Universiteit van Amsterdam
Institute for Logic, Language and Computation

Advanced Topics in Set Theory

2003/2004; 1st Semester
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Homework Set # 5. Deadline: November 18th, 2003

Exercise 18 (Perfect sets and perfect trees).
Let $A \subseteq \mathbb{N}^{\mathbb{N}}$. Show that the following are equivalent:

(1) A is perfect, and
(2) there is a perfect, pruned tree T such that $A = [T]$.

Exercise 19 (The cardinality of the set of perfect trees).
Let $x \in \mathbb{N}^{\mathbb{N}}$ be fixed and $S \subseteq \mathbb{N}$. Let

$$P_S := \{t\,;\, t{\restriction}\mathbb{N}\backslash S = x{\restriction}\mathbb{N}\backslash S\}.$$

Show:

(1) If S is infinite, then P_S is a perfect tree.
(2) If $S \neq S^*$, then $[P_S] \neq [P_{S^*}]$.

Exercise 20 ("All sets are Borel").
Assume that $\mathbb{N}^{\mathbb{N}}$ is the countable union of countable sets.
Show that every set of reals is $\mathbf{\Delta}^0_4$. Why is this not in conflict with the construction of a $\mathbf{\Pi}^0_4$-universal set?

Remark. Note that "For every countable family $\langle X_i\,;\, i \in \mathbb{N}\rangle$ of countable sets there is a family $\langle x_{ij}\,;\, i,j \in \mathbb{N}\rangle$ listing all elements of $\bigcup_{i\in I} X_i$" is a particular instance of the Axiom of Choice. If the reals are a countable union of countable sets, this statement must be false. (Thus, our assumption is a very blatant violation of the Axiom of Choice – be extremely careful to avoid any use of the Axiom of Choice in this exercise.)
It is a result of Feferman and Lévy that "$\mathbb{N}^{\mathbb{N}}$ is the countable union of countable sets" is consistent with ZF.