Polarized Partitions on the Second Level of the Projective Hierarchy.

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Abstract

A subset A of the Baire space ω^ω satisfies the *polarized partition property* if there is an infinite sequence $\langle H_i \mid i \in \omega \rangle$ of finite subsets of ω, with $|H_i| \geq 2$, such that $\prod_i H_i \subseteq A$ or $\prod_i H_i \cap A = \varnothing$. It satisfies the *bounded polarized partition property* if, in addition, the H_i are bounded by some pre-determined recursive function. DiPrisco and Todorčević [6] proved that both partition properties are true for analytic sets A. In this paper we investigate these properties on the $\mathbf{\Delta}^1_2$- and $\mathbf{\Sigma}^1_2$-levels of the projective hierarchy, i.e., we investigate the strength of the statements "all $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2$ sets satisfy the (bounded) polarized partition property" and compare it to similar statements involving other well-known regularity properties.

Keywords: Polarized partitions, projective hierarchy, descriptive set theory.

1. Introduction.

The property studied in this paper is motivated by the following combinatorial question: suppose we are given a partition of the Baire space ω^ω into two pieces, say A and $\omega^\omega \setminus A$, and an infinite sequence $\langle m_i \mid i < \omega \rangle$ of integers ≥ 2. Can we find an infinite sequence $\langle H_i \mid i < \omega \rangle$ of subsets of ω, with $|H_i| = m_i$,

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which is homogeneous for the partition, i.e., such that the product $\prod_i H_i$ is completely contained in A or completely disjoint from A?

As with other questions of this type, the solution depends on the complexity of the partition. It is easy to see that the above property cannot be fulfilled for *all* partitions of the Baire space. For instance, if $\preceq$ is a well-ordering of ω^ω and if for every x we denote by y_x the $\preceq$-least real eventually equal to x, then the following set is a counterexample:

$$A := \{x \mid |\{n \mid x(n) \neq y_x(n)\}| \text{ is even } \}.$$

This is because if there were a sequence $\langle H_i \mid i \leq \omega \rangle$ with $|H_i| \geq 2$ such that, say, $\prod_i H_i \subseteq A$, then any $x \in \prod_i H_i$ could be changed to $x' \in \prod_i H_i$ by altering just one digit, so that $y_x = y_{x'}$ but $|\{n \mid x'(n) \neq y_{x'}(n)\}|$ is odd, yielding a contradiction.

The natural approach of descriptive set theory is to consider partitions of limited complexity. For instance, Silver's theorem—the statement that all analytic sets are Ramsey—implies a positive solution to our question if we consider analytic partitions only. The same holds if we replace "analytic" by "having complexity $\mathbf{\Gamma}$", for any projective pointclass $\mathbf{\Gamma}$:

1.1 Lemma. (Folklore) *Let $\mathbf{\Gamma}$ be any projective pointclass and assume that all sets in $\mathbf{\Gamma}$ are Ramsey. Then our partition problem has a positive solution for all partitions in $\mathbf{\Gamma}$.*

Proof. Suppose $A \subseteq \omega^\omega$ is a given set of complexity $\mathbf{\Gamma}$, and $m_0, m_1, \dots$ are integers ≥ 2. Let $\omega^{\uparrow\omega}$ denote the space of strictly increasing sequences from ω to ω, which we can identify with infinite subsets of ω via increasing enumerations. Set $A' := A \cap \omega^{\uparrow\omega}$. Since A' is still in $\mathbf{\Gamma}$, by assumption there is an $x \in \omega^{\uparrow\omega}$ which is homogeneous for A', i.e., such that $x^{\uparrow\omega} := \{y \in \omega^{\uparrow\omega} \mid \text{ran}(y) \subseteq \text{ran}(x)\} \subseteq A'$ or $x^{\uparrow\omega} \cap A' = \varnothing$. Now, simply take as H_0 the first m_0 values of x, as H_1 the next m_1 values of x, and so on. Clearly, for every $y \in \prod_i H_i$ we have $\text{ran}(y) \subseteq \text{ran}(x)$ and hence either $\prod_i H_i \subseteq A' \subseteq A$ or $\prod_i H_i \cap A' = \varnothing$. Since $\prod_i H_i$ only contains increasing sequences, the latter case implies $\prod_i H_i \cap A = \varnothing$. □

The homogeneous $x \in \omega^{\uparrow\omega}$ obtained from this proof can grow quite rapidly, and in general there is no upper bound on its rate of growth. Hence the homogeneous sequence $\langle H_i \mid i \leq \omega \rangle$ obtained from x is also potentially unbounded. We could ask what happens if we tighten the conditions of the original question so as to rule out these "unbounded" solutions. Suppose that, this time, we are given a partition A and two sequences of integers ≥ 2: $m_0, m_1, \dots$ and $n_0, n_1, \dots$. Can we find $\langle H_i \mid i < \omega \rangle$ such that $|H_i| = m_i$ and $H_i \subseteq n_i$ which is homogeneous for A? Here, we want the n_i to increase at a much quicker rate then the m_i, since otherwise this property will fail even for very simple partitions (e.g., closed).

In [6], DiPrisco and Todorčević first computed explicit upper bounds $\vec{n}$ as a function of $\vec{m}$ and proved that with these bounds the problem has a positive solution for analytic partitions. The techniques used there were fundamentally different from the unbounded case and did not invoke Silver's theorem or the Ramsey property. The computation of $\vec{n}$ in terms of $\vec{m}$ used a recursive but non-primitive-recursive function (an Ackermann-style function) which was improved by Shelah and Zapletal [17] to a direct, primitive-recursive computation using the methods of creature forcing. The computation was improved even further in the recent book [18] by Todorčević.

In this paper we will look at both of the partition problems mentioned above and investigate what happens at the next level of the projective hierarchy: the $\mathbf{\Sigma}^1_2$- and $\mathbf{\Delta}^1_2$-level. First we need to introduce some notation and give precise definitions.

1.2 Definition.

1. We will refer to infinite sequences by $H = \langle H_i \mid i \in \omega \rangle$ and use the shorthand notation $[H]$ instead of $\prod_i H_i$. This corresponds to identifying the sequence H with a finitely branching uniform perfect tree, so that $[H]$ is the set of branches through this tree.
2. Let $m_0, m_1, \dots$ be fixed integers. A set $A \subseteq \omega^\omega$ satisfies the (*unbounded*) *polarized partition property*
$$\begin{pmatrix} \omega \\ \omega \\ \dots \end{pmatrix} \rightarrow \begin{pmatrix} m_0 \\ m_1 \\ \dots \end{pmatrix}$$
if there is an $H = \langle H_i \mid i \in \omega \rangle$ with $|H_i| = m_i$, such that $[H] \subseteq A$ or $[H] \cap A = \varnothing$.
3. Let $m_0, m_1, \dots$ and $n_0, n_1, \dots$ be fixed integers ≥ 2 such that the n_i's are recursive in the m_i's. A set $A \subseteq \omega^\omega$ (or $\subseteq \prod_i n_i$) satisfies the *bounded polarized partition property*
$$\begin{pmatrix} n_0 \\ n_1 \\ \dots \end{pmatrix} \rightarrow \begin{pmatrix} m_0 \\ m_1 \\ \dots \end{pmatrix}$$
if there is an $H = \langle H_i \mid i \in \omega \rangle$ with $|H_i| = m_i$ and $H_i \subseteq n_i$, such that $[H] \subseteq A$ or $[H] \cap A = \varnothing$.
4. Let $\mathbf{\Gamma}$ be a projective pointclass. The notations $\mathbf{\Gamma}(\vec{\omega} \rightarrow \vec{m})$ and $\mathbf{\Gamma}(\vec{n} \rightarrow \vec{m})$ abbreviate the statements "every A in $\mathbf{\Gamma}$ satisfies the partition property $(\vec{\omega} \rightarrow \vec{m})$", respectively "$(\vec{n} \rightarrow \vec{m})$". Similarly, if Φ is some other regularity property for subsets of the Baire or Cantor space then $\mathbf{\Gamma}(\Phi)$ is an abbreviation of "every A in $\mathbf{\Gamma}$ satisfies property Φ".

Our first observation is that as long as we are only interested in solutions within a projective pointclass, the precise value of the right-hand-side integers $m_0, m_1, \dots$ is irrelevant:

1.3 Lemma. *Let $\mathbf{\Gamma}$ be a pointclass and $m_0, m_1, \dots$ and $m'_0, m'_1, \dots$ two sequences of integers ≥ 2. Then*

1. $\mathbf{\Gamma}(\vec{\omega} \to \vec{m})$ *holds if and only if* $\mathbf{\Gamma}(\vec{\omega} \to \vec{m}')$ *holds.*
2. *If* $\mathbf{\Gamma}(\vec{n} \to \vec{m})$ *holds for some (sufficiently large)* $\vec{n}$*, then there are* $\vec{n'}$ *such that* $\mathbf{\Gamma}(\vec{n}' \to \vec{m}')$ *holds.*

Proof.

1. It is clear that decreasing any of the m_i's only makes the partition property easier to satisfy. Suppose we know $\mathbf{\Gamma}(\vec{\omega} \to \vec{m})$ and we are given $\vec{m}'$. Find $0 = k_{-1} < k_0 < k_1 < \dots$ such that for all i we have $m_{k_{i-1}} \cdot m_{k_{i-1}+1} \cdot \dots \cdot m_{k_i - 1} \geq m'_i$:

$$(\overbrace{m_0, m_1, \dots, m_{k_0-1}}^{\text{product is } \geq m'_0}\overbrace{m_{k_0}, m_{k_0+1}, \dots, m_{k_1-1}}^{\text{product is } \geq m'_1}\overbrace{m_{k_1}, m_{k_1+1}, \dots, m_{k_2-1}}^{\text{product is } \geq m'_2}\dots).$$

   Now let $\varphi : \omega^\omega \longrightarrow \omega^\omega$ be the continuous function given by

$$\varphi(x) := (\langle x(0), \dots, x(k_0 - 1)\rangle, \langle x(k_0), \dots, x(k_1 - 1)\rangle, \dots)$$

   where $\langle \dots \rangle$ is the canonical (recursive) bijection between ω and $\omega^{k_i - k_{i-1}}$, for the respective i. Let $A \subseteq \omega^\omega$ be a set in $\mathbf{\Gamma}$. Then $A' := \varphi^{-1}[A]$ is in $\mathbf{\Gamma}$ so by assumption there is an H' such that $\forall i\ (|H'_i| = m_i)$ and $[H'] \subseteq A'$ or $[H'] \cap A' = \varnothing$. Define H by $H_i := \{\langle r_0, \dots, r_{(k_i - k_{i-1})-1}\rangle \mid r_j \in H'_{k_{i-1}+j}\}$. Then clearly $|H_i| = m_{k_{i-1}} \cdot \dots \cdot m_{k_i-1} \geq m'_i$ and it only remains to show that $[H] = \varphi``[H']$. But that follows immediately from the definition of φ.
2. Here, use the same function φ but now note that we may choose H' to be bounded by $\vec{n}$, so that each $H'_{k_{i-1}+j}$ is bounded by $n_{k_{i-1}+j}$. Therefore the possible elements of H_i are bounded by $\langle n_{k_{i-1}}, n_{k_{i-1}+1}, \dots, n_{k_i-1}\rangle$ (assuming that the coding is monotonous). □

We will frequently use the generic notations $(\vec{\omega} \to \vec{m})$ and $(\vec{n} \to \vec{m})$ to refer to the unbounded resp. bounded partition properties, leaving $\vec{n}$ and $\vec{m}$ unspecified if it is irrelevant.

The results of [6] and [17] cover analytic partitions. On the next level in the projective hierarchy things start getting tricky: typically, when studying regularity properties for sets of reals (e.g. Lebesgue measurability, Baire property, Ramsey property), the assertion that all $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2$ sets are regular is independent of ZFC. For instance, an early theorem of Judah and Shelah [10] states that all $\mathbf{\Delta}^1_2$ sets have the Baire property if and only if for every $a \in \omega^\omega$ there is a Cohen real over $L[a]$. As a consequence, $\mathbf{\Delta}^1_2$(Baire) is false in L but true in the iterated Cohen model (i.e., the model obtained by an ω_1-iteration of Cohen forcing with finite support, starting from L). For Lebesgue measurability and random-generic reals analogous results hold. Multiple other studies have been carried out pursuing the connection between regularity properties on the second level and assertions about "transcendence over L", notably [10, 9, 4, 3].

In [11] an abstract version of these results is proved based on the concept of quasigenericity.

1.4 Definition. Let I be a σ-ideal on ω^ω. If M is a model of set theory, an $x \in \omega^\omega$ is said to be *I-quasigeneric over M* if for every Borel set $B \in I$ with Borel code in M, $x \notin B$.

Subsuming Cohen reals, random reals, as well as dominating reals, unbounded reals etc., quasigenericity is a very natural transcendence property. Ikegami showed that for a wide class of proper forcing notions $\mathbb{P}$ called *strongly arboreal forcings*, one can canonically define $\mathbb{P}$-measurability and a σ-ideal $I_\mathbb{P}$ such that $\mathbb{P}$ densely embeds into $\mathsf{BOREL}(\omega^\omega)/I_\mathbb{P}$, and so that (under certain conditions) the following are equivalent:

1. all $\mathbf{\Delta}^1_2$ sets are $\mathbb{P}$-measurable, and
2. $\forall a \, \exists x$ (x is $I_\mathbb{P}$-quasigeneric over $L[a]$).

This was generalized further by the second author in [13] to cover all *idealized forcings*, i.e., forcing notions of the form $\mathsf{BOREL}(\omega^\omega)/I_\mathbb{P}$.

Since transcendence assertions can, to some degree, be controlled by forcing, characterizations like these are extremely useful for building models in which specific regularity properties hold on the $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2$-level while others fail. Consider two regularity properties: Reg_1 and Reg_2. Does $\mathbf{\Delta}^1_2(\mathsf{Reg}_1)$ imply $\mathbf{\Delta}^1_2(\mathsf{Reg}_2)$, or is there a model (obtained by iterated forcing starting from L) in which the former holds but the latter fails? The same can be asked for $\mathbf{\Sigma}^1_2$ sets. One of the earliest results in this direction was a theorem due to Raisonnier and Stern [15], or independently to Bartoszyński [1], stating that if all $\mathbf{\Sigma}^1_2$ sets are Lebesgue-measurable then all $\mathbf{\Sigma}^1_2$ sets have the property of Baire. The converse is not true (see [2, Theorem 9.3.5. and 9.3.6.]), and neither is the analogue of this statement for $\mathbf{\Delta}^1_2$ sets (the iterated random model is a counterexample). More theorems of this kind can be found in the papers quoted above, and a survey including many regularity properties is contained in [5]. See also [14] for more on this topic.

In all the above-mentioned results, the regularity property in question is naturally connected to a forcing notion, and is often actually derived from it. The property we are interested in arises from a natural combinatorial question and is not a priori related to any forcing. As a matter of fact, the most difficult part of our task proved to be finding an adequate forcing that would allow us to build models for the partition property. Moreover, the best candidate for such a forcing notion (see section 5) is different from those typically encountered in the study of the continuum and does not fall under the category of *strongly arboreal forcings* introduced in [11] or *idealized forcings* developed by Zapletal [19, 20].

We were unable to prove a precise characterization in the style of Judah and Shelah's result. Nevertheless, we prove many non-trivial implications and non-implications which locate the polarized partition property accurately among other well-known regularity properties and transcendence statements.

In section 2 we prove a connection with eventually different reals and in section 3 we do the same for E_0-quasigeneric reals. In section 4 we look at some non-implications, and in section 5 we construct a forcing notion which forces $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ without adding unbounded or splitting reals.

2. Eventually different reals

Two reals x and y are called *eventually different* if $\forall^\infty n\ (x(n) \neq y(n))$. We say that a real x is eventually different over $L[a]$ if for every $y \in \omega^\omega \cap L[a]$, x is eventually different from y. By a theorem of Bartoszyński [2, Theorem 2.4.7.] this is equivalent to saying that the reals of $L[a]$ are meager.

By a *bounded eventually different real* over $L[a]$ we mean a real x which is eventually different over $L[a]$ and moreover there exists a $y \in \omega^\omega \cap L[a]$ such that $x \leq y$.

2.1 Theorem.

1. $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m}) \Longrightarrow \forall a\ \exists x$ *(x is eventually different over $L[a]$).*
2. $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m}) \Longrightarrow \forall a\ \exists x$ *(x is bounded eventually different over $L[a]$).*

Proof. 1. Suppose, towards contradiction, that there is an a such that for all x, there is a $y \in L[a]$ such that $\exists^\infty n\ (x(n) = y(n))$.

Claim. *For all x, there is also a $y \in L[a]$ such that $\exists^\infty n\ (x(n) = y(n) \wedge x(n+1) = y(n+1))$.*

Proof. Given x, let $x' := (\langle x(0), x(1)\rangle, \langle x(2), x(3)\rangle, \dots)$. Let $y' \in L[a]$ be such that $\exists^\infty n\ (x'(n) = y'(n))$. Now let y be such that $y' = (\langle y(0), y(1)\rangle, \langle y(2), y(3)\rangle, \dots)$. Since we use *recursive* coding, y is also in $L[a]$. Now it is clear that y is as required. □(claim)

For each x, let y_x denote the $<_{L[a]}$-least real in $L[a]$ such that $\exists^\infty n\ (x(n) = y_x(n) \wedge x(n+1) = y_x(n+1))$. Now define the following set:

$$A := \{x \mid \text{least } n \text{ s.t. } x(n) = y_x(n) \text{ is even}\}.$$

To see that A is $\Delta^1_2(a)$ we use a standard tool. We write: $x \in A$ iff

> $\exists M\ \exists y \in M$ [M countable, well-founded and $M \models$"$\chi(a) \wedge y$ is the $\Psi(a)$-least real s.t. $\exists^\infty n\ (x(n) = y(n) \wedge x(n+1) = y(n+1))$ and the first n s.t. $x(n) = y(n)$ is even"]

where $\chi(a)$ is a formula stating that M is an initial segment of $L[a]$ and $\Psi(a)$ defining an initial segment of $<_{L[a]}$. This sentence is $\Sigma^1_2(a)$. Similarly, $x \notin A$ can be written in the same form but with "even" replaced by "odd", thus showing that A is $\Delta^1_2(a)$.

Now we show that A is indeed a counterexample. Suppose there is an H such that $[H] \subseteq A$ or $[H] \cap A = \varnothing$, w.l.o.g. the former. Let $x \in [H] \subseteq A$. Since x and y_x coincide on two consecutive digits somewhere, we can easily alter x to x' by changing only finitely many digits, so that still $x' \in [H]$ but the first n for which $x'(n) = y_x(n)$ is odd. Since x and x' are eventually equal, $y_x = y_{x'}$ and therefore $x' \notin A$, which is a contradiction.

2. Using an analogous proof, we will show that x can in fact be bounded by the real $\vec{n}' := (\langle n_0, n_1\rangle, \langle n_2, n_3\rangle, \dots)$ which is clearly in $L[a]$. Assume towards

contradiction that for all x bounded by $\vec{n}'$ there is a $y \in L[a]$ infinitely equal to it. Using the same method as before, it follows that for every x bounded by $\vec{n}$, there is a $y \in L[a]$ infinitely equal on two consecutive digits. The rest of the proof proceeds analogously except that this time we define

$$A := \{x \in \prod_i n_i \mid \text{least } n \text{ s.t. } x(n) = y_x(n) \text{ is even}\}$$

and use the fact that the H given by $(\vec{n} \to \vec{m})$ is contained within $\prod_i n_i$. $\square$

3. E_0-quasigenerics

For the next result we require several definitions.

3.1 Definition. Let E_0 be the equivalence relation on 2^ω given by xE_0y iff $\forall^\infty n\ (x(n) = y(n))$. A *partial E_0-transversal* is a set A which contains at most one element from each E_0-equivalence class, in other words, $\forall x, y \in A$: if $x \neq y$ then $\exists^\infty n\ (x(n) \neq y(n))$. Let I_{E_0} be the σ-ideal generated by Borel partial E_0-transversals.

The Borel equivalence relation E_0 is well-known in descriptive set theory and played a key role in the study of the Glimm-Effros dichotomy in [8]. The ideal I_{E_0} was investigated by Zapletal [19, 20] who, among other things, isolated the notion of an E_0-*tree.*

3.2 Definition. (Zapletal) An E_0-tree is a perfect tree $T \subseteq 2^{<\omega}$ such that

1. there is a stem s_0 with $|s_0| = k_0$, and
2. there are numbers $k_0 < k_1 < k_2 < \dots$ and for each i exactly two sequences $s^i_0, s^i_1 \in {}^{[k_i, k_{i+1})}2$, such that

$$[T] = \{s_0{}^\frown s^0_{z(0)}{}^\frown s^1_{z(1)}{}^\frown \cdots \mid z \in 2^\omega\}.$$

Based on results from [8], Zapletal proved the following dichotomy: every Borel (even analytic) set is either in I_{E_0} or contains $[T]$ for some E_0-tree T. It follows that the collection of E_0-trees ordered by inclusion forms a proper forcing notion densely embeddable into $\mathsf{BOREL}(2^\omega)/I_{E_0}$.

Recall the notion of a quasigeneric real from Definition 1.4. From the above consideration, the existence of I_{E_0}-quasigenerics is an interesting transcendence property from the forcing point of view. It is known that sets in I_{E_0} are meager, so I_{E_0}-quasigenerics can certainly exist, in particular Cohen reals are such.

3.3 Theorem. $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m}) \Longrightarrow \forall a\ \exists x$ *(x is I_{E_0}-quasigeneric over $L[a]$).*

Proof. First, we define an auxiliary equivalence relation E_0^ω, which is just like E_0 but on Baire space rather than Cantor space, i.e., for $x, y \in \omega^\omega$ we define $xE_0^\omega y$ iff $\forall^\infty n\ (x(n) = y(n))$. The notions of a partial E_0^ω-transversal as well as the σ-ideal $I_{E_0^\omega}$ are defined analogously.

For a real x and a Borel set B, we say that x *is eventually in* B if there is a $y \in B$ such that $\forall^\infty n\ (x(n) = y(n))$. We denote this by $x \in^* B$.

We will first show that if $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ holds then there is an $I_{E_0^\omega}$-quasigeneric over each $L[a]$. Towards contradiction, suppose a is such that for every x there is $B \in I_{E_0^\omega}$ coded in $L[a]$ with $x \in B$. It is not hard to see that for Borel sets, membership in $I_{E_0^\omega}$ is a $\mathbf{\Sigma}^1_2$-statement and hence absolute. Therefore, for each x there is a Borel partial E_0^ω-transversal B, with code in $L[a]$ and $x \in B$. In particular, there is also such a B with $x \in^* B$. Let B_x be the $<_{L[a]}$-least such Borel set (i.e., the Borel set with the $<_{L[a]}$-least Borel code). Now form the following two sets:

$$A_0 := \{x \mid \exists y \in B_x \text{ s.t. } |\{n \mid x(n) \neq y(n)\}| \text{ is finite and even}\},$$

$$A_1 := \{x \mid \exists y \in B_x \text{ s.t. } |\{n \mid x(n) \neq y(n)\}| \text{ is finite and odd}\}.$$

The key observation here is that A_0 and A_1 form a disjoint partition of ω^ω. The fact that $A_0 \cup A_1 = \omega^\omega$ follows immediately from $x \in^* B_x$, and if there were an $x \in A_0 \cap A_1$, then there would be two distinct $y, y' \in B_x$ both eventually equal to x. But then y and y' would also be eventually equal to each other, contradicting the fact that B_x is a partial E_0^ω-transversal. Hence $A_0 \cap A_1 = \varnothing$.

To see that A_0 is $\Sigma^1_2(a)$ we use the same tool as in the proof of Theorem 2.1, namely $x \in A_0$ holds iff

> $\exists M\ \exists c \in M$ [M countable, well-founded and $M \models$"$\chi(a) \wedge B_c$ is a partial E_0-transversal, and $\exists y \in B_c$ such that $|\{n \mid x(n) \neq y(n)\}|$ is finite and even, and $\forall d\ (\Psi(d, c, a) \wedge B_d$ is a partial E_0-transversal $\to\ x \notin^* B_d)$"]

Again $\chi(a)$ states that M is an initial segment of $L[a]$ and $\Psi(x, y, a)$ defines an initial segment of $<_{L[a]}$. B_c denotes the Borel set coded by the real c.

So A_0 is $\Sigma^1_2(a)$, and an analogous argument with "even" replaced by "odd" shows that A_1 is $\Sigma^1_2(a)$, so in fact both are $\Delta^1_2(a)$. It remains to show that they are counterexamples to $(\vec{\omega} \to \vec{m})$. Suppose there is an H with $[H] \subseteq A_0$ (w.l.o.g.) and let $x \in [H]$. Let $y \in B_x$ be such that $|\{n \mid x(n) \neq y(n)\}|$ is finite and even. Change just one digit of x to form $x' \in [H]$, so that $|\{n \mid x'(n) \neq y(n)\}|$ is still finite but odd. Note that $x' \in^* B_x$ still holds, hence $B_x = B_{x'}$. Therefore $x' \in A_1$, a contradiction.

We have now proved that there exists an $I_{E_0^\omega}$-quasigeneric over each $L[a]$, but we must still make the move to Cantor space. Consider the following continuous function $f : \omega^\omega \to 2^\omega$: for each x, let

$$f(x)(n) := \begin{cases} 1 & \text{if } x((n)_0) = (n)_1 \\ 0 & \text{otherwise} \end{cases}$$

where $n = \langle (n)_0, (n)_1 \rangle$ is the canonical coding. In other words, f sends every x to the characteristic function of the (encoded) graph of x. It is easy to see that for all $x, y \in \omega^\omega$ we have $x E_0^\omega y \iff f(x) E_0 f(y)$. It follows that if x is

$I_{E_0^\omega}$-quasigeneric over $L[a]$, then $f(x)$ is I_{E_0}-quasigeneric over $L[a]$, as we had to show. □

It would have been desirable to extract a stronger transcendence property from $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$, in the same vein as Theorem 2.1 (2). Although we can easily prove that $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ implies the existence of a bounded $I_{E_0^\omega}$-quasigeneric in the Baire space (by the same reasoning), it is not clear what implications this has for the I_{E_0}-quasigeneric.

On the other hand, we can take a closer look at the $(\vec{\omega} \to \vec{m})$ property on the $\mathbf{\Sigma}^1_2$-level and, this time, get a slightly stronger result. Recall that for many tree-like forcing notions $\mathbb{P}$, one can define an ideal $\mathcal{N}_\mathbb{P}$ as follows:

$$A \in \mathcal{N}_\mathbb{P} \;:\Longleftrightarrow\; \forall p \in \mathbb{P}\; \exists q \leq p\; ([q] \cap A = \varnothing).$$

Let $\mathcal{N}_{E_0}$ be the ideal derived from E_0-trees. By Zapletal's dichotomy, it follows that every Borel set is in $\mathcal{N}_{E_0}$ if and only if it is in I_{E_0}, although in general the two ideals are not the same. We show that $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m})$ implies the existence of co-$\mathcal{N}_{E_0}$-many I_{E_0}-quasigenerics.

3.4 Theorem. $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m}) \Longrightarrow \forall a\; (\{x \mid x$ *not* I_{E_0}*-quasigeneric*$/L[a]\} \in \mathcal{N}_{E_0})$

Proof. Again, we first focus on the relation E_0^ω on the Baire space. The instrumental Lemma is the following:

3.5 Lemma. $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m}) \implies \forall a\; \exists H = \langle H_i \mid i < \omega\rangle$, *each* $|H_i| \geq 2$, *such that* $\forall x \in [H]$ $(x$ *is* E_0^ω*-quasigeneric over* $L[a])$.

Proof. This Lemma is proved similarly to Theorem 3.3. Towards contradiction, suppose a is such that for every H with $|H_i| \geq 2$ there is $x \in [H]$ and $B \in I_{E_0^\omega}$ coded in $L[a]$, such that $x \in B$. As before, this means there is a partial E_0^ω-transversal B coded in $L[a]$ with $x \in^* B$. Let B_x be the $<_{L[a]}$-least such Borel set, *if it exists*. Now form the following two sets:

$$\begin{aligned} A_0 := \{x \mid x \text{ is not } I^\omega_{E_0}\text{-quasigeneric over } L[a] \text{ and } \exists y \in B_x \text{ s.t.} \\ |\{n \mid x(n) \neq y(n)\}| \text{ is finite and even}\}, \\ A_1 := \{x \mid x \text{ is not } I^\omega_{E_0}\text{-quasigeneric over } L[a] \text{ and } \exists y \in B_x \text{ s.t.} \\ |\{n \mid x(n) \neq y(n)\}| \text{ is finite and odd}\}. \end{aligned}$$

The same proof as in Theorem 3.3 shows that both A_0 and A_1 are $\Sigma^1_2(a)$. However, while before the two sets were complements of each other, here A_0 and A_1 only form a partition of $\{x \mid x$ not E_0^ω-quasigeneric$\}$. Therefore we cannot, in general, conclude that A_0 and A_1 are $\Delta^1_2(a)$.

Now, if $[H] \subseteq A_0$, then by assumption we can find $x \in [H]$ which is non-$I^\omega_{E_0}$-quasigeneric, and then proceed as before to alter one digit of x, produce $x' \in [H]$ which is still non-$I^\omega_{E_0}$-quasigeneric but $x' \in A_1$, giving a contradiction.

On the other hand, if $[H] \cap A_0 = \varnothing$ we again pick an $x \in [H]$ which is non-$I_{E_0}^{\omega}$-quasigeneric. But then $x \in A_1$, and the contradiction proceeds analogously. This completes the proof of the Lemma. $\Box$

To finish the proof of Theorem 3.4, let T be an arbitrary E_0-tree. We have to find an E_0-tree $S \leq T$ such that $[S]$ contains exclusively E_0-quasigenerics. Let g be the natural bijection between 2^ω and T, i.e., the map which sends every z to $s_0 {}^\frown s^0_{z(0)} {}^\frown s^1_{z(1)} {}^\frown \dots$, where s^i_j are as in Definition 3.2. It is immediate that g preserves E_0. Let f be the mapping between ω^ω and 2^ω from the proof of Theorem 3.3, i.e., the continuous function such that $x E_0^\omega y$ iff $f(x) E_0 f(y)$. Let $[H]$ be the product containing exclusively E_0^ω-quasigenerics which exists by Lemma 3.5. Shrink H, if necessary, so that $|H_i| = 2$ for every i. Identifying $[H]$ with the Cantor space, we can easily see that the relation E_0^ω restricted to $[H]$ is isomorphic to E_0, and it follows that $[H]$ must be $I_{E_0^\omega}$-positive. Since $(g \circ f)$ preserves E_0^ω relative to E_0, $(g \circ f)``[H]$ is an $I_{E_0}$-positive Borel subset of $[T]$ containing exclusively $E_0$-quasigenerics. By Zapletal's dichotomy, there exists an $E_0$-tree $S$ with $[S] \subseteq (g \circ f)``[H]$, so we are done. $\Box$

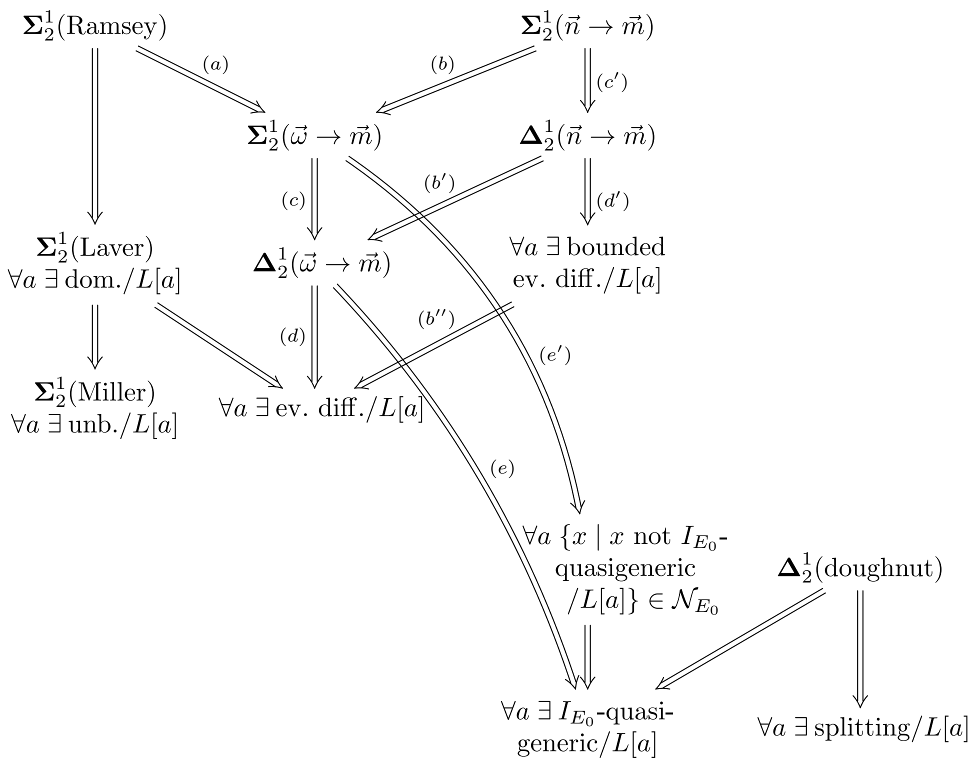
4. Implications and non-implications.

Let us sum up everything we have proved so far in a diagram. In addition to the properties already mentioned, we consider Miller- and Laver-measurability, the doughnut property and splitting reals.

4.1 Definition.

1. A set $A \subseteq \omega^\omega$ is *Miller-measurable* if for every Miller (super-perfect) tree T there is a Miller tree $S \leq T$ such that $[S] \subseteq A$ or $[S] \cap A = \varnothing$. Similarly, A is *Laver-measurable* if the same holds for Laver trees.
2. Let $a \subseteq b \subseteq \omega$ be such that $|b \setminus a| = \omega$. Then $[a, b]^\omega := \{x \in [\omega]^\omega \mid a \subseteq x \subseteq b\}$ is called a *doughnut*. A set $A \subseteq [\omega]^\omega$ has the *doughnut property* if there exists a doughnut which is either completely contained in A or completely disjoint from A.
3. A real $x \in [\omega]^\omega$ is called a *splitting real* over $L[a]$ if for all $y \in [\omega]^\omega \cap L[a]$, both $y \cap x$ and $y \setminus x$ are infinite.

Miller- and Laver-measurability were studied in [4] where it was proved that all $\mathbf{\Sigma}^1_2$ sets are Miller-measurable iff all $\mathbf{\Delta}^1_2$ sets are Miller-measurable iff $\forall a \, \exists x$ (x is unbounded over $L[a]$), and the same for Laver-measurability and dominating reals. The doughnut property is a generalization of the Ramsey property; for more about it and the implications involving it, see [3].



In this diagram, the implications (b), (b'), (b''), (c) and (c') are trivial and (a) is because of Lemma 1.1. The arrows (d) and (d') are Theorem 2.1, and (e) and (e') are Theorems 3.3 and 3.4, respectively.

We are now interested whether the implications in this diagram are the only possible ones. In particular, we would like to prove that all the new implications are strict and cannot be reversed (i.e., they are not equivalences). We start by looking at (e) and (e').

4.2 Lemma. *In the Cohen model, i.e., the model obtained by an ω_1-iteration of Cohen forcing with finite support starting from L, $\mathbf{\Delta}^1_2$(doughnut) holds, $\{x \mid x$ not I_{E_0}-quasigeneric over $L[a]\} \in \mathcal{N}_{E_0}$ holds for every a, but $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ and $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ fail.*

Proof. It is well-known that Cohen forcing does not add eventually different reals, so in the ω_1-iteration both polarized partition properties fail on the $\mathbf{\Delta}^1_2$-level. On the other hand, by [3, Proposition 3.7] all $\mathbf{\Delta}^1_2$ sets (in fact all projective sets and even all sets in $L(\mathbb{R})$) have the doughnut property. Moreover, it is easy to see that Cohen forcing adds an E_0-tree of Cohen reals, and thus an E_0-tree of I_{E_0}-quasigenerics. Using the natural homeomorphisms between E_0-trees and 2^ω, which preserve the E_0-relation, it can easily be seen that in fact Cohen forcing adds co-$\mathcal{N}_{E_0}$-many I_{E_0}-quasigenerics. Therefore the arrows (e) and (e') cannot be reversed. $\square$

Next, we turn to the arrows (b), (b') and (b'')—is the bounded partition property really stronger then the unbounded one? Recall the following properties of forcings:

4.3 Definition. A forcing $\mathbb{P}$ has the

1. *Laver property* if for every $p \in \mathbb{P}$ and every name for a real $\dot{x}$ such that for some y we have $p \Vdash \dot{x} \leq \check{y}$, there is an infinite sequence $S = \langle S_n \mid n < \omega \rangle$ with $\forall n \ (|S_n| \leq 2^n)$, and some $q \leq p$ such that $q \Vdash \dot{x} \in [\check{S}]$.
2. *weak Laver property* if for every $p \in \mathbb{P}$ and every name for a real $\dot{x}$ such that for some y we have $p \Vdash \dot{x} \leq \check{y}$, there is an infinite sequence $S = \langle S_n \mid n < \omega \rangle$ with $\forall n \ (|S_n| \leq 2^n)$, and some $q \leq p$ such that $q \Vdash \exists^\infty n \ (\dot{x}(n) \in \check{S}_n)$.

In fact the weak Laver property has a simpler characterization:

4.4 Lemma. *A forcing $\mathbb{P}$ has the weak Laver property iff it does not add a bounded eventually different real.*

Proof. Throughout the proof, let V be the ground model and $V^{\mathbb{P}}$ the extension. Clearly, if for every bounded real x in $V^{\mathbb{P}}$ there is $y \in V$ infinitely equal to x, then there is also a sequence $S \in V$ with the same property—any S containing y will do. So it remains to prove the converse: let $x \in V^{\mathbb{P}}$ be a real bounded by $y \in V$. Partition ω into $\{B^n \mid n \in \omega\}$ by letting $B^0 := \{0\}, B^1 := \{1,2\}, B^2 := \{3,4,5,6\}$ and so on with $|B^n| = 2^n$. For convenience enumerate $B^n = \{b^n_0, \dots, b^n_{2^n-1}\}$. Let φ be the continuous function defined by $\varphi(x)(n) = \left\langle x(b^n_0), \dots, x(b^n_{2^n-1}) \right\rangle$.

Clearly $x' := \varphi(x)$ is bounded by $\varphi(y) \in V$. Let $S \in V$ be a seqeunce satisfying $\forall n \ (|S_n| \leq 2^n)$ and $\exists^\infty n \ (x'(n) \in S_n)$. Enumerate every S_n as $\{a^n_0, \dots, a^n_{2^n-1}\}$. Now, let $\{s^n_0, \dots, s^n_{2^n-1}\}$ be members of ${}^{B^n}\omega$ such that $\left\langle s^n_j(b^n_0), \dots, s^n_j(b^n_{2^n-1}) \right\rangle = a^n_j$ for every j. Then from the definition of φ it follows that for every n, if $x'(n) \in S_n$ then $x \restriction B^n = s^n_j$ for one of the j's. Hence $\exists^\infty n \ (x \restriction B^n = s^n_j$ for some $j)$. But then we can define a new real z by "diagonalizing" all the possible s^n_j's, that is, $z(b^n_i) := s^n_i(b^n_i)$. Then x is infinitely equal to z, and since z has been explicitly constructed from S, it follows that $z \in V$. This completes the proof. $\square$

4.5 Corollary. *The Mathias model, i.e., the model obtained by an ω_1-iteration of Mathias forcing with countable support starting from L, $\mathbf{\Sigma}^1_2$(Ramsey) holds while $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ fails.*

Proof. It is well-known that $\mathbf{\Sigma}^1_2$(Ramsey) holds in the iterated Mathias model. However, it is also known that Mathias forcing satisfies the Laver property (cf. [2, Section 7.4]), and that this is preserved by the ω_1-iteration. Therefore the iteration certainly also has the weak Laver property. By the above Lemma that implies that in the Mathias model there are no bounded eventually different reals and therefore $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ fails. $\square$

So the arrows (b), (b') and (b'') are also irreversible. The nature of implications (c), (c'), (d) and (d') is still unknown. We conjecture that (d) and (d') are strict implications but efforts to prove this have so far been unsuccessful.

In the next section we prove a strong result which, in particular, will show that the arrow (a) is irreversible.

5. A fat creature forcing

We will now construct a forcing notion $\mathbb{P}$ which yields $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ without adding unbounded or splitting reals. This forcing can be seen as a hybrid of two forcing notions already existing in the literature: the one used by DiPrisco and Todorčević in [6] to prove the original result $\mathbf{\Sigma}^1_1(\vec{n} \to \vec{m})$ in ZFC, and a creature forcing developed by Shelah and Zapletal in [17] and Kellner and Shelah in [12]. The latter forcing does not add unbounded or splitting reals by [17] and can be applied directly to yield $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$, but seems insufficient for $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$. The DiPrisco-Todorčević forcing, on the other hand, does yield $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ but it is so combinatorially complex that it is difficult to prove preservation theorems about it, such as being ω^ω-bounding or not adding splitting reals. That is why we choose a "hybrid" solution.

We start with the following consideration: it is easy to compute integers $M_0, M_1, \dots$ such that the partition

$$\begin{pmatrix} M_0 \\ M_2 \\ \dots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \dots \end{pmatrix}$$

holds for *closed* partitions. For a proof, see [7, Theorem 1] or use an argument like in the proof of Theorem 5.7 (1). We fix such integers M_i for the rest of this section. The next definition and the Lemma following it are instrumental in our approach to constructing a model of $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$.

5.1 Definition. Let M be a model of set theory and H an infinite sequence. We say that H has the *clopification property* with respect to M if for every Borel set B with Borel code in M, the set $B \cap [H]$ is clopen relative to $[H]$ (i.e., in the subset topology on $[H]$ inherited from the standard topology on ω^ω).

5.2 Lemma. *If for every $a \in \omega^\omega$ there is an H with $|H_i| = M_i$ having the clopification property with respect to $L[a]$, then $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{2})$ holds. If, moreover, H is bounded by some recursive $\langle n_i \mid i < \omega \rangle$, then $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{2})$ holds.*

Proof. Let A be $\Sigma^1_2(a)$. If for some a, $\omega_1^{L[a]} < \omega_1$, then $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ in fact follows directly, by a standard argument using e.g. the forcing from [17]. So we may assume that $\forall a\ (\omega_1^{L[a]} = \omega_1)$. Then, using Shoenfield's classical analysis of $\mathbf{\Sigma}^1_2$ sets we can write A as $\bigcup_{\alpha < \omega_1} B_\alpha$ where each B_α is a Borel set coded in $L[a]$. Let H be the sequence of sets with the clopification property. Then for each α, $B_\alpha \cap [H]$ is clopen relative to $[H]$, so $A \cap [H] = \bigcup_{\alpha<\omega_1}(B_\alpha \cap [H])$ is open relative to $[H]$, so the result follows. The second statement of the theorem is also clear. □

We will construct a forcing notion $\mathbb{P}$ with the following three properties:

1. $\mathbb{P}$ adds a *generic product* H_G, such that $\Vdash_{\mathbb{P}}$"$[\dot{H}_G]$ has the clopification property with respect to the ground model, and is bounded by a recursive sequence $\vec{n}$",
2. $\mathbb{P}$ is proper and ω^ω-bounding (every new real is bounded by a real from the ground model), and
3. $\mathbb{P}$ does not add splitting reals (for every $a \in [\omega]^\omega$ there is $b \in [\omega]^\omega$ in the ground model with $b \subseteq a$ or $b \cap a = \varnothing$.)

It is well-known that being proper and ω^ω-bounding is a property preserved by ω_1-iterations with countable support. The property of not adding splitting reals may not be preserved, however its conjunction with being ω^ω-bounding is, by [20, Corollary 6.3.8., p 290]. So, assuming we are able to construct $\mathbb{P}$ we have the following main result of this section:

5.3 Theorem. *In the model obtained by an ω_1-iteration of $\mathbb{P}$, with countable support, starting from L, $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ holds whereas both $\mathbf{\Sigma}^1_2$(Miller) and $\mathbf{\Delta}^1_2$(doughnut) fail. In particular, implication (a) in the diagram cannot be reversed.*

We now proceed with the construction of $\mathbb{P}$. We start by defining, for each n, a local partial order $(\mathbb{P}_n, \leq_n)$. After that $\mathbb{P}$ will be constructed roughly as a product of the $\mathbb{P}_n$.

5.4 Definition.

- For n, let ϵ_n be a given "small" positive real number, and let X_n be a "large" integer. The precise nature of these two numbers will be determined later. Let $\text{prenorm}_n : \mathscr{P}(X_n) \longrightarrow \omega$ be a function satisfying the following condition:

  > For every $c \subseteq X_n$, if $\text{prenorm}_n(c) \geq 1$ then for every partition of $[c]^{M_n}$ into two parts A_0 and A_1, there exists a $d \subseteq c$ such that $\text{prenorm}_n(d) \geq \text{prenorm}_n(c) - 1$ and $[d]^{M_n}$ is completely contained in A_0 or A_1.

- $\mathbb{P}_n$ consists of tuples (c, k), where $c \subseteq X_n$ and k is a natural number, such that $\text{prenorm}_n(c) \geq k + 1$. The ordering is given by $(c', k') \leq_n (c, k)$ iff $c' \subseteq c$ and $k \leq k'$.

- Let $\text{norm}_n : \mathbb{P}_n \longrightarrow \mathbb{R}$ be any function having the property that for every (c, k) with $\text{norm}_n(c, k) \geq \epsilon_n$, if (d, l) is such that $\text{prenorm}_n(d) - l \geq \frac{1}{2}(\text{prenorm}_n(c) - k)$, then $\text{norm}_n(d, l) \geq \text{norm}_n(c, k) - \epsilon_n$. One particular such function is given by

  $$\text{norm}_n(c, k) := \epsilon_n \cdot \log_2(\text{prenorm}_n(c) - k)$$

  but any other function with this property would suffice, too.

Note that one can have trivial partial orders satisfying the above conditions, for example, by choosing the X_n small and the function prenorm_n to be constantly 0. So we put an additional requirement: for each n, there must be at least one condition $(c, k) \in \mathbb{P}_n$ such that $\text{norm}_n(c, k) \geq n$. This can be accomplished by picking the X_n sufficiently large and using the finite Ramsey theorem to define prenorm_n. In general the value of X_n will depend on ϵ_n, i.e., the smaller the latter is the larger the former must be. If norm_n is defined by the explicit computation above, then X_n must be so large that for at least one $c \subseteq X_n$, $\text{prenorm}_n(c) \geq 2^{(n/\epsilon_n)}$.

5.5 Definition. The forcing notion $\mathbb{P}$ contains conditions p which are functions with domain ω, such that for some $K \in \omega$:

- $\forall n < K : p(n) \subseteq X_n$ and $|p(n)| = M_n$,
- $\forall n \geq K : p(n) \in \mathbb{P}_n$, and
- the function mapping n to $\text{norm}_n(p(n))$ converges to infinity.

K is the *stem-length* of p and $p \restriction K$ is the *stem* of p. For two conditions p and p' with stem-length K and K', the ordering is given by $p' \leq p$ iff

- $\text{stem}(p) \subseteq \text{stem}(p')$,
- $\forall n \in [K, K') : p'(n) \subseteq c$, where $p(n) = (c, k)$, and
- $\forall n \geq K' : p'(n) \leq_n p(n)$.

This forcing is very similar to the creature forcing defined in [12] and [17] and we refer the reader to these papers for some additional discussion about its properties. The main difference is that our forcing notion $\mathbb{P}$ does not just add one generic real, but a whole *generic product* of finite subsets of ω, defined from the generic filter G by

$$H_G := \bigcup \{\text{stem}(p) \mid p \in G\}.$$

By construction $H_G(n) \subseteq X_n$ and $|H_G(n)| = M_n$. Each forcing condition in G contains an initial segment of this generic product, namely the stem, concatenated with a sequence of $\mathbb{P}_n$-conditions with norms converging to infinity. Note that this is only possible because we have chosen the X_n to be increasing sufficiently fast.

Next, let us introduce some notation.

5.6 Notation.

1. If $(c, k) \in \mathbb{P}_n$, we refer to the first coordinate c by "val", i.e., $\text{val}(c, k) = c$. By a slight abuse of notation, if p is a condition with stem-length K we define $\text{val}(p(n)) = p(n)$ for all $n < K$.
2. For $p \in \mathbb{P}$, let $\mathcal{T}(p) := \{s \in \omega^{<\omega} \mid \forall n : \ s(n) \in \text{val}(p(n))\}$.

3. Let $\mathcal{S}eq$ denote the set of all finite initial segments potentially in the generic product, i.e.:

$$\mathcal{S}eq := \{\sigma : m \to \mathscr{P}(\omega) \mid m \in \omega \text{ and } \forall n < m\ (\sigma(n) \subseteq X_n \text{ and } |\sigma(n)| = M_n)\}.$$

For n, let $\mathcal{S}eq_n := \{\sigma \in \mathcal{S}eq \mid |\sigma| = n\}$.
4. For $p \in \mathbb{P}$, let $\mathcal{S}eq(p) := \{\sigma \in \mathcal{S}eq \mid \forall n : \ \sigma(n) \subseteq \mathrm{val}(p(n))\}$ and $\mathcal{S}eq_n(p) := \{\sigma \in \mathcal{S}eq(p) \mid |\sigma| = n\}$.
5. For $\sigma \in \mathcal{S}eq(p)$, let $p \uparrow \sigma$ be the $\mathbb{P}$-condition defined by

$$(p \uparrow \sigma)(n) := \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ p(n) & \text{otherwise} \end{cases}$$

We will use the letters $s, t, \dots$ for elements of $\omega^{<\omega}$ and $\sigma, \tau, \dots$ for elements of $\mathcal{S}eq$.

It is important to note that the forcing $\mathbb{P}$ is not separative. In particular $\mathcal{T}(q) \subseteq \mathcal{T}(p)$ does not imply $q \leq p$. However, if there exists a K such that $\mathcal{T}(q) \restriction K \subseteq \mathcal{T}(p) \restriction K$ and $\forall n \geq K : q(n) \leq_n p(n)$, then q is inseparable from p, and hence forces whatever p forces. We shall need this fact several times in the proofs.

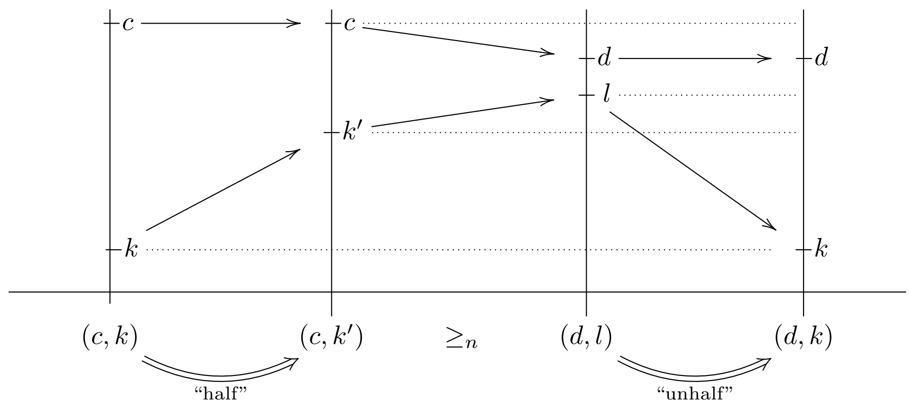
In [12, 17], the main tools for proving results about the forcing notion were so-called ϵ_n*-bigness* and ϵ_n*-halving*. In our setting, the former is significantly stronger although the latter is essentially the same.

- "ϵ_n-bigness" is essentially a re-statement of the definition of prenorm. If $(c, k) \in \mathbb{P}_n$ is any condition with $\mathrm{norm}_n(c, k) \geq \epsilon_n$, then $\mathrm{prenorm}_n(c) - k \geq 2$. In particular, if $[c]^{M_n}$ is partitioned into two pieces A_0 and A_1, then, by the definition of prenorm, there is a $d \subseteq c$ such that $[d]^{M_n}$ is completely contained in A_0 or A_1 and $\mathrm{prenorm}_n(d) \geq \mathrm{prenorm}_n(c) - 1$. In particular, $\mathrm{prenorm}_n(d) - k \geq \mathrm{prenorm}_n(c) - k - 1 \geq \frac{1}{2}(\mathrm{prenorm}_n(c) - k)$, therefore $(d, k) \leq_n (c, k)$ is a valid $\mathbb{P}_n$-condition with $\mathrm{norm}_n(d, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$.

- By "ϵ_n-halving" we mean the following phenomenon: if $(c, k) \in \mathbb{P}_n$ is any condition with $\mathrm{norm}_n(c, k) \geq \epsilon_n$, then let $k' := \lfloor \frac{1}{2}(\mathrm{prenorm}_n(c) + k) \rfloor$. The condition $(c, k') \leq_n (c, k)$ is called the *half of* (c, k), denoted by $\mathrm{half}(c, k)$. It satisfies the following conditions:

  - $\mathrm{norm}_n(c, k') \geq \mathrm{norm}_n(c, k) - \epsilon_n$, and
  - every $(d, l) \leq_n (c, k')$ can be "un-halved" to $(d, k) \leq_n (c, k)$ with $\mathrm{norm}_n(d, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$.

The last inequality holds because $\mathrm{prenorm}_n(d) - k \geq \frac{1}{2}(\mathrm{prenorm}_n(c) - k)$.



5.7 Theorem.

1. *Let $\mathbb{P}$ be the forcing described above, and assume that for all n, $\epsilon_n \leq 1/\left(\prod_{i<n} X_i\right)$. Then $\Vdash_{\mathbb{P}}$ "$[\dot{H}_G]$ has the clopification property w.r.t. the ground model and is bounded".*
2. *Assume that, additionally, for all n, $\epsilon_n \leq 1/\left(\prod_{i<n} \binom{X_i}{M_i}\right)$. Then $\mathbb{P}$ is proper and ω^ω-bounding.*
3. *Assume that, additionally, for all n, $\epsilon_n \leq 1/(\prod_{i<n} \mathrm{prenorm}_i(X_i) \cdot 2^{X_i})$. Then $\mathbb{P}$ does not add splitting reals.*

Recall that the numbers X_n depend on the value of ϵ_n. In this theorem, we require that ϵ_n depends on the previous values of X_i. The combination of these two requirements gives an inductive computation of the numbers X_n which eventually form the upper bound $\vec{n}$ in the partition property ($\vec{n} \to \vec{m}$).

Part 1 of this theorem is loosely based on [6] and Parts 2 and 3 are variations of the proofs in [17]. The rest of this section is devoted to the proof of these three claims.

Before starting on the proofs, let us stipulate how *fusion* works in the case of $\mathbb{P}$: for two conditions p and q and $k \in \omega$, say that $q \leq_{(k)} p$ iff $q \leq p$ and there is a $K \geq k$ such that $p \restriction K = q \restriction K$ and for all $n \geq K : \mathrm{norm}_n(q(n)) \geq k$. It is easy to verify that if $p_0 \geq_{(0)} p_1 \geq_{(1)} p_2 \geq_{(2)} \ldots$ is a fusion sequence, then the natural (pointwise) limit q of this sequence is a $\mathbb{P}$-condition below every p_i.

Proof of 1. For every Borel set B, define $D_B := \{p \in \mathbb{P} \mid B \cap [\mathcal{T}(p)]$ is clopen in $[\mathcal{T}(p)]\}$. Since every $p \in \mathbb{P}$ forces "$[\dot{H}_G] \subseteq [\mathcal{T}(p)]$" it is sufficient to show that every D_B is dense. Define

$$\mathrm{CL} := \{A \subseteq \prod_i X_i \mid \forall p \in \mathbb{P}\, \forall k\, \exists q \leq_{(k)} p\, (A \cap [\mathcal{T}(q)] \text{ is clopen in } [\mathcal{T}(q)])\}.$$

We claim that:

1. A is closed $\implies A \in \mathrm{CL}$,
2. $A \in \mathrm{CL} \implies (\prod_i X_i \setminus A) \in \mathrm{CL}$, and
3. if $A_n \in \mathrm{CL}$ for every n, then $\bigcap_n A_n \in \mathrm{CL}$.

In particular, all Borel sets are in CL and hence every D_B is dense.

Point 2 of the claim follows trivially from the definition of CL. Also, once we have proven point 1, point 3 will follow more or less immediately: by a standard fusion construction $\bigcap_n A_n$ can be rendered relatively closed, and by an application of point 1, it can then be rendered relatively clopen. We leave the details of this construction to the reader and instead focus our efforts on the proof of point 1.

First we need to fix some terminology: let T be any tree, and $X \subseteq T$. For $t \in T$ we say that "the membership of t in X depends only on $t \restriction m$" if

$$t \in X \iff \forall s \in T\ (t \restriction m \subseteq s \ \rightarrow\ s \in X) \text{ and}$$

$$t \notin X \iff \forall s \in T\ (t \restriction m \subseteq s \ \rightarrow\ s \notin X).$$

Let $\mathbb{P} \restriction m := \{p \restriction m \mid p \in \mathbb{P}\}$. If $h \in \mathbb{P} \restriction m$ is such that $h = p \restriction m$, we define $\mathcal{T}(h) := \mathcal{T}(p) \restriction m$, i.e., the tree of finite sequences through h.

Now suppose C is a closed subset of $\prod_i X_i$ and let T_C be the tree of C. Let $p \in \mathbb{P}$ be a condition and $k \in \omega$. Find K such that $\forall n \geq K : \mathrm{norm}_n(p(n)) \geq k+1$. We claim the following:

Subclaim. *For all $m > K$, there is $h \in \mathbb{P} \restriction m$ such that $h \restriction K = p \restriction K$, $\forall n \in [K, m) : \mathrm{norm}_n(h(n)) \geq \mathrm{norm}_n(p(n)) - 1$, and for every $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t \restriction K$.*

Proof. The proof works by backward-induction, from m down to K. First, we set $n := m - 1$. Let $\{s_0, \ldots, s_{l-1}\}$ enumerate $\mathcal{T}(p) \restriction n$. Suppose $p(n) = (c, k)$. We partition c into two parts: $A_0 := \{i \in c \mid s_0{}^\frown \langle i \rangle \in T_C\}$ and $A_1 := c \setminus A_0$. Note that this can be viewed as a partition of $[c]^1$. Our version of "ϵ_n-bigness" is meant to take care of partitions of $[c]^{M_n}$, so it certainly takes care of partitions of $[c]^1$. Therefore, there exists a $(c_0, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_0, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$ and $c_0 \subseteq A_0$ or $c_0 \subseteq A_1$. Now, partition c_0 again into two parts: $A'_0 := \{i \in c_0 \mid s_1{}^\frown \langle i \rangle \in T_C\}$ and $A'_1 := c_0 \setminus A'_0$. Again, ϵ_n-bigness allows us to shrink to a condition $(c_1, k) \leq_n (c_0, k)$ such that $\mathrm{norm}_n(c_1, k) \geq \mathrm{norm}_n(c_0, k) - \epsilon_n$ and $c_1 \subseteq A'_0$ or $c_1 \subseteq A'_1$. We can continue this procedure until we have dealt with all of the s_i. So in the end we have a condition $(c_{l-1}, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_{l-1}, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n \cdot l$ and, if we define $h := p \restriction n{}^\frown \langle (c_{l-1}, k) \rangle$, then for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t \restriction n$. Notice that $l \leq \prod_{i<n} X_i$, so by the assumption on the size of ϵ_n it follows that $\mathrm{norm}_n(c_{l-1}, k) \geq \mathrm{norm}_n(c, k) - 1$.

Now we go one step back, set $n := m-2$, let $\{s_0, \ldots, s_{l-1}\}$ enumerate $\mathcal{T}(p) \restriction n$, and repeat exactly the same procedure. Again, we apply ϵ_n-bigness l times (for the new value of l) and in the end get a new condition, say $h(n)$, such that

$\text{norm}_n(h(n)) \geq \text{norm}_n(p(n)) - 1$ and for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t \restriction n$.

Finally we reach K, and see that we have constructed a partial condition $h \in \mathbb{P} \restriction m$, such that $h \restriction K = p \restriction K$, $\forall n \in [K, m) : \text{norm}_n(h(n)) \geq \text{norm}(p(n)) - 1$ and for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t \restriction K$. □(subclaim.)

Let $\mathfrak{T}$ be the collection of all h that satisfy the statement of the subclaim for some $m > K$, i.e., $\mathfrak{T} := \{h \mid h \in \mathbb{P} \restriction m$ for some $m > K$, $h \restriction K = p \restriction K$, $\forall n \in [K, m) : \text{norm}_n(h(n)) \geq \text{norm}_n(p(n)) - 1$, and for all $t \in \mathcal{T}(h)$, the membership of t in T_C depends only on $t \restriction K\}$. Notice that if $h \in \mathfrak{T}$ and j is an initial segment of h with $|j| > K$, then $j \in \mathfrak{T}$. Therefore $\mathfrak{T}$ is a tree with respect to the ordering of initial segments. It is clearly a *finitely branching* tree, but it is also an *infinite* tree by the subclaim. Therefore, by König's Lemma, $\mathfrak{T}$ has an infinite branch, which we call q. It is now straightforward to verify that $q \restriction K = p \restriction K$, that $\forall n > K : \text{norm}_n(q(n)) \geq \text{norm}_n(p(n)) - 1 \geq k$, and that for every $x \in [\mathcal{T}(q)]$, the membership of x in C depends only on $x \restriction K$. But this is exactly to say that $q \leq_{(k)} p$ and $C \cap [\mathcal{T}(q)]$ is clopen in $[\mathcal{T}(q)]$, thus completing the proof. □

Now we can look at the proof of part 2 of Theorem 5.7.

Proof of 2. Let $\dot{\alpha}$ be a name for an ordinal. If $p \in \mathbb{P}$ is a condition, we say that p *essentially decides* $\dot{\alpha}$ if there is m such that $\forall \sigma \in \mathcal{S}eq_m(p) : p \uparrow \sigma$ decides $\dot{\alpha}$. It is clear that if p essentially decides $\dot{\alpha}$ then p forces $\dot{\alpha}$ into a finite set in the ground model. Therefore, what we must prove is that for each $p \in \mathbb{P}$ and k there is a $q \leq_{(k)} p$ which essentially decides $\dot{\alpha}$—by standard techniques this will allow us to build a fusion sequence showing that $\mathbb{P}$ is proper and ω^ω-bounding.

For a $p \in \mathbb{P}$ and $\sigma \in \mathcal{S}eq(p)$, we call σ *deciding (in p)* if $p \uparrow \sigma$ essentially decides $\dot{\alpha}$, and *bad (in p)* if there is no $p' \leq p \uparrow \sigma$ with $\text{stem}(p') = \sigma$ which essentially decides $\dot{\alpha}$.

5.8 Lemma. *Let $p \in \mathbb{P}$ and $K \in \omega$ be such $\forall n > K : \text{norm}_n(p(n)) \geq N$ for some $N \geq 1$. Then there is a $q \leq p$ such that $q \restriction K = p \restriction K$, $\forall n \geq K : \text{norm}_n(q(n)) \geq N - 1$, and every $\sigma \in \mathcal{S}eq_K(q)$ is either deciding or bad (in q).*

Proof. Let $\{\sigma_0, \dots, \sigma_{l-1}\}$ enumerate $\mathcal{S}eq_K(p)$. Let $p_{-1} := p$ and, by induction, do the following construction: for each i, suppose p_{i-1} has been defined and for all $n \geq K : \text{norm}_n(p_{i-1}(n)) \geq N - \epsilon_n \cdot i$. Then there are two cases:

- Case 1: there is a $p' \leq p_{i-1} \uparrow \sigma_i$ such that $\forall n \geq K : \text{norm}_n(p'(n)) \geq N - \epsilon_n \cdot (i+1)$ and p' essentially decides $\dot{\alpha}$. Let $p_i := p \restriction K^\frown(p' \restriction [K, \infty))$.
- Case 2: it is not possible to find such a p'. Then, define p_i by $p_i \restriction K := p \restriction K$ and $\forall n \geq K : p_i(n) := \text{half}(p_{i-1}(n))$.

Finally let $q := p \restriction K^\frown(p_{l-1} \restriction [K, \infty))$. Clearly $q \leq p$ and for $n \geq K$ we have $\text{norm}_n(q(n)) \geq N - \epsilon_n \cdot l$. Since $l \leq \prod_{i<K} \binom{X_i}{M_i}$, the assumption on the size of ϵ_n implies that $\text{norm}_n(q(n)) \geq N - 1$.

Every σ_i for which Case 1 occurred is clearly deciding (in q). If Case 2 occurred, we will show that σ_i is bad. Suppose not, i.e., suppose there is a $q' \leq q \uparrow \sigma_i$ such that $\mathrm{stem}(q') = \sigma_i$ and q' essentially decides $\dot{\alpha}$. Let $L > K$ be such that $\forall n > L : \mathrm{norm}_n(q'(n)) \geq N - \epsilon_n \cdot (i+1)$. For every $n \in [K, L)$, by assumption $p_i(n) = \mathrm{half}(p_{i-1}(n))$. Since $q'(n) \leq q(n) \leq p_i(n)$, by the property called "ϵ_n-halving" there exists a condition $r(n) \leq p_{i-1}(n)$ such that $\mathrm{norm}_n(r(n)) \geq \mathrm{norm}_n(p_{i-1}(n)) - \epsilon_n$ and $\mathrm{val}(r(n)) = \mathrm{val}(q'(n))$. Define $r' := \sigma_i{}^\frown(r \restriction [K, L)){}^\frown(q' \restriction [L, \infty))$. Then for all $n \geq K$ we have $\mathrm{norm}_n(r'(n)) \geq N - \epsilon_n \cdot (i+1)$. Moreover, $\forall n \leq L$ we know that $\mathrm{val}(r'(n)) = \mathrm{val}(q'(n))$ and $\forall n > L : r'(n) = q'(n)$. As we mentioned before, this implies that r' is inseparable from q', and since q' essentially decides $\dot{\alpha}$, so does r'. But now the condition r' satisfies all the requirements for Case 1 to occur at step i of the induction, which is a contradiction. $\square$

For the next Lemma, we fix the following terminology: let $T \subseteq \mathcal{S}eq$ be a set closed under initial segments and $X \subseteq T$. For $\sigma \in T$ we say that "the membership of σ in X depends only on $\sigma \restriction m$" if

$$\sigma \in X \iff \forall \tau \in T \ (\sigma \restriction m \subseteq \tau \ \rightarrow \ \tau \in X) \text{ and}$$

$$\sigma \notin X \iff \forall \tau \in T \ (\sigma \restriction m \subseteq \tau \ \rightarrow \ \tau \notin X).$$

5.9 Lemma. *Let $p \in \mathbb{P}$ and $K < K'$ be such that $\forall n \in [K, K') : \mathrm{norm}_n(p(n)) \geq 1$. Let $X \subseteq \mathcal{S}eq_{K'}(p)$. Then there exists a $q \leq p$ such that $q \restriction K = p \restriction K$, $q \restriction [K', \infty) = p \restriction [K', \infty)$, for all $n \in [K, K') : \mathrm{norm}_n(q(n)) \geq \mathrm{norm}_n(p(n)) - 1$, and for all $\sigma \in \mathcal{S}eq_{K'}(q)$, the membership of σ in X depends only on $\sigma \restriction K$.*

Proof. This proof works by backward-induction, analogously to the proof of the subclaim in the proof of Theorem 5.7 (1) above. First we set $n := K' - 1$. Let $\{\sigma_0, \dots, \sigma_{l-1}\}$ enumerate $\mathcal{S}eq_n(p)$. Suppose $p(n) = (c, k)$. We partition $[c]^{M_n}$ into two parts: $A_0 := \{b \subseteq c \mid |b| = M_n \text{ and } \sigma_0{}^\frown \langle b \rangle \in X\}$, and $A_1 := [c]^{M_n} \setminus A_0$. By ϵ_n-bigness, there exists a condition $(c_0, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_0, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n$ and $[c_0]^{M_n} \subseteq A_0$ or $[c_0]^{M_n} \subseteq A_1$. Now, partition $[c_0]^{M_n}$ again into two parts: $A'_0 := \{b \subseteq c_0 \mid |b| = M_n \text{ and } \sigma_1{}^\frown \langle b \rangle \in X\}$, and $A'_1 := [c_0]^{M_n} \setminus A'_0$. Again, ϵ_n-bigness allows us to shrink to a condition $(c_1, k) \leq_n (c_0, k)$ such that $\mathrm{norm}_n(c_1, k) \geq \mathrm{norm}_n(c_0, k) - \epsilon_n$ and $[c_1]^{M_n} \subseteq A'_0$ or $[c_1]^{M_n} \subseteq A'_1$, etc. Finally we get a condition $(c_{l-1}, k) \leq_n (c, k)$ such that $\mathrm{norm}_n(c_{l-1}, k) \geq \mathrm{norm}_n(c, k) - \epsilon_n \cdot l$. If we define $p_{K'-1} := p \restriction (K'-1){}^\frown \langle (c_{l-1}, k) \rangle{}^\frown(p \restriction [K', \infty))$, then for all $\tau \in \mathcal{S}eq_{K'}(p_{K'-1})$, the membership of τ in X depends only on $\tau \restriction (K'-1)$. Moreover, $l \leq \prod_{i<K} \binom{X_i}{M_i}$, so by the assumption on the size of ϵ_n it follows that $\mathrm{norm}_n(c_{l-1}, k) \geq \mathrm{norm}_n(c, k) - 1$.

Now we repeat the same procedure for $n := K' - 2$ and find a new condition $p_{K'-2}$, such that $p_{K'-2} \restriction (K'-2) = p \restriction (K'-2)$, $p_{K'-2} \restriction [K', \infty) = p \restriction [K', \infty)$, $\forall n \in \{K'-2, K'-1\} : \mathrm{norm}_n(p_{K'-2}(n)) \geq \mathrm{norm}_n(p(n)) - 1$, and for all $\tau \in \mathcal{S}eq_{K'}(p_{K'-2})$, the membership of τ in X depends only on $\tau \restriction (K'-2)$.

Finally we reach K, and see that we have constructed a condition $q := p_K$ such that $q \restriction K = p \restriction K$, $q \restriction [K', \infty) = p \restriction [K', \infty)$, $\forall n \in [K, K') : \mathrm{norm}_n(q(n)) \geq$

norm$(p(n)) - 1$, and for all $\tau \in \mathcal{S}eq_{K'}(q)$, the membership of τ in X depends only on $\tau \restriction K$. $\square$

We are ready to prove the main result. Let $p \in \mathbb{P}$ and k be given. We must find a $q \leq_{(k)} p$ which essentially decides $\dot{\alpha}$. Find $K \geq k$ such that $\forall n \geq K$: $\mathrm{norm}_n(p(n)) \geq k+2$. Apply Lemma 5.8 with p and K to get a condition $q \leq p$ such that $q \restriction K = p \restriction K$, $\forall n \geq K : \mathrm{norm}_n(q(n)) \geq k+1$ and every $\sigma \in \mathcal{S}eq_K(q)$ is either deciding or bad. If every σ is deciding then q essentially decides $\dot{\alpha}$, and $q \leq_{(k)} p$ holds, so the proof is complete. We will show that this is the only possibility, i.e., that no $\sigma \in \mathcal{S}eq_K(q)$ can be bad.

Towards contradiction, fix some $\sigma \in \mathcal{S}eq_K(q)$ which is bad. By induction, we will construct an increasing sequence of integers $K_0 < K_1 < K_2 < \dots$ and conditions $q_0 \geq q_1 \geq \dots$. We start by setting $K_0 := K$ and $q_0 := q \uparrow \sigma$. The induction hypothesis for stage i says that

1. $\forall n \geq K_i : \mathrm{norm}_n(q_i(n)) \geq k+i+1$, and
2. all $\tau \in \mathcal{S}eq_{K_i}(q_i)$ are bad.

We will also guarantee that $\forall i, \forall j \geq i+1, \forall n \geq K_i : \mathrm{norm}_n(q_j(n)) \geq k+i$.

Clearly, q_0 satisfies the conditions since the only $\tau \in \mathcal{S}eq_K(q)$ is σ. Suppose K_j and q_j have been defined for $j < i$. We describe the i-th induction step. Let K_i be such that $\forall n \geq K_i : \mathrm{norm}_n(q_{i-1}(n)) \geq k+i+2$. Apply Lemma 5.8 with parameters q_{i-1} and K_i to find a condition $q'_i \leq q_{i-1}$ such that $q'_i \restriction K_i = q_{i-1} \restriction K_i$, $\forall n \geq K_i : \mathrm{norm}_n(q'_i(n)) \geq k+i+1$ and every $\tau \in \mathcal{S}eq_{K_i}(q'_i)$ is either deciding or bad. Now apply Lemma 5.9 on the condition q'_i and the interval $[K_{i-1}, K_i)$ to find a condition $q_i \leq q'_i$ such that $q_i \restriction K_{i-1} = q'_i \restriction K_{i-1}$, $q_i \restriction [K_i, \infty) = q'_i \restriction [K_i, \infty)$, for all $n \in [K_{i-1}, K_i) : \mathrm{norm}_n(q(i)) \geq \mathrm{norm}_n(q'_i(n)) - 1 \geq k + (i-1)$, and for all $\tau \in \mathcal{S}eq_{K_i}(q_i)$, whether τ is deciding or bad depends only on $\tau \restriction K_{i-1}$.

If there is any $\tau' \in \mathcal{S}eq_{K_{i-1}}(q_i)$ such that all $\tau \in \mathcal{S}eq_{K_i}(q_i)$ extending τ' are deciding, then τ' itself would be deciding (in q_i), and hence τ' could not be bad in q_{i-1}, contradicting the induction hypothesis. Thus, in fact all $\tau \in \mathcal{S}eq_{K_i}(q_i)$ must be bad, which completes the i-th induction step.

In the end, let q_ω be the limit of this sequence. It is clear that $\forall i\ \forall n \in [K_i, K_{i+1}) : \mathrm{norm}_n(q_\omega(n)) \geq k+i$ and hence q_ω is a valid $\mathbb{P}$-condition. By construction, all $\tau \in \mathcal{S}eq(q_\omega)$ are bad. But there must be some $r \leq q_\omega$ deciding $\dot{\alpha}$, and then stem(r) cannot be bad. This contradiction completes the proof. $\square$

Finally, we turn to the splitting reals.

Proof of 3. Let $\dot{x}$ be a name for an element of 2^ω and p a condition. To show that $\mathbb{P}$ does not add splitting reals, it suffices to find a condition $q \leq p$ such that for infinitely many n, q decides $\dot{x}(n)$. By the previous argument, we can assume, w.l.o.g., that p essentially decides $\dot{x}(i)$ for every i.

Here we need to introduce new notation. For two partial conditions $h, j \in \mathbb{P} \restriction K$, $h \leq j$ is defined as for conditions in $\mathbb{P}$. For every $p \in \mathbb{P}$, let $\mathcal{S}ub_K(p) := \{h \in (\mathbb{P} \restriction K) \mid h \leq p\}$. Consider any $h \in \mathcal{S}ub_K(p)$, where $K > |\mathrm{stem}(p)|$. Call such an h *i-deciding* (in p) if $h^\frown(p \restriction [K, \infty))$ decides $\dot{x}(j)$ for some $j > i$, and

i-*bad* (in p) if there is no $p' \leq p$ such that $p' \restriction K = h$ which decides $\dot{x}(j)$ for any $j > i$.

5.10 Lemma. *Let $p \in \mathbb{P}$ and $K \in \omega$ be such $\forall n > K : \mathrm{norm}_n(p(n)) \geq N$ for some $N \geq 1$. Then for all i, there is a $q \leq p$ such that $q \restriction K = p \restriction K$, $\forall n \geq K$: $\mathrm{norm}_n(q(n)) \geq N - 1$, and every $h \in \mathcal{S}ub_K(q)$ is either i-deciding or i-bad (in q).*

Proof. This is proved exactly as Lemma 5.8. The only difference is that we iterate over $\mathcal{S}ub_K(p)$ instead of $\mathcal{S}eq_K(p)$. Note that for each p and each n, if $p(n) = (c, k)$ then there are at most 2^{X_n} possibilities for values of c and at most $\mathrm{prenorm}_n(X_n)$ possibilities for values of k (here we have assumed, without loss of generality, that the function $\mathrm{prenorm}_n$ is monotone). Therefore, for each p and each K, there are at most $\prod_{i<K} \mathrm{prenorm}_i(X_i) \cdot 2^{X_i}$ members of $\mathcal{S}ub_K(p)$. The definition of ϵ_n compensates for this precisely. □

Now we construct a sequence $p_0 \geq p_1 \geq \dots$ of conditions and a sequence $K_0 < K_1 < \dots$ of integers by the following induction. Let $p_{-1} := p$. For each i, if p_{i-1} has been defined, pick K_i such that $\forall n \geq K_i : \mathrm{norm}_n(p_i(n)) \geq i + 2$. Apply Lemma 5.10 with p_{i-1}, K_i and i-decision/badness, and let p_i be the new condition. It is clear that in this way we get a fusion sequence whose limit $q \leq p$ has the following property: $\forall i \; \forall h \in \mathcal{S}ub_{K_i}(q) : h$ is i-deciding or i-bad. Also note that $\forall n \geq K_0 : \mathrm{norm}_n(q(n)) \geq 1$.

Claim. *For each i, there is a condition $q_i \leq q$ such that $\forall n \geq K_0 : \mathrm{norm}_n(q_i(n)) \geq \mathrm{norm}_n(q(n)) - 1$ and q_i decides $\dot{x}(i)$.*

Proof. Recall that q essentially decides $\dot{x}(i)$, so let m be such that $\forall \sigma \in \mathcal{S}eq_m(q) : q \uparrow \sigma$ decides $\dot{x}(i)$. Label each such σ "positive" or "negative" depending on whether $q \uparrow \sigma \Vdash \dot{x}(i) = 1$ or $q \uparrow \sigma \Vdash \dot{x}(i) = 0$. Apply Lemma 5.9 on the condition q and the interval $[K_0, m)$ to form a new condition q_i' such that $\forall n \in [K_0, m) : \mathrm{norm}_n(q_i'(n)) \geq \mathrm{norm}_n(q(n)) - 1$ and for all $\sigma \in \mathcal{S}eq_m(q_i')$, whether σ is positive or negative depends only on $\sigma \restriction K_0$ (if $m \leq K_0$, skip this step). Now shrink q_i' further down to q_i on the digits $n < K_0$, by whatever means necessary, to make sure that $q_i \Vdash \dot{x}(i) = 0$ or $q_i \Vdash \dot{x}(i) = 1$. □(claim.)

Each forcing condition $p \in \mathbb{P}$ can be viewed as an element in the compact topological space $\mathcal{X} := \prod_n (\mathscr{P}(X_n) \times \mathrm{prenorm}_n(X_n))$. In such a space every infinite sequence has an infinite convergent subsequence, in particular this applies to the sequence $\langle q_i \mid i \in \omega \rangle$. Let $a \subseteq \omega$ be an infinite set such that $\langle q_i \mid i \in a \rangle$ converges to some $r \in \mathcal{X}$. Since for all $n \geq K_0$, $\mathrm{norm}_n(q_i(n))$ is bounded from below by $\mathrm{norm}_n(q(n)) - 1$, the same is true for $r(n)$ which shows that r is a valid $\mathbb{P}$-condition.

But now we see that r decides infinitely many values of $\dot{x}$: for any given i, pick $j \in a$ with $j > i$ so that $q_j \restriction K_i = r \restriction K_i$. Let $h := r \restriction K_i$. Since $q_j \leq q$, $q_j \restriction K_i = h$, and q_j decides $\dot{x}(j)$, h certainly cannot be i-bad in q. So then it must be i-deciding in q, i.e., $h^\frown(q \restriction [K_i, \infty))$ must decide $\dot{x}(k)$ for some $k > i$. But then $r \leq h^\frown(q \restriction [K_i, \infty))$ must do so, too. □

6. Open questions.

We have not been able to understand the nature of the arrows $(c), (c'), (d)$ and (d') in the diagram from section 4. Recall that for the regularity properties of being Ramsey, Miller- and Laver-measurable, the $\mathbf{\Delta}^1_2$-statement is equivalent to the $\mathbf{\Sigma}^1_2$-statement. However, this is not the case for Lebesgue measure, the Baire property and, quite surprisingly, the doughnut property (see [3]). We currently have no intuition as to what the situation is in the case of the polarized partition properties.

Concerning eventually different reals, we believe that the arrows (d) and (d') are irreversible, i.e., that $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ is stronger than the existence of eventually different reals. Indeed, we conjecture the following:

6.1 Conjecture. *In the random model, i.e., the ω_1-iteration of random forcing with finite support starting from L, $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ fails.*

An alternative way to go about this problem would be by searching for a forcing notion which adds eventually different reals but not I_{E_0}-quasigenerics (and the latter is preserved in ω_1-iterations). Random forcing is not one of them, but perhaps a more sophisticated partial order can be found to do the job.

Of course, an even more basic question is whether a characterization theorem for polarized partitions can be found, stating that $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ or $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m})$ is equivalent to a transcendence statement over L. The abstract methods of [11] and [13] do not seem to apply because the fat creature forcing does not fall into the right category. Moreover, by a recent result of Marcin Sabok [16], such methods do not even apply to Mathias forcing (because the associated ideal is not $\mathbf{\Sigma}^1_2$), and our situation is clearly more complicated. However, it is still possible that a characterization result can be achieved by alternative methods.

Finally, we would like to mention that, as an aside, our result answers a question posed in [3, Question 6], namely whether the existence of I_{E_0}-quasigenerics implies $\mathbf{\Delta}^1_2$(doughnut). By Theorem 5.3 and Theorem 3.3, it does not.

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