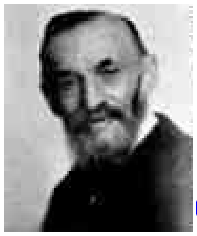
Mathematical Logic.

- From the XIXth century to the 1960s, logic was essentially mathematical.
- Development of first-order logic (1879-1928): Frege, Hilbert, Bernays, Ackermann.
- Development of the fundamental axiom systems for mathematics (1880s-1920s): Cantor, Peano, Zermelo, Fraenkel, Skolem, von Neumann.



Giuseppe Peano (1858-1932)

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- Traditional four areas of mathematical logic:
  - **Set Theory**.
  - **Recursion Theory**.
  - **Proof Theory**.
  - **Model Theory**.

Applications of Set Theory (1).

Measure Theory.

Émile Borel (1871-1956) | Henri Lebesgue (1875-1941) | Giuseppe Vitali (1875-1932)

- **Borel**: A measure for sets of real numbers assigning $b - a$ to $[a, b] := \{x\,;\, a \leq x \leq b\}$ and closed under countable disjoint unions (1894).
- **Lebesgue**: An integral based on Borel measure (1901-1902). *Question:* Is there an extension of Borel measure to all sets of real numbers? **The Measure Problem.**
- **Vitali**: No, if the Axiom of Choice is true (1905).

Applications of Set Theory (2).

Topology.

		
Felix Hausdorff	Nikolai Luzin	Pavel Alexandrov
(1868-1942)	(1883-1950)	(1896-1982)

- **Cantor** (1872): Notion of open and closed sets.
- The **Borel** sets: closure of the open sets under countable unions and complementation.
- **Hausdorff** introduces topological spaces.
- **Lebesgue**'s mistake: "The class of Borel sets is closed under continuous images."
- 1915. **Alexandrov** / **Hausdorff** prove the perfect set theorem for Borel sets.
- $\rightsquigarrow$ "Descriptive Set Theory".

The Axiom of Choice (1).

The Axiom of Choice (AC**).** For every function f defined on some set X with the property that $f(x) \neq \varnothing$ for all x, there is a choice function F defined on X, such that

$$\text{for all } x \in X, \text{ we have } F(x) \in f(x).$$

- Implicitly used in Cantor's work.
- Isolated by Peano (1890) in Peano's Theorem on the existence of solutions of ordinary differential equations.
- **1904.** Zermelo's wellordering theorem.

The Axiom of Choice (2).

A linear order $\langle X, \leq \rangle$ is called a well-order if there is no infinite strictly descending chain, *i.e.*, a sequence

$$x_0 > x_1 > x_2 > \ldots$$

Examples. Finite linear orders, $\langle \mathbb{N}, \leq \rangle$.

Nonexamples. $\langle \mathbb{Z}, \leq \rangle$, $\langle \mathbb{Q}, \leq \rangle$, $\langle \mathbb{R}, \leq \rangle$.

Important: If $\langle X, \leq \rangle$ is not a wellorder, that does not mean that the set X cannot be wellordered.

	-1	-2	-3	-4	-5	...
	0	1	2	3	4	...
$\rightsquigarrow$	0	-1	1	-2	2	...

The Axiom of Choice (3).

0 -1 1 -2 2 -3 3 -4 4 ...

$$z \sqsubseteq z^* :\leftrightarrow |z| \leq |z^*| \;\&\; z \leq z^*$$

There is an isomorphism between $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{Z}, \sqsubseteq \rangle$. The order $\langle \mathbb{Z}, \sqsubseteq \rangle$ is a wellorder, thus $\mathbb{Z}$ is wellorderable.

Question. Are all sets wellorderable?

Theorem (Zermelo's Wellordering Theorem). If AC holds, then all sets are wellorderable.

The Axiomatization of Set Theory (1).

-  **Zermelo** (1908).

  Zermelo Set Theory Z^-. Union Axiom, Pairing Axiom, *Aussonderungsaxiom* (Separation), Power Set Axiom, Axiom of Infinity.

  Zermelo Set Theory with Choice ZC^-. Axiom of Choice.

- **Hausdorff** (1908/1914). *Are there any regular limit cardinals?* "weakly inaccessible cardinals".

  > "The least among them has such an exorbitant magnitude that it will hardly be ever come into consideration for the usual purposes of set theory."

The Axiomatization of Set Theory (2).

- **1911-1913.** Paul Mahlo generalizes Hausdorff's questions in terms of fixed point phenomena ($\rightsquigarrow$ Mahlo cardinals).

-  

  Thoralf Skolem (1887-1963) Abraham Fraenkel (1891-1965)

  1922: *Ersetzungsaxiom* (Replacement) $\rightsquigarrow$ ZF^- and ZFC^-.

- **von Neumann** (1929): Axiom of Foundation $\rightsquigarrow$ Z, ZF and ZFC.

The Axiomatization of Set Theory (3).

- **Zermelo** (1930): ZFC doesn't solve Hausdorff's question (independently proved by Sierpiński and Tarski).
- **Question.** Does ZF prove AC? (Will be discussed later.)

Polish mathematics (1).

“There were no scientifi c problems common to all of [the Polish professors of mathematics in 1911]. ... I thought over this problem and came to the conclusion that this situation must not continue. ... We had Polish mathematicians known abroad from their work, but we had no Polish mathematics. My conclusion was that it would be much better if a greater number of Polish mathematicians worked in one area of research. (Sierpiński).”

Jan Łukasiewicz (1878-1956)

- 1917: Three-valued logic.
- 1919: Polish Minister of Education.
- Founder of the Warsaw School of Logic with Leśniewski.

Stanisław Leśniewski (1886-1939)

- Founder of the Warsaw School of Logic with Łukasiewicz.
- Joint work with Janiszewski and Mazurkiewicz to found the journal *Fundamenta Mathematicae*.

Polish Mathematics (2).

Zygmunt Janiszewski
(1888-1920)

Stefan Mazurkiewicz
(1888-1945)

- 1917: Janiszewski and Mazurkiewicz have a Topology seminar in Warsaw.
- 1918: “On the needs of mathematics in Poland”.

  Polish mathematicians can achieve an independent position for Polish mathematics by concentrating on narrow fi elds in which Polish mathematicians have already made internationally important contributions. These areas include set theory and the foundations of mathematics.

Polish Mathematics (3).

- **Fundamenta Mathematicae** (1920).

Janiszewski
Leśniewski
Mazurkiewicz
Sierpiński

Polish Mathematics (4).

Waclaw Sierpiński (1882-1969)

- Collaboration with Luzin (Descriptive Set Theory)
- Underground Warsaw University: “The proofs of these theorems will appear in *Fundamenta Mathematicae*.”

Kazimierz Kuratowski (1896-1980)

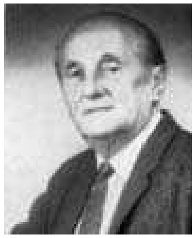
- Standard textbook “*Topologie*”.

The Scottish Café.

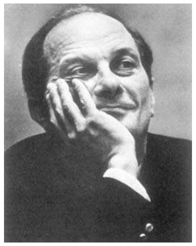
- Lvov, Ukraine.
- Banach, Steinhaus, Ulam.
- Dan **Mauldin**, The Scottish Book, Mathematics from the Scottish Café, 1981.

Polish Mathematics (5).

Stefan Banach
(1892-1945)



Hugo Steinhaus
(1887-1972)



Stanisław Ulam
1909-1984)

The Measure Problem.

- Hausdorff's Paradox: the Banach-Tarski paradox (1926).
- Banach's generalized measure problem (1930): existence of real-valued measurable cardinals.
- Banach connects the existence of real-valued measurable cardinals to Hausdorff's question about inaccessibles: if Banach's measure problem has a solution, then Hausdorff's answer is 'Yes'.
- Ulam's notion of a measurable cardinal in terms of ultrafi lters.

Early large cardinals.

- Weakly inaccessibles (Hausdorff, 1914).
- Inaccessibles (Zermelo, 1930).
- Real-valued measurables (Banach, 1930).
- Measurables (Ulam, 1930).

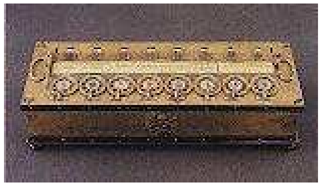
$$
\begin{tikzcd}
& \text{measurable} \arrow[dl] \arrow[dr] & \\
\text{real-valued meas.} \arrow[dr, "\mathrm{GCH}"] & & \text{inaccessible} \arrow[dl] \\
& \text{weakly inaccessible} &
\end{tikzcd}
$$

Question. Are these notions different? Can we prove that the least inaccessible is not the least measurable?

Early History of Computing.

1623. 

Wilhelm Schickard (1592-1635)

1642. 

Blaise Pascal (1623-1662)

1671. 

Gottfried Wilhelm von Leibniz (1646-1716)

Computation beyond numbers.

Charles Babbage (1791-1871)

- Difference Engine (1822)
- Analytical Engine

"Well, Babbage, what are you dreaming about?" — "I am thinking that all these [logarithmic] tables might be calculated by machinery." (c.1812)

Computation beyond numbers.

Charles Babbage (1791-1871)

- Difference Engine (1822)
- Analytical Engine

Ada King, Countess of Lovelace (1815-1852)

- Daughter of Lord Byron
- Collaborated with Babbage and de Morgan

“[The Analytical Engine] might act upon other things besides number, were objects found whose mutual fundamental relations could be expressed by those of the abstract science of operations, and which should be also susceptible of adaptations to the action of the operating notation and mechanism of the engine ... Supposing, for instance, that the fundamental relations of pitched sounds in the science of harmony and of musical composition were susceptible of such expression and adaptations, the engine might compose elaborate and scientifi c pieces of music of any degree of complexity or extent.”

Turing.

Alan Turing (1912-1954)

- **1936**. *On computable numbers.* The Turing Machine.
- **1938**. PhD in Princeton.
- **1939-1942**. Government Code and Cypher School at Bletchley Park.
- *Enigma*.
- **1946**. Automatic Computing Engine (ACE).
- **1948**. Reader in Manchester.
- **1950**. *Computing machinery and intelligence.* The Turing Test.
- **1952**. Arrested for violation of British homosexuality statutes.

Turing Machines (1).

Entscheidungsproblem.

Is there an algorithm that decides whether a given formula of predicate logic is a tautology or not?

Positive answer simple; negative answer hard. Define "algorithm".

Turing Machine. An idealized model of computation: an infinite tape, a finite alphabet Σ of symbols that can be on the tape, a read/write head, a finite set of actions A, a finite set S of states and a function ("programme") $F : \Sigma \times S \to A$. One of the states is designated the HALT state. Write $T := \langle \Sigma, S, A, F \rangle$. There are only countably many Turing machines.

Turing Machines (2).

Turing Machine. An idealized model of computation: an infinite tape, a finite alphabet Σ of symbols that can be on the tape, a read/write head, a finite set of actions A, a finite set S of states and a function ("programme") $F : \Sigma \times S \to A$. One of the states is designated the HALT state. Write $T := \langle \Sigma, S, A, F \rangle$. There are only countably many Turing machines.

- Given some finite string $s \in \Sigma^*$ as input, the machine starts its computation according to F.
- There is a unique defined sequence of states that the computation runs through.
- If one of them is HALT, we say that the machine halts and write $T(s) \downarrow$.
- Otherwise, we say that the machine loops (diverges) and write $T(s) \uparrow$.
- If $T(s) \downarrow$, then the machine outputs the content of the tape. We write $T(s)$ for the output.
- We say that T accepts s if $T(s) \downarrow$ and $T(s) = 1$.
- We say that T rejects s if $T(s) \downarrow$ and $T(s) = 0$.
- A set $X \subseteq \Sigma^*$ is decidable if there is a Turing machine T such that $s \in X$ if and only if T accepts s and $s \notin X$ if and only if T rejects s.

The Universal Turing Machine (1).

Fixing a finite alphabet $\Sigma := \{\sigma_0, ..., \sigma_s\}$ and a finite set of actions $A := \{\alpha_0, ..., \alpha_a\}$, we can list all Turing machines:

If $F : \Sigma \times S \to A$ is a Turing machine programme, we can view it as a partial function $\Phi_F : \{0, ..., s\} \times \{0, ..., n\} \to \{0, ..., a\}$ for some natural number n.

If now $\Phi : \{0, ..., s\} \times \{0, ..., n\} \to \{0, ..., a\}$ is a partial function, we assign a natural number (the "Gödel number of Φ"):

$$\mathrm{G}(\Phi) := \prod_{i \le s, j \le n} \mathrm{prime}_{ij}^{\Phi(i,j)+1}.$$

The Universal Turing Machine (2).

$$\mathrm{G}(\Phi) := \prod_{i \leq s, j \leq n} \mathrm{prime}_{ij}^{\Phi(i,j)+1}.$$

Let

$$T \subseteq \mathbb{N} := \{n\,;\ \exists F\,(\mathrm{G}(\Phi_F) = n\,)\,\}$$

be the set of numbers that are Gödel numbers of some Turing machine. Let t_n be the nth number in T and let T_n be the Turing machine such that $\mathrm{G}(\Phi_{T_n}) = t_n$.

> "It can be shown that a single special machine of that type can be made to do the work of all. It could in fact be made to work as a model of any other machine. The special machine may be called the universal machine. (Turing 1947)."

The Universal Turing Machine (3).

Let T be the set of numbers that are Gödel numbers of some Turing machine. Let t_n be the nth number in T and let T_n be the (a) Turing machine such that $\mathrm{G}(\Phi_{T_n}) = t_n$.

A universal Turing machine is a Turing machine U with alphabet $\{0, 1\}$ such that at input $\langle n, m \rangle$ such that $n \in T$ the following happens:

- If $T_n(m)\uparrow$, then $U(n, m)\uparrow$.
- If $T_n(m)\downarrow= k$, then $U(n, m)\downarrow= k$.

The Halting Problem K is the set

$$K := \{n\,;\, U(n, n)\downarrow\}.$$

The Halting Problem.

Theorem (Turing). The Halting Problem is not decidable.

Proof. Suppose it is decidable. Then there is a Turing machine T such that

$$\begin{array}{ccccc} T(n)\downarrow = 0 & \leftrightarrow & n \in K & \leftrightarrow & U(n,n)\downarrow \\ T(n)\downarrow = 1 & \leftrightarrow & n \notin K & \leftrightarrow & U(n,n)\uparrow \end{array}$$

By universality, there is some $e \in T$ such that $T = T_e$, *i.e.*,

$$\begin{array}{ccccc} T(n)\downarrow = 0 & \leftrightarrow & T_e(n)\downarrow = 0 & \leftrightarrow & U(e,n)\downarrow = 0 \\ T(n)\downarrow = 1 & \leftrightarrow & T_e(n)\downarrow = 1 & \leftrightarrow & U(e,n)\downarrow = 1 \end{array}$$

Substitute $n = e$ in the above equivalences and get:

$$U(e,e)\downarrow = 1 \;\leftrightarrow\; U(e,e)\uparrow .$$

Contradiction! q.e.d.

The *Entscheidungsproblem.*

Theorem (Church). The set of all (codes for) tautologies in predicate logic is undecidable, *i.e.*, there is no Turing machine T such that

$$T(n)\downarrow = 0 \quad \leftrightarrow \quad \varphi_n \text{ is a tautology}$$
$$T(n)\downarrow = 1 \quad \leftrightarrow \quad \varphi_n \text{ is not a tautology.}$$

Alonzo Church (1903-1995)

Alonzo **Church**, An Unsolvable Problem of Elementary Number Theory, **American Journal of Mathematics** 58 (1936), p. 345-363

Church-Turing Thesis. Every algorithm is represented by a Turing machine.