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Differential Topology

Problem Set 3

1. Suppose Σ_1 and Σ_2 are two closed connected oriented surfaces of positive genus g_1 and g_2, respectively. Let $i_1 : S^1 \to \Sigma_1$ and $i_2 : S^1 \to \Sigma_2$ be embeddings of circles. Note that the normal bundles of the images $K_j \subseteq \Sigma_j$ of these embeddings are necessarily trivial (why?).

a) Given the orientations on the K_j induced by the embeddings, there is a unique way to obtain a new closed *oriented* surface by gluing Σ_1 and Σ_2 along the K_j, i.e. cutting out neighborhoods of the K_j and gluing their complements along the resulting boundaries in a way that respects orientations. Prove that the diffeomorphism type of the resulting surface does depend on the chosen embeddings i_j.

b) What properties of the embeddings i_j are relevant in determining the diffeomorphism type of the glued surface? Can you list all the possible outcomes (up to diffeomorphism) of this process?
Try to come up with a clear conjecture and some supporting evidence, even if you cannot give a complete proof.

2. Let p and q be two relatively prime integers with $p \geq 2$, and consider the diffeomorphism of $S^3 \subseteq \mathbb{C}^2$ given as

$$\sigma_{p,q} : S^3 \to S^3$$
$$\sigma_{p,q}(z,w) := (e^{\frac{2\pi \mathbf{i}}{p}} z, e^{\frac{2\pi \mathbf{i} q}{p}} w).$$

a) Prove that $\sigma_{p,q}$ generates an action of $\mathbb{Z}_p$ on S^3 without fixpoints. The quotient space is called *the lens space of type* (p,q) and denoted by $L(p,q)$.

b) Prove that $L(p,q)$ is a closed 3-manifold which inherits a natural orientation from S^3. Note that for $q \equiv q' \mod p$ the spaces $L(p,q)$ and $L(p,q')$ are naturally diffeomorphic.

c) Prove that there is an orientation reversing diffeomorphism between $L(p,q)$ and $L(p,-q)$.

d) Prove that $L(p,q)$ can be obtained from gluing two solid tori $S^1 \times D^2$ along a diffeomorphism of their boundary tori. Can you describe that diffeomorphism in terms of p and q?

e) Can you describe a Morse function on $L(p,q)$ with one critical point for each index $k \in \{0,1,2,3\}$?

3. The goal of this exercise is to show that a map $f : S^n \to S^n$ with $\deg(f) = 0$ is homotopic to a constant map. Observe that the result is trivial for $n = 0$.

a) Prove the result for $n = 1$.

b) Prove that if the result holds for some dimension $n \in \mathbb{N}$, then it also holds for dimension $n+1$, for example by proceeding in the following steps:

Bitte wenden!

(i) Suppose $0 \in \mathbb{R}^n$ is a regular value of a smooth map $g : U \to \mathbb{R}^n$ from an open subset $U \subseteq \mathbb{R}^n$, and $g(x) = 0$. Prove that if we choose $r > 0$ such that $\overline{B(x,r)} \subseteq U$ and $g|_{\overline{B(x,r)}}$ is a diffeomorpism onto a closed neighborhood of 0, the degree of the map $h : \partial\overline{B(x,r)} \to S^n$ given by $h(y) = \frac{g(y)}{\|g(y)\|}$ (computed with respect to the induced boundary orientations) agrees with the local sign $\epsilon_x \in \{\pm 1\}$ of x.

(ii) Given a smooth map $f : S^{n+1} \to S^{n+1}$ of degree 0, choose a pair $p \neq q$ of regular values and a closed ball $B \subseteq S^{n+1}$ with $f^{-1}(p) \subseteq B$ and $B \cap f^{-1}(q) = \varnothing$. Postcomposing with stereographic projection from q and a translation, we can view $f|_B$ as a map $\tilde{f} : B \to \mathbb{R}^{n+1}$ sending $f^{-1}(p)$ to 0. Prove that if $\deg(f) = 0$, then the map $g : \partial B \to S^n$ given by $g(y) = \frac{\tilde{f}(y)}{\|\tilde{f}(y)\|}$ also has degree zero (this is a special case of Problem **4. a)**).

(iii) Now use the induction hypothesis and the fact that any two maps $B \to \mathbb{R}^{n+1}$ sending ∂B to S^n are homotopic relative to the boundary to prove that f can be modified by a homotopy on B to avoid $p \in S^{n+1}$. Why does this complete the induction step?

4. Let W be a compact oriented manifold of dimension $n+1$ with boundary $M = \partial W$.

a) Suppose N is a closed connected oriented n-dimensional manifold. Prove that if a smooth map $f : M \to N$ extends to a map $F : W \to N$, then $\deg(f) = 0$.

b) Find an example where the converse fails, i.e. a situation where a map $f : M \to N$ has degree 0, but does not extend to W?

c) Use the result of Problem **3.** to prove the following partial converse to part **a)**:
If a map $f : M \to S^n$ has degree 0, then it admits an extension $F : M \to S^n$.
Hint: (Obviously?) the crucial step is to reduce the general case to the case $M = S^n$.

d) Conclude Hopf's theorem: Two maps from a closed connected oriented n-manifold M to S^n are homotopic if and only if they have the same degree.

5. The goal of this exercise is to prove that for a closed manifold Q, the sum of the indices of a vector field $V : Q \to TQ$ with isolated zeros is independent of the vector field V.

a) Prove the result under the additional assumption that Q is oriented.

b) Prove that for every manifold Q there is an oriented manifold $\widehat{Q}$ with a two-sheeted covering projection $p : \widehat{Q} \to Q$.

c) Conclude the independence of the index sum from the choice of V for nonorientable closed manifolds by comparing the index sum of V to the index sum of the unique lift $\widehat{V}$ to $\widehat{Q}$.

6. **a)** Give examples of vector fields defined on a neighborhood of $0 \in \mathbb{R}^2$ which have an isolated zero with index $k \neq \pm 1$. Can you achieve every $k \in \mathbb{Z}$?

b) Construct a vector field on S^2 with a single zero. Does your construction generalize to other dimensions?

7. Our definition of the Euler characteristic as the sum of indices of a vector field with isolated zeros can be extended to compact manifolds with boundary by restricting to vector fields which point outwards along the boundary. This exercise discusses some properties of this extended definition.

Siehe nächstes Blatt!

a) Suppose a closed manifold M can be written as $M = A \cup B$, where $A, B \subseteq M$ are compact subdomains of the same dimension with common boundary $\partial A = \partial B = A \cap B$. Prove that $\chi(M) = \chi(A) + \chi(B) - \chi(A \cap B)$.

b) Use this to give an inductive computation of $\chi(\mathbb{R}P^n)$.

c) Prove that if a closed manifold C is the boundary of some compact manifold, then $\chi(C)$ is even.

d) Prove that for even n the manifolds $\mathbb{R}P^n$ and $\mathbb{C}P^n$ cannot be boundaries of compact manifolds.

e) Find a compact 3-dimensional manifold whose boundary is a Klein bottle. This proves that nonorientability is not by itself an obstruction to being the boundary of a compact manifold.

8. Given a C^2-function $f : M \to \mathbb{R}$ and a critical point $p \in M$ (i.e. a point where $df_p = 0$), the Hessian form is defined as

$$\begin{aligned} \mathrm{Hess_p}\, f : T_pM \times T_pM &\to \mathbb{R} \\ (v, w) &\mapsto X(Y(f)) \end{aligned}$$

where X and Y are vector fields defined locally near p with $X(p) = v$ and $Y(p) = w$.

a) Prove that this form is indeed a well-defined and symmetric bilinear form.

b) Prove that a critical point $p \in M$ of f is nondegenerate if and only if the Hessian is non-degenerate, i.e. has all its eigenvalues (in any and hence every coordinate representation) non-zero.

c) Now let g be any Riemannian metric on M, and let X be the gradient of f. Prove that the index of a nondegenerate zero p of the vector field X and the index of p as a critical point of f are related by

$$\mathrm{ind}_p\, X = (-1)^{\mathrm{ind}_p\, f}.$$

d) Conclude that if f is a Morse function on a closed manifold M, then

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^{c_i(f)},$$

where $c_i(f)$ is the number of critical points of f of index i.

9. Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth closed submanifold. For every $v \in S^n \subseteq \mathbb{R}^n$, we define the smooth function

$$f_v : M \to \mathbb{R}, \quad f_v(x) = \langle v, x \rangle,$$

where $\langle\, .\, , .\, \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^{n+1}$. Prove that the set ov $v \in S^n$ such that f_v is a Morse function is open and dense.

10. Suppose that M is a smooth closed n-dimensional manifold which admits a Morse function with exactly two critical points. Prove that M is homeomorphic to S^n.
Remark: A famous theorem of Milnor says that such manifolds need not always be diffeomorphic *to S^n. His original construction gave examples of this phenomenon for $n = 7$.*

11. Consider real projective space $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$, where the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ acts by scaling. Write $[x_0 : \ldots : x_n]$ for the point in $\mathbb{R}P^n$ corresponding to $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. Now consider the function $f : \mathbb{R}P^n \to \mathbb{R}$ given by

$$f([x_0 : \ldots : x_n]) := \frac{1}{\|x\|^2} \sum_{k=0}^{n} k \cdot x_k^2.$$

a) Prove that f is a Morse function on $\mathbb{R}P^n$ and determine the critical points as well as their Morse indices.

b) Let $p : S^n \to \mathbb{R}P^n$, $p(x) = [x]$ be the standard double cover obtained from the embedding $S^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}$. Sketch the critical points and the qualitative behaviour of the gradient flow with respect to the round metric on S^n of the lift $\widetilde{f} = f \circ p : S^n \to \mathbb{R}$ of f for $n = 1$ and $n = 2$.

12. Let M be a connected compact manifold with boundary and let $f : M \to [a, b]$ be a Morse function with the following properties:

- a and b are regular values of f, and $\partial M = f^{-1}(a) \cup f^{-1}(b)$ (a priori one or both of $f^{-1}(a)$ and $f^{-1}(b)$ could be empty)
- f has exactly three critical points, two of index 0 and one of index 1.

a) Prove that in fact $f^{-1}(a)$ must be empty, and M must be diffeomorphic to a closed ball whose boundary is a regular level set. In particular, there is a Morse function $g : M \to [a, b]$ with a single local minimum such that $g \equiv f$ near ∂M.

b) Deduce that every closed connected manifold M admits a Morse function with a single local minimum and a single local maximum.

13. The goal of this exercise is to classify closed connected oriented surfaces. Recall that a 3-dimensional handle body with g handles is obtained by attaching $g \geq 0$ 1-handles to B^3. We call the boundary of a handle body with g handles an *oriented surface of genus g*. Our goal is to show that every closed connected oriented surface is diffeomorphic to a unique oriented surface of genus g for some $g \geq 0$.

a) Compute the Euler characteristic of an oriented surfaces of genus g and conclude that two such surfaces of different genus are not diffeomorphic.

b) Suppose that Σ is a connected oriented surface with boundary and $f : \Sigma \to \mathbb{R}$ is a Morse function with a single critical point of index 1 and such that $\partial\Sigma$ is a union of regular level sets. Prove that Σ is diffeomorphic to a pair of pants, i.e. a sphere with three open disks (with disjoint closures) removed.

c) Now let Σ be a closed, connected oriented surface and $f : \Sigma \to \mathbb{R}$ a Morse function. Assume that the critical values are distinct for distinct critical points, and let $p \in \Sigma$ be the index 1 critical point with the lowest value $f(p)$. Classify the topological types that the component $\Sigma_0 \subseteq \Sigma^{\leq f(p)+\varepsilon}$ containing p can have. *Hint: There are only two options.*

d) Now argue by induction on the number n of critical points of index 1 of a Morse function $f : \Sigma \to \mathbb{R}$ that every closed connected oriented surface is diffeomorphic to a surface of some genus g with $0 \leq 2g \leq n$. *Hint: Problem 12. might be useful here.*