both rays have tails in the same component of $G - U$. If a ray lies in ε , then we call it an ε -ray. An end ε is *thick* if there are infinitely many pairwise disjoint ε -rays [Hal65].

2.2. Distance and balls. Let G be a graph. We write $d_G(v, u)$ for the distance of the two vertices v and u in G . For two sets U and U' of vertices of G , we write $d_G(U, U')$ for the minimum distance of two elements of U and U' , respectively. If one of U or U' is just a singleton, then we omit the braces, writing $d_G(v, U') := d_G(\{v\}, U')$ for $v \in V(G)$. If X is a subgraph of G , then we abbreviate $d_G(U, V(X))$ as $d_G(U, X)$.

Given a set U of vertices of G , the *ball (in G) around U of radius $r \in \mathbb{N}$* , denoted by $B_G(U, r)$, is the set of all vertices in G of distance at most r from U in G . If $U = \{v\}$ for some $v \in V(G)$, then we omit the braces, writing $B_G(v, r)$ for the ball (in G) around v of radius r . Additionally, we abbreviate the induced subgraph on $B_G(U, r)$ of G with $G[U, r] := G[B_G(U, r)]$. If X is a subgraph of G , then we abbreviate $B_G(V(X), r)$ and $G[V(X), r]$ as $B_G(X, r)$ and $G[X, r]$, respectively.

2.3. Fat, asymptotic and diverging minors. Let G, X be graphs. A *(minor-)model* $(\mathcal{V}, \mathcal{E})$ of X in G is a collection $\mathcal{V}$ of disjoint sets $V_x \subseteq V(G)$ for vertices x of X such that each $G[V_x]$ is connected, and a collection $\mathcal{E}$ of internally disjoint $V_{x_0} - V_{x_1}$ paths E_e for edges $e = x_0 x_1$ of X which are disjoint from every V_x with $x \neq x_0, x_1$. The V_x are its *branch sets* and the E_e are its *branch paths*.

Let $K \in \mathbb{N}$. A model $(\mathcal{V}, \mathcal{E})$ of X in G is *K -fat* if $d_G(Y, Z) \geq K$ for every two distinct $Y, Z \in \mathcal{V} \cup \mathcal{E}$ unless $Y = E_e$ and $Z = V_x$ for some vertex $x \in V(X)$ incident to $e \in E(X)$, or vice versa. The graph X is a *K -fat minor* of G , denoted by $X \prec_K G$, if G contains a K -fat model of X . Moreover, X is an *asymptotic minor* of G , denoted by $X \prec_\infty G$, if X is a K -fat minor of G for all $K \in \mathbb{N}$.

Let ε be an end of G . If X is a one-ended graph, and $(\mathcal{V}, \mathcal{E})$ is a model of X in G , then we say that *all rays of $(\mathcal{V}, \mathcal{E})$ are ε -rays* if every ray in $\bigcup_{x \in V(X)} G[V_x] \cup \bigcup_{e \in E(X)} E_e$ is an ε -ray. We write $X \prec_K^\varepsilon G$ if G contains a K -fat model $(\mathcal{V}, \mathcal{E})$ of X whose rays are all ε -rays, and $X \prec_\infty^\varepsilon G$ if $X \prec_K^\varepsilon G$ for all $K \in \mathbb{N}$.

Let $(\mathcal{V}, \mathcal{E})$ be a model of a graph X in a graph G . Then $(\mathcal{V}, \mathcal{E})$ *diverges* (in G) if for every two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of vertices and/or edges of X such that $d_X(x_n, y_n) \rightarrow \infty$, we have $d_G(U_n, W_n) \rightarrow \infty$ where $U_n := V_{x_n}$ if $x_n \in V(X)$ and $U_n := V(E_{x_n})$ if $x_n \in E(X)$ and analogously $W_n := V_{y_n}$ or $W_n := V(E_{y_n})$.

2.4. Ultra-fat half-grid minors. Let $(\mathcal{V}, \mathcal{E})$ be a model of the half-grid in a graph G . Denote by $(\mathcal{V}_K, \mathcal{E}_K)$ the submodel of $(\mathcal{V}, \mathcal{E})$ that consists precisely of all branch sets in $\mathcal{V}$ which correspond to vertices (n, m) of the half-grid with $n \geq K$ or $m \geq K$ and of precisely all branch paths in $\mathcal{E}$ which correspond to edges $(n, m)(n', m')$ of the half-grid with $n, n' \geq K$ or $m, m' \geq K$, ie. $(\mathcal{V}_K, \mathcal{E}_K)$ corresponds to the induced subgraph of the half-grid with vertex set $\mathbb{N}^2 \setminus [K]^2$. We say that $(\mathcal{V}, \mathcal{E})$ is *ultra-fat* if for every $K \in \mathbb{N}$ the model $(\mathcal{V}_K, \mathcal{E}_K)$ is K -fat. The half-grid is an *ultra-fat minor* of G , denoted by $HG \prec_{UF} G$, if G contains an ultra-fat model of the half-grid. For an end ε of G , we write $HG \prec_{UF}^\varepsilon G$ if G contains an ultra-fat model $(\mathcal{V}, \mathcal{E})$ of HG whose rays are all ε -rays.

The following observation is immediate from the definitions:

Observation 2.1. *Let ε be an end of a graph G . If $HG \prec_{UF}^\varepsilon G$, then $HG \prec_\infty^\varepsilon G$.* □

Moreover, we have the following proposition:

Proposition 2.2. *Let G be a graph, and let $(\mathcal{V}, \mathcal{E})$ be an ultra-fat model of the half-grid in G . If all its branch sets are finite, then $(\mathcal{V}, \mathcal{E})$ diverges.*

Proof. The proof is analogous to [AH24, Proposition 3.2], we repeat the details here for clarity.

Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ with $x_n, y_n \in V(HG) \cup E(HG)$ for all $n \in \mathbb{N}$ such that $d_{HG}(x_n, y_n) \rightarrow \infty$. Let U_n, W_n be the branch sets or paths of $(\mathcal{V}, \mathcal{E})$ corresponding to x_n, y_n , respectively. Now suppose for a contradiction that $d_G(U_n, W_n)$ does not tend to infinity. Then we may assume, by restricting to subsequences, that $d_G(U_n, W_n) = K$ for some $K \in \mathbb{N}$ and all $n \in \mathbb{N}$. Since $(\mathcal{V}, \mathcal{E})$ is ultra-fat, this implies that, for all $n \in \mathbb{N}$, at least one of x_n, y_n is contained in $HG[[K]^2]$. Hence, one of $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$, say $(x_n)_{n \in \mathbb{N}}$, is eventually constant, so we may assume that $x_n = z$ for all $n \in \mathbb{N}$ and some $z \in V(HG) \cup E(HG)$. Since $d_G(x_n, y_n) \rightarrow \infty$, it follows (after possibly restricting to a subsequence of $(y_n)_{n \in \mathbb{N}}$) that the y_n are pairwise distinct. Set $U_z := V_z$ if $z \in V(HG)$ or $U_z := V(E_z)$ if $z \in E(HG)$. As U_z is finite and $d_G(U_z, W_n) = K$ for all $n \in \mathbb{N}$, there is some $v \in U_z$ and an infinite index set $I \subseteq \mathbb{N}$ such that $d_G(v, W_n) = K$ for all $n \in I$. Hence, $d_G(W_n, W_m) \leq 2K$ for all $n, m \in \mathbb{N}$. But since I is infinite and the W_n are pairwise distinct, this contradicts that $(\mathcal{V}, \mathcal{E})$ is ultra-fat. $\square$

3. PROOF OF COARSE HALIN GRID THEOREM

In this section we prove two coarse versions of [Halin's Grid Theorem](#). Let us first state them for locally finite graphs.

Theorem 3.1. *Let ε be an end of a locally finite graph G .*

- (1) *If $\omega \cdot R \prec_{2K-1}^\varepsilon G$ for some $K \in \mathbb{N}$, then $HG \prec_K^\varepsilon G$. In particular, if $\omega \cdot R \prec_\infty^\varepsilon G$, then $HG \prec_\infty^\varepsilon G$.*
- (2) *If G contains ε -rays $R_0, R_1, \dots$ such that $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j$, then $HG \prec_{UF}^\varepsilon G$.*

We remark that we show in [Example 4.2](#) that the $2K - 1$ in (1) is necessary.

The first part of [Theorem 3.1](#) immediately implies [Theorem 4](#).

Proof of Theorem 4 given Theorem 3.1. Since G is one-ended, $\omega \cdot R \prec_\infty G$ implies $\omega \cdot R \prec_\infty^\varepsilon G$ for the unique end ε of G . The assertion now follows from [Theorem 3.1](#) (1). $\square$

In fact, we prove a more general version of [Theorem 3.1](#) (see [Theorems 3.2](#) and [3.3](#) below) for not necessarily locally finite graphs, where we impose a further condition on the rays in the fat model of $\omega \cdot G$.

Since a K -fat model of the half-grid contains between any two of its rays infinitely many paths that are pairwise at least K apart, a necessary condition for containing a K -fat model of the half-grid whose rays are all ε -rays is that there are infinitely many ε -rays which cannot be separated by removing finitely many sets of vertices that all have diameter at most $K - 1$. We will present in [Example 4.1](#) a graph that verifies this statement, showing that [Theorem 3.1](#) is false for arbitrary (not locally finite) graphs without this additional assumption. Conversely, we will prove that a slightly stronger condition already suffices to guarantee a K -fat half-grid minor.

Before we can state our two coarse versions of [Halin's Grid Theorem](#) for arbitrary graphs, we need another definition. We say that a ray $R = r_0 r_1 \dots$ in a graph G is *K -fat* for some $K \in \mathbb{N}$ if for every $n \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that $d_G(Rr_n, r_N R) \geq K$. It is easy to see that this is the case if and only if there exist (finite) sets $B_i \subseteq V(R)$ such that $\bigcup_{i \in \mathbb{N}} B_i = V(R)$ and such that the B_i with even i form the branch sets and the B_i with odd i form the branch paths of a K -fat model of the ray (cf. [\[GP25, Lemma 6.2\]](#)). Moreover, every ray in a locally finite graph is K -fat for every $K \in \mathbb{N}$.

Theorem 3.2. *Let ε be an end of a graph G , and let $K \in \mathbb{N}$. Suppose that there is an infinite collection $\mathcal{R}$ of $(2K - 1)$ -fat ε -rays in G that are pairwise at least $2K - 1$ apart and such that no two rays in $\mathcal{R}$ can be separated in G by removing finitely many balls of radius at most $4K - 4$ in G . Then $HG \prec_K^\varepsilon G$.*

Theorem 3.3. *Let ε be an end of a graph G , and suppose that there are ε -rays $R_0, R_1, \dots$ in G that are K -fat for all $K \in \mathbb{N}$ and such that $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$ and such that no two rays R_i, R_j can be separated in G by removing finitely many balls of finite radius. Then $HG \prec_{U_F}^\varepsilon G$.*

Let us first show that Theorems 3.2 and 3.3 imply Theorem 3.1.

Proof of Theorem 3.1 given Theorems 3.2 and 3.3. (1): Take any model $(\mathcal{V}, \mathcal{E})$ of $\omega \cdot R$ that witnesses $\omega \cdot R \prec_{2K-1}^\varepsilon G$. Since every vertex of $\omega \cdot R$ has degree at most 2, we may assume that the branch sets in $\mathcal{V}$ are finite paths. Then $(\mathcal{V}, \mathcal{E})$ yields an infinite collection $\mathcal{R}$ of $(2K - 1)$ -fat ε -rays that are pairwise at least $2K - 1$ apart. Since all rays in $\mathcal{R}$ are ε -rays, no two of them can be separated by removing finitely many vertices. Hence, as G is locally finite, they can neither be separated by removing finitely many balls of radius at most $4K - 4$. Thus, $\mathcal{R}$ satisfies the premise of Theorem 3.2, and so $HG \prec_K^\varepsilon G$ as desired.

(2): Since G is locally finite, every ray R_i is K -fat for every $K \in \mathbb{N}$. Moreover, since all rays R_i are ε -rays, no two of them can be separated by removing finitely many vertices. As G is locally finite, they can neither be separated by removing finitely many balls of finite radius. Thus, the R_i satisfy the premise of Theorem 3.3, and so $HG \prec_{U_F}^\varepsilon G$ as desired. $\square$

The remainder of this section is devoted to the proofs of Theorems 3.2 and 3.3. The *hexagonal half-grid* is obtained from the half-grid by deleting every other rung, as shown in Figure 3.1. We call the rays S_i of the (hexagonal) half-grid its *vertical rays* and the edges e_i its *horizontal edges* (see Figure 3.1).

A *subdivision* of a graph H is a graph which arises from H by replacing every edge in H by a new path between its endvertices such that no new path has an inner vertex in $V(H)$ or on any other new path. The original vertices of H are the *branch vertices* of the subdivision and the new paths are its *branch paths*. If H is a subdivision of the (hexagonal) half-grid, we call its branch paths corresponding to horizontal edges *horizontal paths*.

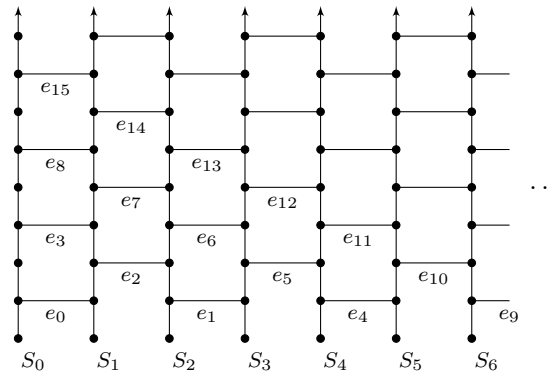


FIGURE 3.1. The hexagonal half-grid with vertical rays S_i and horizontal edges e_i .

For the proofs of Theorems 3.2 and 3.3, it will be more convenient to first construct a subdivision of the hexagonal half-grid and then contract it into a model of the half-grid. Since Theorems 3.2 and 3.3 yield fat models of the half-grid, we want to find a subdivision of the hexagonal half-grid whose vertical

rays S_i are fat and pairwise far apart and whose horizontal paths are far apart from all other horizontal paths and from all vertical rays except those two rays that contain its endvertices.

For this, consider the following strengthening of [Halin's Grid Theorem](#) by Kurkofka, Melcher and Pitz.

Theorem 3.4 ([KMP22, Theorem 1]). *For every infinite collection $\mathcal{R}$ of disjoint equivalent rays in a graph G there is a subdivision of the hexagonal half-grid in G such that all its vertical rays belong to $\mathcal{R}$.*

[Theorem 3.4](#) comes somewhat close to what we want: If we apply [Theorem 3.4](#) to the collection $\mathcal{R}$ of ε -rays provided by the premise of [Theorem 3.2](#) or [3.3](#), then we obtain a subdivision H of the hexagonal half-grid in G whose vertical rays belong to $\mathcal{R}$, and which are thus fat and pairwise far apart.

However, we also need that the horizontal paths P_i of H are pairwise far apart and that they are far apart from all vertical rays S_i (except those two containing their endvertices). While the former property is easily achievable by restricting to a subset of the P_i at least for locally finite graphs, the latter property cannot be achieved by post-processing the given subdivision H . Indeed, it might even happen that for all $i < j \in \mathbb{N}$ all S_i – S_{i+1} paths P_k in H have distance 1 from S_j , so we cannot obtain a hexagonal half-grid with the desired distance constraints by removing some of the paths and rays from H .

To solve this problem, we apply [Theorem 3.4](#) to a suitable auxiliary graph G' . Let us briefly sketch the idea for locally finite graphs. In that case, we may take as G' the graph obtained from G by ‘contracting’ large-radius balls around the rays $R \in \mathcal{R}$ onto the rays R . We then apply [Theorem 3.4](#) to $\mathcal{R}$ in G' , which yields a subdivision H' of the hexagonal half-grid in G' . By the definition of G' , each horizontal path of H' ‘lifts’ to a path in G that is far apart from all vertical rays of H' except for the two which it links. To obtain the desired subdivision H of the hexagonal half-grid in G , we then only need to choose the horizontal paths for H among the ‘lifts’ of the horizontal paths of H' so that they are pairwise far apart, which we can easily do recursively in locally finite graphs. To obtain [Theorems 3.2](#) and [3.3](#) also for graphs that are not locally finite, we still follow this approach, but we need to define G' more carefully.

We will find the subdivisions of the hexagonal half-grid for [Theorems 3.2](#) and [3.3](#) simultaneously, by showing the following lemma ([Lemma 3.5](#)). For the proofs of [Theorems 3.2](#) and [3.3](#), it then suffices to apply [Lemma 3.5](#) with $K_i := K$ for all $i \in \mathbb{N}$ or with $K_i := i$ for all $i \in \mathbb{N}$, and then contract the given subdivision of the hexagonal half-grid appropriately into a half-grid.

Lemma 3.5. *Let $K_0 \leq K_1 \leq \dots \in \mathbb{N}$, and let $R_0, R_1, \dots$ be rays in a graph G that are $(2K_i - 1)$ -fat for all $i \in \mathbb{N}$ such that $d_G(R_i, R_j) \geq K_i + K_j - 1$ for all $i \neq j \in \mathbb{N}$ and such that no two rays R_i, R_j can be separated in G by deleting finitely many balls of radius $4K_k - 4$ for any $k \in \mathbb{N}$. Then there exists a subdivision H of the hexagonal half-grid in G such that*

- (i) *the set $\mathcal{S} = \{S_0, S_1, \dots\}$ of vertical rays in H is a subset of $\{R_0, R_1, \dots\}$,*
- (ii) *every horizontal path P in H with endvertices in S_i, S_{i+1} has distance at least K_k from S_k in G for all rays $S_k \in \mathcal{S} \setminus \{S_i, S_{i+1}\}$,*
- (iii) *for every horizontal path P in H with an endvertex in S_i , the subgraph $P \cap G[S_i, K_i - 1]$ is a path of length $K_i - 1$, and*
- (iv) *every two distinct horizontal paths in H with endvertices in S_i, S_{i+1} and S_j, S_{j+1} , respectively, are at least K_n apart in G where $n := \min\{K_i, K_j\}$.*

Proof. We first construct an auxiliary graph G' , to which we will apply [Theorem 3.4](#). For this, we recursively choose R_{k_i} – R_{ℓ_i} paths P_i in G , where $((k_i, \ell_i))_{i \in \mathbb{N}}$ is some sequence in $\mathbb{N}^2$ containing every $(n, m) \in \mathbb{N}^2$ with $n \neq m$ infinitely often, such that for all $i < j \in \mathbb{N}$

- (1) $d_G(P_i, P_j) \geq 4K_j - 3$,
- (2) if $d_G(P_i, r), d_G(P_j, s) < K_k$ for some $k \in \mathbb{N}$ and $r, s \in V(R_k)$, then s appears on R_k after r .

Let $j \in \mathbb{N}$ and assume that we have already chosen P_i for all $i < j$. Since $d_G(R_k, R_\ell) \geq K_k + K_\ell - 1$ for all $k \neq \ell \in \mathbb{N}$, there exists some $M \in \mathbb{N}$ such that no ray R_k with $k > M$ has distance less than K_k to any P_i with $i < j$. As the rays R_k are $(2K_k - 1)$ -fat, there is for every $k \leq M$ some vertex r_k such that $r_k R_k$ has distance at least K_k from all P_i with $i < j$. Since deleting finitely many balls of radius $4K_n - 4$ does not separate R_{k_j} and R_{ℓ_j} in G for any $n \in \mathbb{N}$ by assumption, there is an R_{k_j} – R_{ℓ_j} path P_j in G which avoids $\bigcup_{i < j} B_G(P_i, 4K_j - 4)$ and $\bigcup_{k \leq M} B_G(R_k r_k, K_k - 1)$. Clearly, P_j satisfies (1) and (2).

We now define an auxiliary graph G' . Its vertex set is $V(G') := \bigcup_{i \in \mathbb{N}} (V(R_i) \cup V(P'_i))$ where $P'_i := P_i - \bigcup_{k \in \mathbb{N}} B_G(R_k, K_k - 1)$ for all $i \in \mathbb{N}$. We put an edge uv between $u, v \in V(G')$ in the following three cases: 1) if $uv \in E(G)$, 2) if $u \in V(R_k)$ and $v \in V(P'_i)$ for some $k, i \in \mathbb{N}$ and $d_G(u, v) = K_k$, and 3) if $u \in V(R_k)$ and $v \in V(R_\ell)$ for some $k \neq \ell \in \mathbb{N}$ and some $B_G(R_k, K_k - 1)$ – $B_G(R_\ell, K_\ell - 1)$ edge is contained in P_i for some $i \in \mathbb{N}$.

By definition, $R_i \subseteq G'$ for all $i \in \mathbb{N}$. We claim that the R_i are still equivalent in G' . Indeed, let R_k, R_ℓ be given, and let P_i, P_j , with $i < j$, be R_k – R_ℓ paths. Let $I := \{n \in \mathbb{N} : d_G(R_n, P_i) < K_n\}$. Since P_i is connected, it follows by the definition of G' that $G'_i := G'[V(P'_i) \cup \bigcup_{n \in I} V(R_n)]$ is connected. Let Q_i be an R_k – R_ℓ path in G'_i , and let Q_j be defined analogously for P_j . Then Q_i, Q_j are disjoint. Indeed, P'_i and P'_j are disjoint by (1), and any subpath of Q_i in some R_n is disjoint from any subpath of Q_j in R_n since the endvertices of edges of G'_j incident with R_n appear on R_n after the endvertices of edges of G'_i incident with R_n by (2). Since infinitely many P_i are R_k – R_ℓ paths, R_k, R_ℓ are equivalent in G' .

We now apply [Theorem 3.4](#) in G' to $\mathcal{R} := \{R_0, R_1, \dots\}$, which yields a subdivision H' of the hexagonal half-grid in G' . By [Theorem 3.4](#), the vertical rays of H' are a subset $\mathcal{R}'$ of $\mathcal{R}$. We first choose rays $R_{i_0}, R_{i_1}, \dots \in \mathcal{R}'$ such that $i_0 < i_1 < \dots$ and such that, for all $j < k \in \mathbb{N}$, the ray R_{i_k} appears in H' as a vertical ray after R_{i_j} . Set $S_j := R_{i_j}$ for all $j \in \mathbb{N}$, and let $\mathcal{S} := \{S_0, S_1, \dots\}$. Further, set $K'_j := K_{i_j}$ for all $j \in \mathbb{N}$, and note that $K'_j \geq K_j$. It is now straightforward to find a subdivision $H'' \subseteq H' (\subseteq G')$ of the hexagonal half-grid whose i -th vertical is S_i for every $i \in \mathbb{N}$.

We now turn H'' into a subdivision H of the hexagonal half-grid in G . Its vertical rays will be precisely the rays in $\mathcal{S}$ (in that order), and hence H will satisfy (i). Before we choose suitable horizontal paths for H , we first turn every horizontal path in H'' into a path in G . For this, let $W'_i = w'_0 \dots w'_n$ be a horizontal path of H'' with endvertices in $S_{k_i}, S_{k_i+1} \in \mathcal{S}$. By the definition of G' , every edge f of W'_i which is not an edge of G is either of the form uv for some vertex v in some $R_j \in \mathcal{R}$ and a vertex $u \in B_G(v, K_j) \setminus B_G(v, K_j - 1)$ or of the form uv for vertices u, v on distinct rays $R_j, R_k \in \mathcal{R}$ such that $B_G(u, K_j - 1)$ and $B_G(v, K_k - 1)$ are joined by an edge e in some P_ℓ . In the former case we replace uv by a u – v path in G of length K_j , and in the latter case we replace uv by a u – v path in G of length $K_j + K_k - 1$ containing e . We then let W_i be some w'_0 – w'_n path in the resulting w'_0 – w'_n walk in G .

Since W'_i is a horizontal path of H'' , it avoids all rays $S_m \in \mathcal{S}$ except for S_{k_i}, S_{k_i+1} . By the definition of G' , it follows that $V(W'_i)$ has distance at least K'_m in G from all $S_m \in \mathcal{S}$ with $S_m \neq S_{k_i}, S_{k_i+1}$. As $d_G(R_j, R_k) \geq K_j + K_k - 1$ for all $R_j \neq R_k \in \mathcal{R}$, and every vertex of $W_i - V(W'_i)$ has distance at most

$K'_{k_i} - 1$ or $K'_{k_i+1} - 1$ from S_{k_i}, S_{k_i+1} , respectively, or distance at most $K_j - 1$ from some ray $R_j \in \mathcal{R} \setminus \mathcal{S}$, the path W_i in G has distance at least K'_j from S_j for every $S_j \in \mathcal{S} \setminus \{S_{k_i}, S_{k_i+1}\}$.

We now choose for H for every horizontal edge e_i of the hexagonal half-grid a branch path E_i , in the order $e_0, e_1, e_2, \dots$ indicated in Figure 3.1. In fact, we will choose E_i among the paths W_j joining the respective vertical rays of H . By the construction above and because $K'_i \geq K_i$ for all $i \in \mathbb{N}$, this immediately implies that H will satisfy (ii) and (iii).

So let $j \geq 0$ be given, and assume that for every $i < j$, we have already chosen some W_{ℓ_i} to be the branch path E_i corresponding to the edge e_i of the hexagonal half-grid. Let $k_j \in \mathbb{N}$ such that S_{k_j}, S_{k_j+1} correspond to the vertical rays of the hexagonal half-grid that are joined by e_j . Since $d_G(P_i, P_k) \geq 4K_{k_j} - 3$ for all $i, k \geq k_j$ by (1), every vertex of some E_i has distance less than $2K_{k_j} - 1$ to at most one path P_k with $k \geq k_j$. As $\bigcup_{i < j} V(E_i)$ is finite, it follows that at most finitely many P_k have distance less than $2K_{k_j} - 1$ from some E_i with $i < j$. Since the paths W'_ℓ are disjoint, all but finitely many of them avoid these finitely many P_k . Hence, by the definition of the W_ℓ , all but finitely many of the paths W_ℓ that link S_{k_j} and S_{k_j+1} in H'' satisfy the following condition:

(a) For every $k \in \mathbb{N}$, if $P_k \cap W_\ell \neq \emptyset$, then $d_G(P_k, E_i) \geq 2K_{k_j} - 1$ for all $i < j$.

As $d_G(R_i, R_k) \geq K_i + K_k - 1$ for all $i \neq k \in \mathbb{N}$, and because the E_i are finite, there is some $L \in \mathbb{N}$ such that $d_G(R_k, E_i) \geq K_{k_j}$ for all $k > L$. Since the R_k are $(2K_{k_j} - 1)$ -fat, at most finitely many vertices of R_k have distance less than K_{k_j} from some E_i . Hence, and again because the R_k are $(2K_k - 1)$ -fat, there exists for every $k \in \mathbb{N}$ some $r_k \in V(R_k)$ such that, for every $i < j$, the tail $r_k R_k$ has distance at least K_{k_j} from E_i , and distance at least $2K_k - 1$ from all vertices of R_k that have distance less than K_k from E_i . Then $\bigcup_{k < L} R_k r_k$ is finite, and hence, as the paths W'_ℓ are disjoint, all but finitely many of them avoid $\bigcup_{k < L} R_k r_k$. Thus, by the definition of the W_ℓ , all but finitely many of the paths W_ℓ that link S_{k_j} and S_{k_j+1} in H'' satisfy the following conditions:

(b) For every $k \in \mathbb{N}$, if $R_k \cap W_\ell \neq \emptyset$, then $d_G(R_k \cap W_\ell, E_i) \geq K_{k_j}$ for all $i < j$.

(c) For every $i, k \in \mathbb{N}$ with $i < j$, and every $r, s \in V(R_k)$, if $B_G(r, K_k - 1) \cap V(W_\ell) \neq \emptyset$ and $B_G(s, K_k - 1) \cap V(E_i) \neq \emptyset$, then $d_G(r, s) \geq 2K_k - 1$.

Since, for all three conditions (a) to (c) separately, all but finitely many of the paths W_ℓ that link S_{k_j} and S_{k_j+1} in H'' satisfy the respective condition, there is some such W_ℓ that satisfies (a) to (c) simultaneously, and whose endvertices in S_{k_j}, S_{k_j+1} appear on S_{k_j}, S_{k_j+1} , respectively, after all endvertices of branch paths E_i with $i < j$. We set $\ell_j := \ell$ and $E_j := W_\ell$.

In this way, we construct a subdivision H of the hexagonal half-grid in G that satisfies (i) to (iii). We claim that H also satisfies (iv). Indeed, let $i < j$ be given, and let $v_j \in V(E_j)$. If $v_j \in B_G(P_k, K_{k_j} - 1)$ for some $k \in \mathbb{N}$, then $d_G(v_j, E_i) \geq K_{k_j}$ follows by (a). Hence, as $E_j = W_{\ell_j}$ and by the definition of W_{ℓ_j} , every other vertex $v_j \in V(E_j)$ is contained in $B_G(R_k, K_k - K_{k_j})$ for some $k \in \mathbb{N}$. If $v_j \in V(R_k)$, then $d_G(v_j, E_i) \geq K_{k_j}$ follows by (b). Hence, we may assume $v_j \in B_G(R_k, K_k - K_{k_j}) \setminus V(R_k)$, which in particular implies that $K_{k_j} < K_k$. Let also $v_i \in V(E_i)$. If $v_i \notin B_G(R_k, K_k - 1)$, then $d_G(v_i, v_j) \geq K_{k_j}$ follows since $v_j \in B_G(R_k, K_k - K_{k_j})$ and $K_k > K_{k_j}$. Hence, we may assume that $v_i \in B_G(R_k, K_k - 1)$. Let $r, s \in V(R_k)$ such that $v_i \in B_G(r, K_k - 1)$ and $v_j \in B_G(s, K_k - K_{k_j})$. Then $d_G(r, s) \geq 2K_k - 1$ by (c), and thus $d_G(v_i, v_j) \geq d_G(r, s) - d_G(v_i, r) - d_G(v_j, s) \geq (2K_k - 1) - (K_k - 1) - (K_k - K_{k_j}) = K_{k_j}$. This completes the proof that H satisfies (iv), and hence concludes the proof. $\square$

We can now prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. Without loss of generality assume that the collection $\mathcal{R}$ is countably infinite and enumerated as $\mathcal{R} := \{R_0, R_1, \dots\}$. By the assumptions on $\mathcal{R}$, the rays R_i satisfy the premise of Lemma 3.5 with $K_i := K$ for all $i \in \mathbb{N}$. Hence, applying Lemma 3.5 yields a subdivision H of the hexagonal half-grid in G that satisfies (i) to (iv). It is now straight forward to check that H can be contracted into a K -fat model $(\mathcal{V}, \mathcal{E})$ of the half-grid (where the branch paths in $\mathcal{E}$ corresponding to horizontal edges of the half-grid are a subset of the horizontal paths of H , and the branch sets and paths corresponding to vertices and edges of the i -th vertical ray $\{i\} \times \mathbb{N}$ of the half-grid are suitable subpaths of S_i). Indeed, (i) combined with $d_G(R_i, R_j) \geq 2K - 1 \geq K$ for all $R_i \neq R_j \in \mathcal{R}$ ensures that branch sets and paths of vertices and edges of the half-grid on distinct vertical rays have distance at least K in G . Moreover, (iv) ensures that every two distinct horizontal branch paths have distance at least K , and (ii) yields that horizontal branch paths have distance at least K from branch sets and paths on vertical rays that are not joined by that branch path. Finally, (i) together with the rays in $\mathcal{R}$ being $(2K - 1)$ -fat ensures that we may choose $(\mathcal{V}, \mathcal{E})$ so that branch sets and paths on the same ray S_i have distance at least K from each other, and by (iii) we can also choose them so that they have distance at least K from branch paths corresponding to non-incident horizontal edges that have an endvertex on the same vertical ray. $\square$

Proof of Theorem 3.3. By restricting the sequence $(R_i)_{i \in \mathbb{N}}$ to those rays R_i whose index is even and renumbering them (i.e. ' $R_i := R_{2i}$ '), we may assume that $d_G(R_i, R_j) \geq \max\{2i, 2j\} \geq i + j - 1$. Then the R_i satisfy the premise of Lemma 3.5 with $K_i := i$ for all $i \in \mathbb{N}$. Hence, applying Lemma 3.5 yields a subdivision H in G of the hexagonal half-grid that satisfies (i) to (iv). Analogously to the previous proof (of Theorem 3.2), it is straightforward to check that H can be contracted into a model $(\mathcal{V}, \mathcal{E})$ of the half-grid where the fact that $K_i = i$ instead of $K_i = K$ and that the rays R_i are K_i -fat for all $i \in \mathbb{N}$ yields that for every $i \in \mathbb{N}$, the submodel of $(\mathcal{V}, \mathcal{E})$ corresponding to $\mathbb{N}_{\geq i} \times \mathbb{N}$ is i -fat.

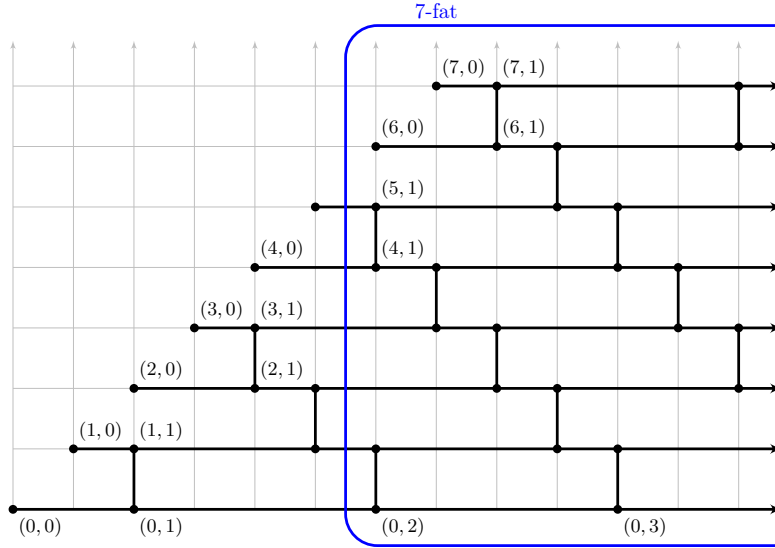


FIGURE 3.2. The model $(\mathcal{V}', \mathcal{E}')$ of the hexagonal half-grid in the proof of Theorem 3.3.

We now use $(\mathcal{V}, \mathcal{E})$ to find an ultra-fat model of the half-grid. For this, we first define a model $(\mathcal{V}', \mathcal{E}')$ of the hexagonal half-grid as indicated in Figure 3.2. For every vertex $(n, m) \in \mathbb{N}^2$, its branch set $V'_{(n,m)}$ is equal to $V_{(j,n)}$ for some suitable $j \in \mathbb{N}$, every horizontal branch path $E'_{(n,m)(n+1,m)}$ is equal to some

‘vertical’ branch path of the form $E_{(i,j)(i,j+1)}$, and every ‘vertical’ branch path $E'_{(n,m)(n,m+1)}$ is contained in $\bigcup_{j \in \mathbb{N}} (G[V_{(j,n)}] \cup E_{(j,n)(j+1,n)})$ (see Figure 3.2). As the submodel of $(\mathcal{V}, \mathcal{E})$ corresponding to $\mathbb{N}_{\geq K} \times \mathbb{N}$ is K -fat for all K , the submodel of $(\mathcal{V}', \mathcal{E}')$ corresponding to $\mathbb{N}^2 \setminus [K]^2$ is K -fat for all $K \in \mathbb{N}$. It is now easy to check that $(\mathcal{V}', \mathcal{E}')$ can be contracted into a model of the half-grid which still has this property, and which is hence ultra-fat. $\square$

4. COUNTEREXAMPLES REGARDING COARSE VERSIONS OF HALIN’S GRID THEOREM

In this section, we present two examples regarding coarse versions of Halin’s Grid Theorem. The first example shows that Problem 1.2 is false in general for graphs that are not locally finite. The second example shows that the number $2K - 1$ in Theorem 3.1 is best possible.

Example 4.1. *There is a one-ended graph G such that $\omega \cdot R \prec_{\infty} G$ but $HG \not\prec_2 G$.*

Proof. Let G be the graph obtained from the disjoint union of infinitely many rays $R_0, R_1, R_2, \dots$ and a $K_{\aleph_0}$ by joining the i -th vertices of the R_j with the i -th vertex of $K_{\aleph_0}$ by paths $P_{i,j}$ of length j for all $i, j \in \mathbb{N}$. Then for all $K \in \mathbb{N}$ the rays $R_K, R_{K+1}, \dots$ are $2K$ -fat and pairwise $2K$ apart, and thus can be contracted into a $2K$ -fat model of $\omega \cdot R$. In particular, as G is one-ended, we have $\omega \cdot R \prec_{\infty}^{\varepsilon} G$ for the unique end ε of G .

We claim that the half-grid is not a 2-fat minor of G . Indeed, suppose for a contradiction that G contains a 2-fat model $(\mathcal{V}, \mathcal{E})$ of the half-grid. As the $K_{\aleph_0}$ has diameter 1, at most one branch set in $\mathcal{V}$ and at most one branch path in $\mathcal{E}$ can contain a vertex from $K_{\aleph_0}$. But since the half-grid still contains a half-grid after removing an arbitrary vertex and an arbitrary edge, and because removing the $K_{\aleph_0}$ from G leaves only combs $C_j = R_j \cup \bigcup_{i \in \mathbb{N}} P_{i,j}$ as components, one of the combs C_j would contain a K -fat model of the half-grid. But as combs are trees, they not even contain a 0-fat model of the half-grid. This yields the desired contradiction. $\square$

Example 4.2. *For all $K \in \mathbb{N}$, there is a one-ended, locally finite graph G such that $\omega \cdot R \prec_{2K-2} G$ but $HG \not\prec_K G$.*

Proof. Let G be the graph obtained from the disjoint union of infinitely many rays $T, R^0, R^1, R^2, \dots$ by joining the vertex r_i^j of $R^j = r_0^j r_1^j \dots$ and the vertex t_{i+j} of $T = t_0 t_1 \dots$ by a path $P_{i,j}$ of length $K - 1$ for all $i, j \in \mathbb{N}$. Then T and all rays R^j are equivalent, and G is one-ended. Note that G is locally finite since $\deg_G(r_i^j) \leq 3$ for all $i, j \in \mathbb{N}$, since all internal vertices of the paths $P_{i,j}$ have degree 2 in G , and since

$$\deg_G(t_i) \leq |\{(j, k) \in \mathbb{N}^2 \mid j + k = i\}| + 2 \leq |\{(j, k) \in \mathbb{N}^2 \mid j, k \leq i\}| + 2 = (i + 1)^2 + 2$$

for all $i \in \mathbb{N}$. Since $d_G(R^i, R^j) = 2K - 2$ by construction, and since the rays R_i are $(2K - 2)$ -fat as G is locally finite, the rays R^i can be contracted into a $(2K - 2)$ -fat model of $\omega \cdot R$.

We claim that the half-grid is not a K -fat minor of G . For this, suppose towards a contradiction that there is a K -fat model $(\mathcal{V}, \mathcal{E})$ of the half-grid in G . Let S'_i be the induced subgraph of G on those vertices that lie in some branch set or branch path in $\mathcal{V} \cup \mathcal{E}$ belonging to the vertical ray S_i of the half-grid. Let further $U_n := \{t_n\} \cup \{r_i^j : i, j \in \mathbb{N}, i + j = n\}$, for $n \in \mathbb{N}$, be the n -th ‘level’ of G , and note that $G - U_n$ has precisely one infinite component C_n , and that $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$. As the S'_i are connected and $\bigcap_{n \in \mathbb{N}} C_n = \emptyset$, there exists for every S'_i some $n \in \mathbb{N}$ such that S'_i meets all ‘levels’ U_m with $m \geq n$. Hence, there is some $n \in \mathbb{N}$ such that at least three distinct S'_i meet every U_m for $m \geq n$. Then for every

$m \geq n$ the vertex t_m in T cannot be in any S'_i since it has distance less than K to all the other vertices in U_m . But similarly, t_m cannot lie in a branch path in $\mathcal{E}$ representing a horizontal edge of the half-grid since t_m has distance less than K from at least three S'_i . So $(\mathcal{V}, \mathcal{E})$ avoids the ray $T_{\geq n}$, and hence at most n branch sets and paths of $(\mathcal{V}, \mathcal{E})$ contain a vertex of T . Since the half-grid still contains the half-grid as a minor after deleting finitely many vertices and edges, and because the half-grid is connected, there is some component C of $G - T$ which contains the half-grid as a minor. But the components of $G - T$ are all combs of the form $R^i \cup \bigcup_{j \in \mathbb{N}} P_{ij}$, and hence not even contain a 0-fat model of the half-grid. This yields the desired contradiction. $\square$

5. HALF-GRID MINORS IN ONE-ENDED QUASI-TRANSITIVE GRAPHS

In this section we apply one of our coarse versions of Halin's Grid Theorem to prove [Theorem 1](#), which we restate here for convenience:

Theorem 1. *Every one-ended, quasi-transitive, locally finite graph G contains the half-grid as an ultra-fat minor. In particular, the half-grid is an asymptotic minor and a diverging minor of G .*

A graph G is *quasi-transitive* if the automorphism group of G acts on $V(G)$ with only finitely many orbits, that is, if $V(G)$ can be partitioned into finitely many sets $U_0, \dots, U_n$ such that for all $i \in \{0, \dots, n\}$ and $u, v \in U_i$ there exists an automorphism φ of G such that $\varphi(u) = v$.

Given an automorphism φ of a graph G , a set $U \subseteq V(G)$ is *φ -invariant* if $\varphi(U) = U$. A subgraph X of G is *φ -invariant* if $V(X)$ is φ -invariant. An automorphism φ of a graph G is *elliptic* if G contains a finite φ -invariant set of vertices and *non-elliptic* otherwise. If R is a φ -invariant double ray and φ is non-elliptic, then we say that φ *acts as a translation* on R .

The main effort of this section goes into showing that every graph G as in [Theorem 1](#) contains infinitely many rays $R_0, R_1, \dots$ such that $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$. [Theorem 1](#) then follows from [Theorem 3.1 \(2\)](#). For this, we distinguish two cases. In the first case, in [Section 5.1](#) below, we assume that some automorphism of G is non-elliptic. The second case, in [Section 5.2](#) below, deals with the remaining case that all automorphisms of G are elliptic. The formal proof of [Theorem 1](#), which collects the results from [Sections 5.1](#) and [5.2](#), can be found in [Section 5.3](#).

5.1. Some automorphism of G is non-elliptic. We now show that every one-ended, connected, locally finite graph G with a non-elliptic automorphism contains infinitely many rays that are pairwise far apart. For this, we need the following result of Halin:

Lemma 5.1 ([\[Hal73, Theorem 7\]](#)). *For every non-elliptic automorphism φ of a connected graph G there exists a φ -invariant double ray.*

Let G and H be graphs. A map $f: V(H) \rightarrow V(G)$ is a *coarse embedding*, if there exist functions $\rho^-: [0, \infty) \rightarrow [0, \infty)$ and $\rho^+: [0, \infty) \rightarrow [0, \infty)$ such that $\rho^-(a) \rightarrow \infty$ for $a \rightarrow \infty$ and

$$\rho^-(d_H(u, v)) \leq d_G(f(u), f(v)) \leq \rho^+(d_H(u, v))$$

for all $u, v \in V(H)$. If H is a subgraph of G , then H is *coarsely embedded* in G if the inclusion map $\iota: V(H) \rightarrow V(G)$ is a coarse embedding.

The next lemma guarantees that the double ray from [Lemma 5.1](#) is coarsely embedded in G .

Lemma 5.2. *Let G be a locally finite graph. Let φ be a non-elliptic automorphism of G , and let R be a φ -invariant double ray. Then R is coarsely embedded in G .*

Proof. For $L \geq 0$, we set $K_L := \min\{d_G(x, y) \mid x, y \in V(R), d_R(x, y) = L\}$, and $\rho: \mathbb{N} \rightarrow \mathbb{N}, L \mapsto K_L$. Then

$$\rho(d_R(x, y)) = K_{d_R(x, y)} \leq d_G(x, y) \leq d_R(x, y).$$

Hence, it remains to show that K_L tends to infinity for $L \rightarrow \infty$. For this, let $n \in \mathbb{N}$ be given, and let $x_1, \dots, x_m$ be representatives of the $\langle \varphi \rangle$ -orbits on R . Since G is locally finite, the balls $B_G(x_i, n)$ are finite. As there are only finitely many $\langle \varphi \rangle$ -orbits on R , this implies that there exists some N_n such that $d_R(x_i, r) \leq N_n$ for all $i \leq m$ and $r \in B_G(x_i, n) \cap V(R)$. In particular, as $x_1, \dots, x_m$ are representatives of the $\langle \varphi \rangle$ -orbits on R , we have $d_R(r, s) \leq N_n$ for all $r, s \in V(R)$ with $d_G(r, s) \leq n$. This implies that $K_{N_n+1} > n$, and thus K_L tends to infinity for $L \rightarrow \infty$, as desired. $\square$

Combining Lemmas 5.1 and 5.2 we obtain that every locally finite, connected graph with a non-elliptic automorphism contains a φ -invariant double ray R , and R is coarsely embedded in G . The next two lemmas show that if G is one-ended, and hence all tails of R lie in the same end, then we can find infinitely many rays R_i in G in ‘thickened cylinders’ around R of the form $G[R, L_{i+1}] - B_G(R, L_i + i + 1)$ for suitable integers $L_0 < L_1 < \dots \in \mathbb{N}$ with $L_{i+1} - L_i \geq i + 1$. Hence, $d_G(R_i, R_j) \geq |L_j - L_i| \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$ as desired.

Let G be a graph, and let R be a double ray in G . For $K \in \mathbb{N}$, let C be a component of $G - B_G(R, K)$. We say that C is *half-long* if there are infinitely many $r \in V(R)$ such that C has a neighbour in $B_G(r, K)$. Further, C is *half-thick* if, for every $L \geq K$, some half-long component of $G - B_G(R, L)$ is contained in C .

Lemma 5.3. *Let G be a one-ended, locally finite graph. Let φ be a non-elliptic automorphism of G , and let R be a φ -invariant double ray. Then $G - B_G(R, L)$ has at least one half-thick component for every $L \in \mathbb{N}$.*

Proof. Set $R =: \dots r_{-1}r_0r_1\dots$, and let $L \in \mathbb{N}$. Let us first show that $G[R, K] \neq G$ for all $K \in \mathbb{N}$. As R is coarsely embedded in G by Lemma 5.2, there is some K' such that $R_{\leq 0}$ and $R_{\geq K'}$ have distance at least $2K + 2$ from each other, which implies that the sets $B_G(R_{\leq 0}, K)$ and $B_G(R_{\geq K'}, K)$ are disjoint and not joined by an edge. Hence, every $B_G(R_{\leq 0}, K) - B_G(R_{\geq K'}, K)$ path meets either $B_G(r_1Rr_{K'-1}, K)$ or $G - G[R, K]$. But since both $R_{\leq 0}$ and $R_{\geq K'}$ lie in the unique end of G , there are infinitely many disjoint such paths, of which at most finitely many can meet the set $B_G(r_1Rr_{K'-1}, K)$, which is finite since G is locally finite. Hence, $G - G[R, K]$ is non-empty for all $K \in \mathbb{N}$.

Thus, there exist vertices $x_1, x_2, \dots$ such that $d(x_i, R) \geq i$ for all $i \in \mathbb{N}$. Let $y_i \in V(R)$ such that $d(x_i, R) = d(x_i, y_i)$. Since φ acts on R as a translation, there are only finitely many $\langle \varphi \rangle$ -orbits on R . Hence, there is an infinite subset $I \subseteq \mathbb{N}$ such that all y_i for $i \in I$ lie in the same $\langle \varphi \rangle$ -orbit. We may assume $1 \in I$. For every $i \in I$, let P_i be a shortest $y_i - x_i$ path and let $\varphi_i \in \langle \varphi \rangle$ such that $\varphi_i(y_i) = y_1$. Then, since G is locally finite, there exists an infinite index set $I_1 \subseteq I$ such that $\varphi_i(P_i) \cap G[y_1, 1]$ coincides for all $i \in I_1$, amongst which we again find an infinite index set $I_2 \subseteq I_1$ such that $\varphi_i(P_i) \cap G[y_1, 2]$ coincides for all $i \in I_2$ and so on. This results in a geodesic ray S that starts at y_1 and satisfies $d_G(u, R) = d_S(u, y_1)$ for all $u \in V(S)$. In particular, S has a tail in $G - B_G(R, K)$ for all $K \in \mathbb{N}$. Since G is one-ended, there are infinitely many pairwise disjoint $S - R$ paths. This shows that the components of $G - B_G(R, K)$

that contain a tail of S are half-long, and hence the component of $G - B_G(R, L)$ containing a tail of S is half-thick. $\square$

Lemma 5.4. *Let G be a one-ended, connected, locally finite graph with a non-elliptic automorphism. Then there are infinitely many rays $R_0, R_1, \dots$ such that $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$.*

Proof. Let φ be a non-elliptic automorphism of G . By Lemma 5.1, there exists a φ -invariant double ray R in G . By Lemma 5.3, some component C of $G - B_G(R, 0)$ is half-thick. Hence, there exists for every $i \in \mathbb{N}$ a half-long component $C_i \subseteq C$ of $G - B_G(R, i)$. We will construct an increasing sequence $(K_i)_{i \in \mathbb{N}}$ of integers $K_i \in \mathbb{N}$ and a sequence $(R_i)_{i \in \mathbb{N}}$ of rays such that R_i lies in $G[R, K_i] - B_G(R, K_{i-1} + i)$. Then the rays $R_0, R_1, \dots$ clearly satisfy $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$ and hence the assertion follows.

Let $j \in \mathbb{N}$ and assume that we have constructed all K_i and R_i for $i < j$. Since $C_{K_{j-1}+j}$ is half-long, there are infinitely many distinct vertices $x_1, x_2, \dots$ in R with $d(x_i, C_{K_{j-1}+j}) = K_{j-1} + j + 1$. As $\langle \varphi \rangle$ acts as a translation on R , infinitely many of the x_i lie in the same $\langle \varphi \rangle$ -orbit. Hence, we may assume that all x_i lie in the same $\langle \varphi \rangle$ -orbit. For every $i \in \mathbb{N}$, let $\varphi_i \in \langle \varphi \rangle$ such that $\varphi_i(x_i) = x_1$ and let $y_i \in V(C_{K_{j-1}+j})$ with $d_G(x_i, y_i) = d_G(x_i, C_{K_{j-1}+j})$. Since G is locally finite, the ball $B_G(x_1, K_{j-1} + j + 1)$ is finite, and hence there are distinct i, ℓ such that $\varphi_i(y_i) = \varphi_\ell(y_\ell)$. As $\varphi_i(R) = R = \varphi_\ell(R)$ since $\varphi_i, \varphi_\ell \in \langle \varphi \rangle$, and because $\varphi_i(y_i) = \varphi_\ell(y_\ell)$ and $y_i, y_\ell \in V(C_{K_{j-1}+j})$, it follows that $\varphi_i(C_{K_{j-1}+j}) = \varphi_\ell(C_{K_{j-1}+j})$.

Again because $\varphi_i, \varphi_\ell \in \langle \varphi \rangle$, there is $n \in \mathbb{Z} \setminus \{0\}$ such that $\varphi_i^{-1}\varphi_\ell = \varphi^n$. Then, $\varphi^n(C_{K_{j-1}+j}) = \varphi_i^{-1}(\varphi_\ell(C_{K_{j-1}+j})) = C_{K_{j-1}+j}$; in particular, $\varphi^{kn}(y_1) \in C_{K_{j-1}+j}$ for all $k \in \mathbb{Z}$. Thus, $\varphi^{kn}(x_1)$ has distance $K_{j-1} + j + 1$ to $C_{K_{j-1}+j}$ for every $k \in \mathbb{Z}$. Let P be a y_1 - $\varphi^n(y_1)$ path in $C_{K_{j-1}+j}$. Then $\bigcup_{k \in \mathbb{Z}} \varphi^{kn}(P)$ is connected and contained in $G[R, K_j] - B_G(R, K_{j-1} + j)$ where K_j is equal to the length of P plus $K_{j-1} + j + 1$. Moreover, $\bigcup_{k \in \mathbb{Z}} \varphi^{kn}(P)$ is connected, and it is infinite since φ is not elliptic. Hence, as G is locally finite, $\bigcup_{k \in \mathbb{Z}} \varphi^{kn}(P)$ contains a ray R_j (see e.g. [Die24, Proposition 8.2.1]), which thus lies in $G[R, K_j] - B_G(R, K_{j-1} + j) \supseteq \bigcup_{k \in \mathbb{Z}} \varphi^{kn}(P)$. $\square$

5.2. All automorphisms of G are elliptic. We show that every one-ended, quasi-transitive, locally finite graph G whose automorphisms are all elliptic contains infinitely many rays that are pairwise far apart. For this, we first show in Lemma 5.5 below that if an elliptic automorphism φ maps the first vertex r_0 of a ‘geodesic’ ray R to another vertex of R far enough away from r_0 , then $\varphi(R)$ has a tail that is far apart from R . We then apply this result infinitely often to find infinitely many rays in G such that, for every $n \in \mathbb{N}$, the n -th ray has a tail with distance at least n from all previously chosen rays.

A subgraph X of G is *geodesic (in G)* if for every two vertices $u, v \in V(X)$ we have $d_X(u, v) = d_G(u, v)$.

Lemma 5.5. *Let $K \in \mathbb{N}$, let G be a locally finite graph, and let $R = r_0 r_1 \dots$ be a geodesic ray in G . Let φ be an elliptic automorphism of G such that $\varphi(r_0) = r_d$ for some $d \geq 4K$. Then there exists $j \in \mathbb{N}$ such that $d_G(R_{\geq j}, \varphi(R)) \geq K$.*

Proof. Let us suppose for a contradiction that $d_G(R_{\geq j}, \varphi(R)) < K$ for all $j \in \mathbb{N}$. We show that $d_G(r_0, \varphi^n(r_0))$ tends to infinity for $n \rightarrow \infty$. This then yields a contradiction as φ is elliptic, and hence the φ -orbit of r_0 is finite.

Set $i_0 := 0$. Since $d_G(R_{\geq j}, \varphi(R)) < K$ for all $j \in \mathbb{N}$, there exist $d := i_1 < i_2 < \dots \in \mathbb{N}$ such that $d_G(r_{i_j}, \varphi(R)) < K$ for all $j \in \mathbb{N}_{\geq 1}$. Set $\ell_j := i_{j+1} - i_j$ for all $j \in \mathbb{N}$. By restricting to a subsequence of $(i_j)_{j \in \mathbb{N}}$ if necessary, we may assume that $\ell_j \geq d + 2K$ for all $j \in \mathbb{N}_{\geq 1}$. For $j \in \mathbb{N}_{\geq 1}$, let y_j be in $\varphi(R)$

with $d_G(r_{i_j}, \varphi(R)) = d_G(r_{i_j}, y_j)$, and set $\ell'_j := d_G(\varphi(r_{i_j}), y_{j+1})$ for all $j \geq 1$. Then

$$(5.1) \quad \begin{aligned} \ell'_j + d_G(y_j, \varphi(r_{i_j})) &= d_G(\varphi(r_{i_j}), y_{j+1}) + d_G(y_j, \varphi(r_{i_j})) = d_G(y_j, y_{j+1}) \\ &\leq d_G(y_j, r_{i_j}) + d_G(r_{i_j}, r_{i_{j+1}}) + d_G(r_{i_{j+1}}, y_{j+1}) < K + \ell_j + K = \ell_j + 2K, \end{aligned}$$

where we used for the second equality that R is geodesic. Since $\varphi(r_0) = r_{i_1} = r_d$, we have

$$\begin{aligned} i_j &= d_G(r_0, r_{i_j}) = d_G(\varphi(r_0), \varphi(r_{i_j})) = d_G(r_{i_1}, \varphi(r_{i_j})) \\ &\leq d_G(r_{i_1}, r_{i_j}) + d_G(r_{i_j}, y_j) + d_G(y_j, \varphi(r_{i_j})) < (i_j - d) + K + d_G(y_j, \varphi(r_{i_j})), \end{aligned}$$

where we again used that R is geodesic. Hence, $d_G(y_j, \varphi(r_{i_j})) > d - K$. Combining this with (5.1) yields

$$\ell'_j < \ell_j + 2K - d_G(y_j, \varphi(r_{i_j})) < \ell_j + 3K - d,$$

and hence

$$(5.2) \quad d_G(\varphi(r_{i_j}), r_{i_{j+1}}) \leq d_G(\varphi(r_{i_j}), y_{j+1}) + d_G(y_{j+1}, r_{i_{j+1}}) < \ell'_j + K < (\ell_j + 3K - d) + K \leq \ell_j$$

as $d \geq 4K$. For the next sequence of inequalities, we recall that $\varphi^0 = \text{id}$; in particular $\varphi^0(r_{i_n}) = r_{i_n}$. Since $d_G(r_0, \varphi^n(r_0)) + d_G(\varphi^n(r_0), r_{i_n}) \geq d_G(r_0, r_{i_n}) = i_n$, we conclude

$$d_G(r_0, \varphi^n(r_0)) \geq i_n - d_G(\varphi^n(r_0), r_{i_n}) \geq i_n - \sum_{j=0}^{n-1} d_G(\varphi^{j+1}(r_{i_{n-j-1}}), \varphi^j(r_{i_{n-j}}))$$

(where we used the generalised triangle inequality for the second inequality)

$$= i_n - \sum_{j=0}^{n-1} d_G(\varphi(r_{i_{n-j-1}}), r_{i_{n-j}}) \geq i_n - \sum_{j=0}^{n-1} (\ell_{n-j-1} - 1)$$

(where we used for the first (in)equality that automorphisms preserve distances and (5.2) for the second)

$$= i_n - \sum_{j=0}^{n-1} (i_{n-j} - i_{n-j-1} - 1) = i_n - (i_n - i_0 - n) = n.$$

Since this last expression tends to infinity for $n \rightarrow \infty$, this contradicts φ being elliptic. $\square$

Lemma 5.6. *Let $K \in \mathbb{N}$ and let G be an infinite, connected, quasi-transitive, locally finite graph all of whose automorphisms are elliptic. Then there are infinitely many geodesic rays $R_0, R_1, \dots$ such that $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$.*

Proof. Since G is infinite, connected, and locally finite, a standard compactness argument yields a geodesic ray $R = r_0 r_1 \dots$ in G . Since G is quasi-transitive, some orbit under the automorphism group of G contains infinitely many vertices of R . We may assume that r_0 lies in this orbit. Then there is an automorphism φ_1 of G such that $\varphi_1(r_0) = r_{i_1}$ for some $i_1 \geq 4$. For general j , we choose φ_j such that $\varphi_j(r_{i_{j-1}}) = r_{i_j}$ for some $i_j \in \mathbb{N}$ with $i_j - i_{j-1} \geq 4j$. Set $R'_0 := R$, and $R'_j := \varphi_1^{-1} \circ \dots \circ \varphi_j^{-1}(R_{\geq i_j})$ for all $j \geq 1$. We claim that for every $k < j$ the ray R'_j has a tail R_j^k that has distance at least j from R'_k . The assertion then follows by setting $R_j := \bigcap_{k < j} R_j^k$, which is still a ray as j is finite.

So let $k < j$ be given. Since all automorphisms of G are elliptic, also $\varphi := \varphi_j \circ \dots \circ \varphi_{k+1}$ is elliptic. As also $\varphi(r_{i_k}) = r_{i_j}$ and $i_j - i_k \geq i_j - i_{j-1} \geq 4j$, applying Lemma 5.5 to $R_{\geq i_k}$, φ and $K := j$ shows that there is some $n \geq i_j$ such that $d_G(R_{\geq n}, \varphi(R_{\geq i_k})) \geq j$. Set $R_j^k := \varphi_1^{-1} \circ \dots \circ \varphi_j^{-1}(R_{\geq n})$. As $R'_k = \varphi_1^{-1} \circ \dots \circ \varphi_k^{-1}(R_{\geq i_k}) = \varphi_1^{-1} \circ \dots \circ \varphi_j^{-1}(\varphi(R_{\geq i_k}))$ and because automorphisms preserve distances, it

follows that $d_G(R_j^k, R'_k) = d_G(R_{\geq n}, \varphi(R_{\geq i_k})) \geq j$. Since $R'_j = \varphi_1^{-1} \circ \dots \circ \varphi_j^{-1}(R_{\geq i_j})$ and $n \geq i_j$, we have $R_j^k \subseteq R'_j$, and hence R_j^k is as desired. $\square$

5.3. Proof of Theorem 1. We can now prove Theorem 1, by combining Lemmas 5.4 and 5.6 with Theorem 3.3.

Proof of Theorem 1. If some automorphism of G is non-elliptic, then Lemma 5.4 implies that G contains infinitely many rays $R_0, R_1, \dots$ such that $d_G(R_i, R_j) \geq \max\{i, j\}$ for all $i \neq j \in \mathbb{N}$. If all automorphisms of G are elliptic, then we obtain the same by Lemma 5.6. Thus, Theorem 3.1 (2) implies that G contains an ultra-fat model of the half-grid.

The ‘in particular’-part follows by Observation 2.1 and Proposition 2.2. $\square$

6. APPLICATIONS OF THEOREM 1

In this section we present some applications of Theorem 1; in particular, we show Corollaries 2 and 3.

6.1. Half-grid minors in multi-ended, quasi-transitive graphs. As a corollary of Theorem 1, we obtain two results on quasi-transitive, locally finite graphs that need not be one-ended.

A finite set $U \subseteq V(G)$ *distinguishes* two ends $\varepsilon, \varepsilon'$ of G if no component of $G - U$ contains rays from both ε and ε' . An end ε of G is *accessible* if there exists some $n \in \mathbb{N}$ such that every other end of G can be distinguished from ε by a set of at most n vertices of G . A graph G is *accessible* if there exists some $n \in \mathbb{N}$ such that every two distinct ends of G can be distinguished by a set of at most n vertices of G .

Corollary 6.1. *Let ε be an accessible, thick end of a quasi-transitive, locally finite graph G . Then G contains an ultra-fat model of the half-grid all of whose rays are ε -rays.*

Proof. By the same argument as in [AH24, Lemma 4.6], there exists a connected, one-ended, quasi-geodesic,⁷ quasi-transitive subgraph X of G such that every ray in X is an ε -ray in G . By Theorem 1, there is an ultra-fat model $(\mathcal{V}, \mathcal{E})$ of the half-grid in X . Since X is c -quasi-geodesic for some $c \in \mathbb{N}$, the submodel of $(\mathcal{V}, \mathcal{E})$ corresponding to $\mathbb{N}^2 \setminus [cK]^2$ is K -fat in G for all $K \in \mathbb{N}$. It is now straightforward to turn $(\mathcal{V}, \mathcal{E})$ into an ultra-fat model of the half-grid in G . In fact, as every ray in X is an ε -ray in G , we find $HG \prec_{UF}^\varepsilon G$. $\square$

Corollary 6.2. *Let G be an accessible, quasi-transitive, locally finite graph with a thick end. Then the half-grid is an ultra-fat minor of G .*

Proof. Since all ends in an accessible, quasi-transitive, locally finite graph are accessible, the claim follows directly from Corollary 6.1. $\square$

For two graphs G and H , a map $f: V(H) \rightarrow V(G)$ is a *quasi-isometry* if there exist $M \geq 1$ and $A \geq 0$ such that

- $M^{-1} \cdot d_H(u, v) - A \leq d_G(f(u), f(v)) \leq M \cdot d_H(u, v) + A$ for all $u, v \in V(H)$, and
- $d_G(f(V(H)), w) \leq A$ for all $w \in V(G)$.

Two graphs are *quasi-isometric* if there exists a quasi-isometry between them.

Finally, we can prove Corollary 2. In fact, we prove the following more detailed version:

⁷A subgraph X of G is *quasi-geodesic* in G if there exists some $c \in \mathbb{N}$ such that $d_X(u, v) \leq c \cdot d_G(u, v)$ for all $u, v \in V(X)$.

Corollary 2'. *Let G be an accessible, connected, quasi-transitive, locally finite graph. Then the following are equivalent:*

- (i) G has a thick end.
- (ii) The half-grid is an ultra-fat minor of G .
- (iii) The half-grid is an asymptotic minor of G .
- (iv) The half-grid is a diverging minor of G .
- (v) G is not quasi-isometric to a tree.

Proof. (i) $\Leftrightarrow$ (v) is a result by Krön and Möller [KM08, Theorem 5.5]. (i) $\Rightarrow$ (ii) is Corollary 6.2. (ii) $\Rightarrow$ (iii) is Observation 2.1, and (ii) $\Rightarrow$ (iv) is Proposition 2.2. Finally, (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) hold because the rays in a model of the half-grid are all equivalent, and thus the end of G that contains them must be thick. $\square$

We conjecture that Corollary 2' holds for all connected, quasi-transitive, locally finite graphs (even for those that are not accessible). To prove this, it would suffice to show the following:

Conjecture 6.3. *Every connected, quasi-transitive, locally finite graph with a thick end contains an ultra-fat model of the half-grid.*

By Corollary 2' and because every inaccessible, quasi-transitive, locally finite graph has a thick end, it remains to show for a proof of Conjecture 6.3 that every inaccessible, quasi-transitive, locally finite graph contains an ultra-fat model of the half-grid.

6.2. Diverging rays in quasi-transitive graphs. In this section we apply Theorem 1 to prove Corollary 3, which we restate here for convenience:

Corollary 3. *Every quasi-transitive, locally finite graph that contains infinitely many pairwise disjoint rays also contains infinitely many pairwise diverging rays.*

Recall that two rays R, S in a graph G *diverge* if for every $K \in \mathbb{N}$ they have tails $R' \subseteq R$, $S' \subseteq S$ satisfying $d_G(R', S') > K$.

Proof. Let G be a quasi-transitive, locally finite graph with infinitely many pairwise disjoint rays. If G has infinitely many ends, then we may take an infinite set of rays no two of which lie in the same end. Indeed, let R_1 and R_2 be rays from distinct ends and let S be a finite vertex set such that R_1 and R_2 have tails in distinct components C_1 and C_2 of $G - S$, respectively. Since G is locally finite, each R_i has a tail in $C_i - B_G(S, n)$ for every $n \in \mathbb{N}$. Thus, R_1 and R_2 diverge.

So, we may assume that G has only finitely many ends. Then infinitely many of the pairwise disjoint rays lie in the same end, which hence must be thick. As every graph with only finitely many ends is accessible, Corollary 6.2 yields that G contains an ultra-fat model $(\mathcal{V}, \mathcal{E})$ of the half-grid. Without loss of generality we may assume that the branch sets are finite. For every $i \in \mathbb{N}$, there exists a ray R_i in G that is contained in the union of the branch sets and paths of $(\mathcal{V}, \mathcal{E})$ corresponding to the i -th vertical ray S_i of the half-grid. Since all branch sets and paths are finite, R_i has a tail outside of the branch sets and paths corresponding to the first K vertices and edges of S_i . Since $(\mathcal{V}, \mathcal{E})$ is ultra-fat, it follows that the rays R_i diverge pairwise. $\square$

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