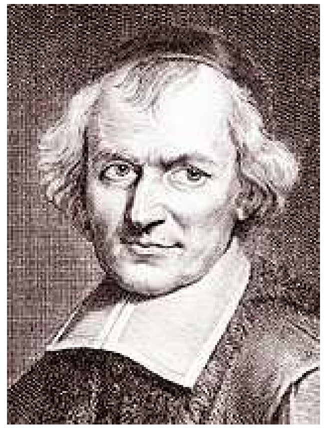
Port Royal.



1662.

“We do need another type of logic book for those whose culture is of a literary-historical type, who need logic as much as scientists and mathematicians do but not entirely the same bits of it and not in entirely the same way. There ought to be some modern counterpart of the Port-Royal Logic.” (Arthur Prior 1958)

Comprehension vs *Extension*

Comprehension vs Extension.

- *Comprehension of X.* The set of properties that x has to have in order to be an X.
- *Extension of X.* The set of all X.
- *An example.*
  - Universe of Discourse: $U = \{a, A, A, B, b\}$
  - Properties: $\mathbf{Consonant}$, $\mathbf{Capital}$, $\mathbf{Blue}$.
  - Extensions:
    - $\mathbf{Consonant} \rightsquigarrow \{B, b\}$
    - $\mathbf{Capital} \rightsquigarrow \{A, A, B\}$
    - $\mathbf{Blue} \rightsquigarrow \{a, B, b\}$
  - The Comprehension of $\mathbf{Consonant}$ in this universe of discourse includes the property $\mathbf{blue}$.

Leibniz (1).

Gottfried Wilhelm von Leibniz (1646-1716)

- Work on philosophy, mathematics, law, alchemy, theology, physics, engineering, geology, history.
- Diplomatic tasks (1672).
- Attempts to build a calculating machine (1672).

Leibniz (2).

- **1673-1677**: Invented calculus independently of Sir Isaac Newton (1643-1727).
- **1679**: Binary numbers.
- **1684**: Determinant theory.
- Research politics; foundation of Academies: Brandenburg, Dresden, Vienna, and St Petersburg.
- **1710**: *Théodicée*. “The best of all possible worlds”.

Leibniz (3).

Properties.

- *Identity of Indiscernibles:* If $\{\Phi\,;\,\Phi(x)\} = \{\Phi\,;\,\Phi(y)\}$, then $x = y$.
- Primary substances (“Plato”, “Socrates”) can be expressed in terms of properties: a uniform language of predication.
- Connected to Leibniz’ monadology (1714).

Relations.

- Call for an analysis of relations.
- Attempt to reduce relations to unary predicates:

“Plato is taller than Socrates”	$\mathbf{Taller}(\mathrm{Pla}, \mathrm{Soc})$
“Plato is tall *in as much as* Socrates is short”	$\mathbf{Tall}(\mathrm{Pla}) \oplus \mathbf{Short}(\mathrm{Soc})$

Calculemus!

“quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: ***calculemus****.”*

⇝ Arithmetization of Language and Automatization of Reasoning

Arithmetization of Language (1).

- *characteristica universalis*: general notation system for everything, based on the unanalyzable basics.
- *calculus ratiocinator*: formal system with a mechanizable deduction system.
- "*calculus de continentibus et contentis est species quaedam calculi de combinationibus*"
- The properties correspond to the natural numbers $n > 1$. The unanalyzable properties correspond to the prime numbers.
- **Example.** If *animal* corresponds to 2, and *rationalis* corresponds to 3, then *homo* would correspond to 6. If *philosophicus* corresponds to 5, then *philosophus* = *homo philosophicus* would be 30.

Arithmetization of Language (2).

animal $\rightsquigarrow 2$, *rationalis* $\rightsquigarrow 3$, *homo* $\rightsquigarrow 6$, *philosophicus* $\rightsquigarrow 5$, *philosophus* $\rightsquigarrow 30$.

- All individuals are determined by their properties, so *Socrates* is represented by a number n. Since Socrates is a philosopher, $30|n$.
- In general, "the individual represented by n has the property represented by m" is rendered as $m|n$.
- Now we can formalize $A\mathrm{a}B$ and $A\mathrm{i}B$. Let n_A and n_B be the numbers representing A and B, respectively.
  - $A\mathrm{a}B$: $n_A|n_B$.

    "Every human is an animal": $2|6$.
  - $A\mathrm{i}B$: $\exists k(n_A|k \cdot n_B)$.

    "Some human is a philosopher": $30|5 \cdot 6$.

Arithmetization of Language (3).

$A\mathrm{a}B$: $n_A | n_B$; $A\mathrm{i}B$: $\exists k(n_A | k \cdot n_B)$.

- **Barbara** becomes: “If $n|m$ and $m|k$, then $n|k$.” So, the laws of arithmetic prove **Barbara**.
- **Darii** becomes: “If $n|m$ and there is some w such that $m|w \cdot k$, then there is some w^* such that $n|w^* \cdot k$.” (Let $w^* := w$.)
- **But:** $A\mathrm{i}B$ is always true, as $n|n \cdot m$ for all n and m.
- If n represents *homo* and m represents *asinus*, then $n \cdot m$ would be a “man with the added property of being a donkey”.
- This simple calculus is not able to deal with negative propositions.

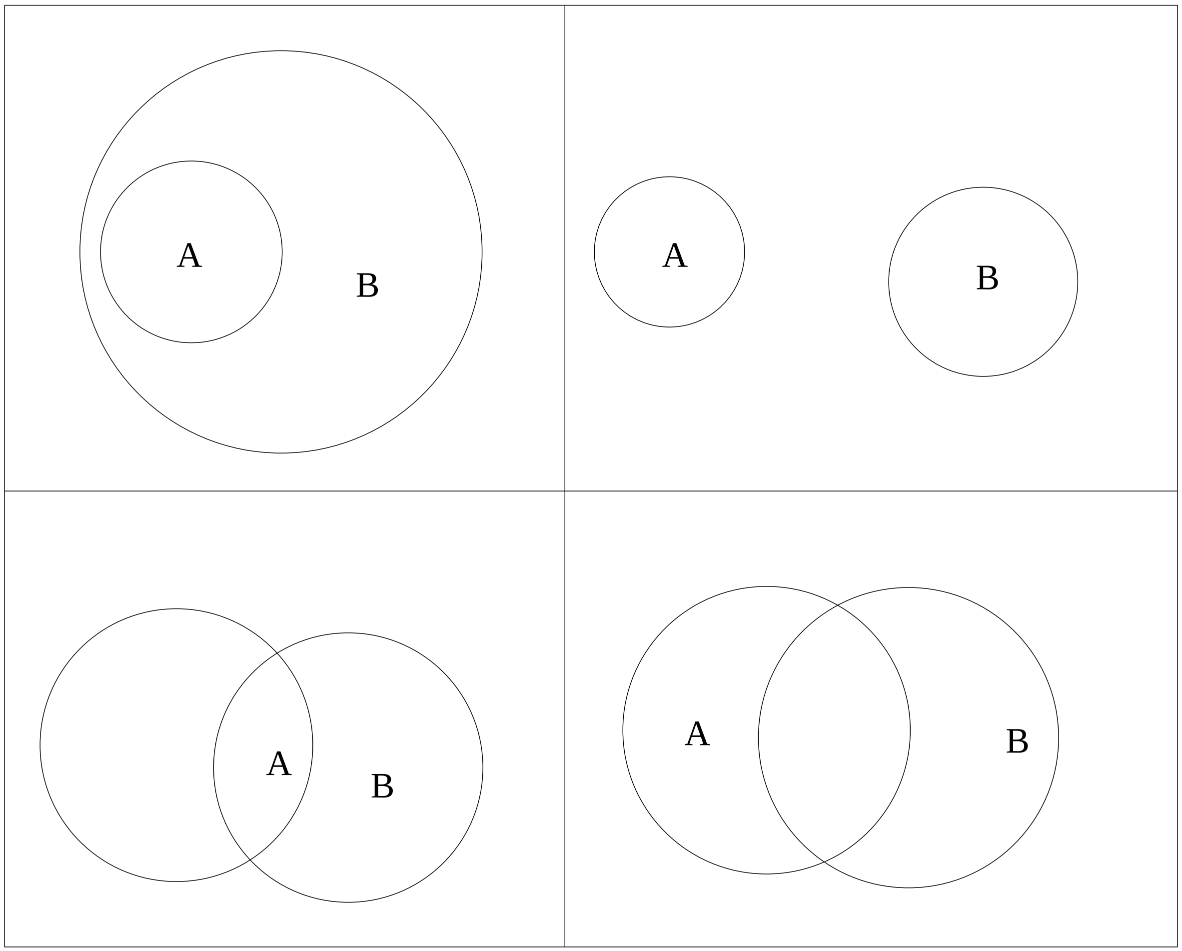
Euler.

Leonhard Euler (1707-1783)

- Member of the newly founded St. Petersburg Academy of Sciences (1727).
- 1741-1766: Director of Mathematics, later inofficial head of the Berlin Academy.

Euler diagrams.

Lettres à une Princesse d'Allemagne (1768-72).



"Every A is B."

"No A is B."

"Some (but only some) A is B."

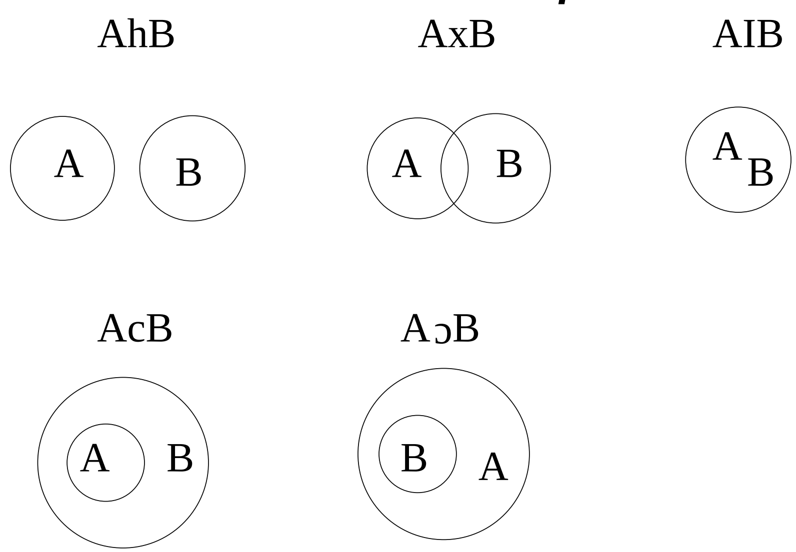
"Some (but only some) A is not B."

(Diagrams with Existential import!)

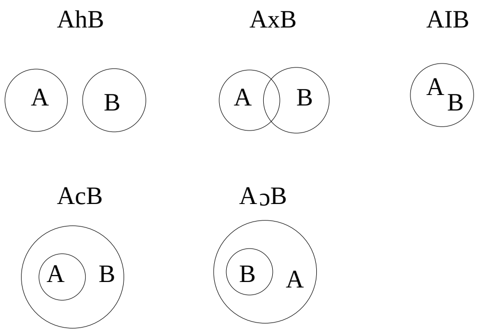
Gergonne (1).

Joseph Diaz Gergonne (1771-1859).

- Very active in the wars after the French revolution.
- Discoverer of the duality principle in geometry.
- *Essais de dialectique rationnelle* (1816-1817):



Gergonne (2).



Any two non-empty extensions ("sets") A and B are in exactly one of Gergonne's five relations:

h	*est hors de*
x	*s'entre-croise avec*
I	*est identique à*
c	*est contenue dans*
ɔ	*contiens*

Gergonne (3).

Syllogisms of the first figure: $A \bullet_0 B, B \bullet_1 C : A \bullet_2 C$.

	h	x	I	c	ɔ
h		¬I,¬ɔ	h	¬I,¬ɔ	h
x	¬I,¬c		x	¬h,¬I,¬ɔ	¬I,¬c
I	h	x	I	c	ɔ
c	h	¬I,¬ɔ	c	c	
ɔ	¬I,¬c	¬h,¬I,¬c	ɔ	¬h	ɔ

If $A\mathrm{x}B$ and $B\mathrm{c}C$, then $\neg A\mathrm{I}C$ and $\neg A$ɔC.

De Morgan.

Augustus de Morgan (1806-1871).

- Professor of Mathematics at UCL (1828).
- Corresponded with Charles Babbage (1791-1871) and William Rowan Hamilton (1805-1865).
- **1866.** First president of the London Mathematical Society.
- $x = 43$, $x^2 = 1849$. $y = 45$, $y^2 = 2025$.

De Morgan rules.

$$\neg(\Phi \wedge \Psi) \equiv \neg\Phi \vee \neg\Psi$$
$$\neg(\Phi \vee \Psi) \equiv \neg\Phi \wedge \neg\Psi$$

Boole (1).

George Boole (1815-1864).

- School teacher in Doncaster, Liverpool, Waddington (1831-1849).
- Correspondence with de Morgan.
- Professor of Mathematics at Cork (1849).

- Developed an algebra of logic based on the idea of taking the extensions of predicates as objects of the algebra.
- 1 is the “universe of discourse”, 0 is the empty extension.

Boole (2).

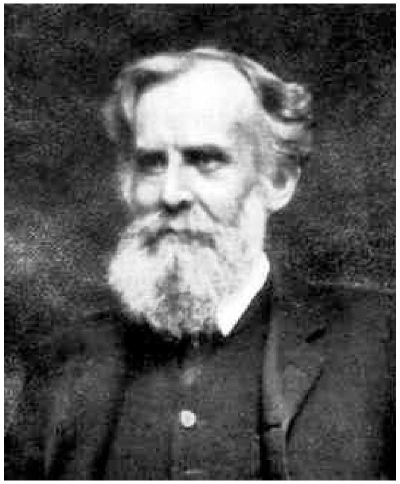
"No B is an A"	$ba = \mathbf{0}$.
"Some B is an A"	$ba \neq \mathbf{0}$.
"All B are A"	$b(\mathbf{1} - a) = \mathbf{0}$.
"Some B is not an A"	$b(\mathbf{1} - a) \neq \mathbf{0}$.

Celarent.

- *We assume:* $ba = \mathbf{0}$ and $c(\mathbf{1} - b) = \mathbf{0}$.
- *We have to show:* $ca = \mathbf{0}$.
- $ba = \mathbf{0}$ implies that $cba = c\mathbf{0} = \mathbf{0}$.
- $ca = ca - \mathbf{0} = ca - cba = a(c - bc) = a(c(\mathbf{1} - b)) = ac\mathbf{0} = \mathbf{0}$.

Venn.

John Venn (1834-1923).



- Lecturer in Moral Science at Cambridge (1862).
- Area of interest: logic and probability theory.
- *Symbolic Logic* (1881).
- *The Principles of Empirical Logic* (1889).
- *Alumni Cantabrigienses*.

Venn diagrams.

Boolean Algebras (1).

A structure $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ is a Boolean algebra if

- B is a set with $0, 1 \in B$.
- $+$ and $\cdot$ are binary operations on B satisfying the commutative and associative laws.
- $-$ is a unary operation on B.
- $+$ distributes over $\cdot$ and *vice versa*: $x + (y \cdot z) = (x + y) \cdot (x + z)$ and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
- $x \cdot x = x + x = x$ (idempotence), $- - x = x$.
- $-(x \cdot y) = (-x) + (-y)$, $-(x + y) = (-x) \cdot (-y)$ (de Morgan's laws).
- $x \cdot (-x) = 0$, $x + (-x) = 1$, $x \cdot 1 = x$, $x + 0 = x$, $x \cdot 0 = 0$, $x + 1 = 1$.
- $-1 = 0$, $-0 = 1$.

Example. $B = \{0, 1\}$.

$\cdot$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	1

Boolean Algebras (2).

$X := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}, \text{Themistokles}\}$

$\mathbf{Phil} := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}\}$

$\mathbf{Rhet} := \{\text{Themistokles}\}$

$\mathbf{Acad} := \{\text{Platon}, \text{Speusippus}\}$

$\mathbf{Peri} := \{\text{Aristotle}\}$

For predicates $\mathbf{P}$, $\mathbf{Q}$, defi ne

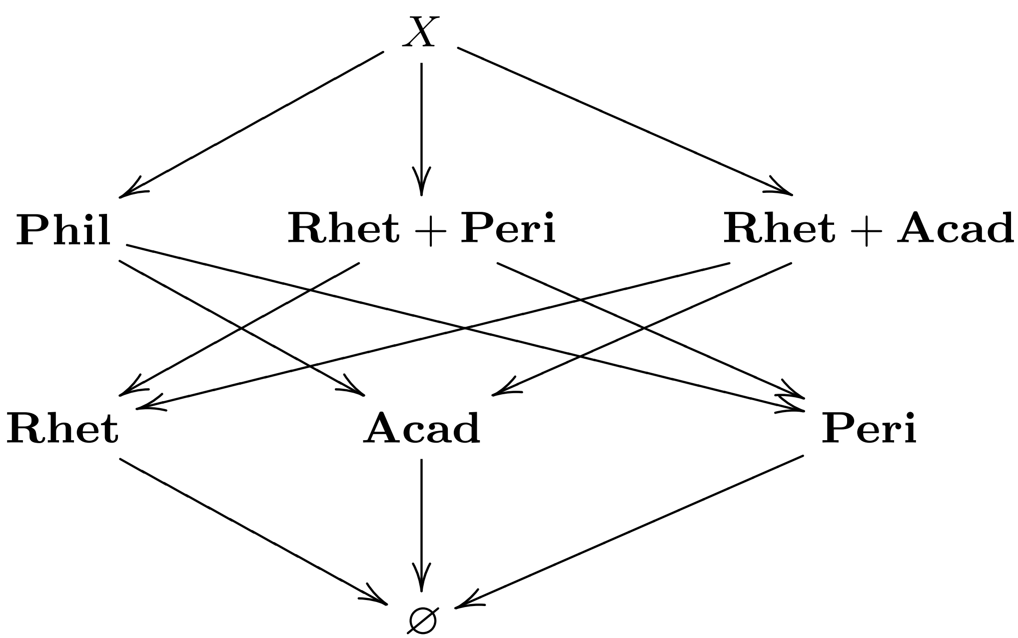
$\mathbf{P} \cdot \mathbf{Q} := \mathbf{P} \cap \mathbf{Q}$	$\mathbf{Peri} + \mathbf{Acad} = \mathbf{Phil}$
$\mathbf{P} + \mathbf{Q} := \mathbf{P} \cup \mathbf{Q}$	$\mathbf{Peri} \cdot \mathbf{Acad} = \mathbf{0}$
$\mathbf{0} := \varnothing$	$\mathbf{Phil} + \mathbf{Rhet} = \mathbf{1}$
$\mathbf{1} := X$	$-\mathbf{Phil} = \mathbf{Rhet}$
$-\mathbf{P} := X \backslash \mathbf{P}$	$-\mathbf{Peri} = ?$

$B := \{\varnothing, X, \mathbf{Phil}, \mathbf{Rhet}, \mathbf{Acad}, \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Acad}\}.$

Boolean Algebras (2).

$X := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}, \text{Themistokles}\}$
$\mathbf{Phil} := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}\}$
$\mathbf{Rhet} := \{\text{Themistokles}\}$
$\mathbf{Acad} := \{\text{Platon}, \text{Speusippus}\}$
$\mathbf{Peri} := \{\text{Aristotle}\}$

$B := \{\varnothing, X, \mathbf{Phil}, \mathbf{Rhet}, \mathbf{Acad}, \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Acad}\}$.



Boolean Algebras (3).

If X is a set, let $\wp(X)$ be the power set of X, *i.e.*, the set of all subsets of X.
For $A, B \in \wp(X)$, we can define

- $A \cdot B := A \cap B$,
- $A + B := A \cup B$,
- $0 := \varnothing$,
- $1 := X$,
- $-A := X \backslash A$.

Then $\langle \wp(X), 0, 1, +, \cdot, - \rangle$ is a Boolean algebra, denoted by $\mathbf{Pow}(X)$.

Boolean Algebras (4).

Define the notion of isomorphism of Boolean algebras: Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ and $\mathbf{C} = \langle C, \bot, \top, \oplus, \otimes, \ominus \rangle$ be Boolean algebras. A function $f : B \to C$ is a Boolean isomorphism if

- f is a bijection,
- for all $x, y \in B$, we have $f(x+y) = f(x) \oplus f(y)$, $f(x \cdot y) = f(x) \otimes f(y)$, $f(-x) = \ominus f(x)$, $f(0) = \bot$, $f(1) = \top$.

Stone Representation Theorem. If $\mathbf{B}$ is a Boolean algebra, then there is some set X such that $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{Pow}(X)$.

Circuits.

- $+$ corresponds to having two switches in parallel: if either (or both) of the switches are **ON**, then the current can flow.
- $\cdot$ corresponds to having two switches in series: if either (or both) of the switches are **OFF**, then the current is blocked.

Mathematics and real content.

Mathematics getting more abstract...

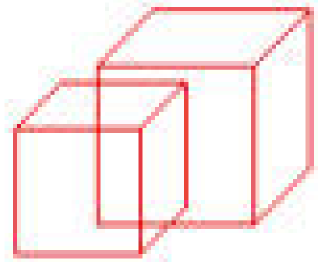
Imaginary numbers.

Nicolo Tartaglia	Girolamo Cardano
(1499-1557)	(1501-1576)

Carl Friedrich Gauss (1777-1855)

Ideal elements in number theory.
Richard Dedekind (1831-1916)

The Delic problem (1).



If a cube has height, width and depth 1, then its volume is $1 \times 1 \times 1 = 1^3 = 1$.

If a cube has height, width and depth 2, then its volume is $2 \times 2 \times 2 = 2^3 = 8$.

In order to have volume 2, the height, width and depth of the cube must be $\sqrt[3]{2}$:

$$\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = (\sqrt[3]{2})^3 = 2.$$

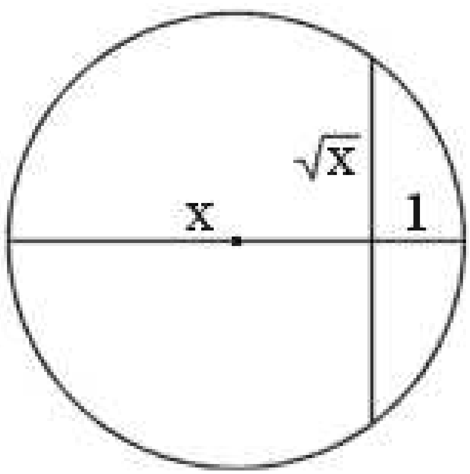
The Delic problem (2).

Question. Given a compass and a ruler that has only integer values on it, can you give a geometric construction of $\sqrt[3]{2}$?

Example. If x is a number that is constructible with ruler and compass, then $\sqrt{x}$ is constructible.

Proof.

If x is the sum of two squares (*i.e.*, $x = n^2 + m^2$), then this is easy by Pythagoras. In general:



The Delic problem (3).

It is easy to see what a positive solution to the Delic problem would be. But a negative solution would require reasoning about all possible geometric constructions.