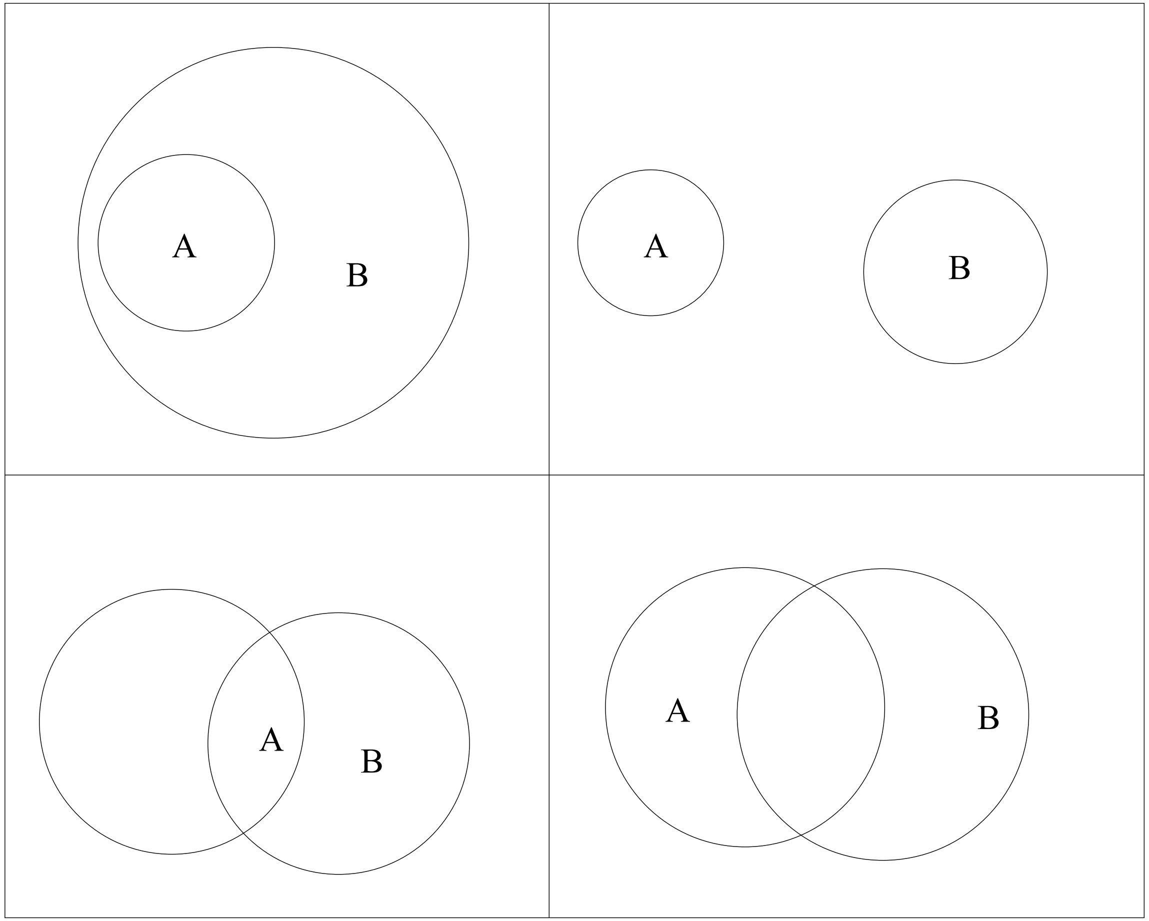
Euler.

Leonhard Euler (1707-1783)

- Member of the newly founded St. Petersburg Academy of Sciences (1727).
- 1741-1766: Director of Mathematics, later inofficial head of the Berlin Academy.

Euler diagrams.

Lettres à une Princesse d'Allemagne (1768-72).



"Every A is B."

"No A is B."

"Some (but only some) A is B."

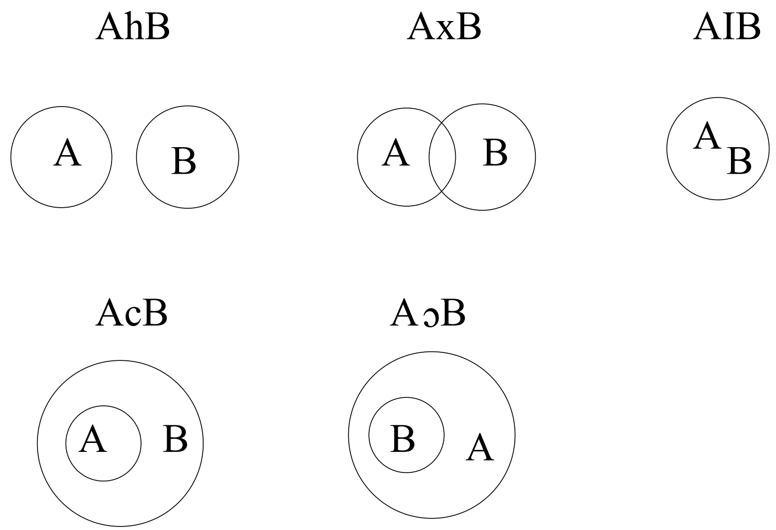
"Some (but only some) A is not B."

(Diagrams with Existential import!)

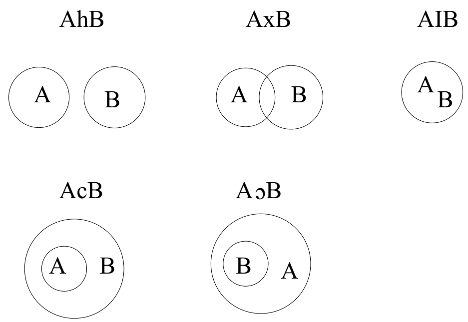
Gergonne (1).

Joseph Diaz Gergonne (1771-1859).

- Very active in the wars after the French revolution.
- Discoverer of the duality principle in geometry.
- *Essais de dialectique rationnelle* (1816-1817):



Gergonne (2).



Any two non-empty extensions ("sets") A and B are in exactly one of Gergonne's five relations:

h *est hors de*

x *s'entre-croise avec*

I *est identique à*

c *est contenue dans*

ɔ *contiens*

Gergonne (3).

Syllogisms of the first figure: $A \bullet_0 B, B \bullet_1 C : A \bullet_2 C$.

	h	x	I	c	ɔ
h		¬I,¬ɔ	h	¬I,¬ɔ	h
x	¬I,¬c		x	¬h,¬I,¬ɔ	¬I,¬c
I	h	x	I	c	ɔ
c	h	¬I,¬ɔ	c	c	
ɔ	¬I,¬c	¬h,¬I,¬c	ɔ	¬h	ɔ

If $A\mathrm{x}B$ and $B\mathrm{c}C$, then $\neg A\mathrm{I}C$ and $\neg A\text{ɔ}C$.

De Morgan.

Augustus de Morgan (1806-1871).

- Professor of Mathematics at UCL (1828).
- Corresponded with Charles Babbage (1791-1871) and William Rowan Hamilton (1805-1865).
- **1866.** First president of the London Mathematical Society.
- $x = 43$, $x^2 = 1849$. $y = 45$, $y^2 = 2025$.

De Morgan rules.

$$\neg(\Phi \wedge \Psi) \equiv \neg\Phi \vee \neg\Psi$$

$$\neg(\Phi \vee \Psi) \equiv \neg\Phi \wedge \neg\Psi$$

Boole (1).

George Boole (1815-1864).

- School teacher in Doncaster, Liverpool, Waddington (1831-1849).
- Correspondence with de Morgan.
- Professor of Mathematics at Cork (1849).

- Developed an algebra of logic based on the idea of taking the extensions of predicates as objects of the algebra.
- 1 is the “universe of discourse”, 0 is the empty extension.

Boole (2).

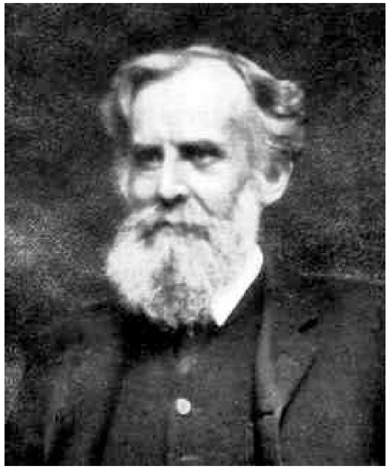
"No B is an A"	$ba = \mathbf{0}$.
"Some B is an A"	$ba \neq \mathbf{0}$.
"All B are A"	$b(\mathbf{1} - a) = \mathbf{0}$.
"Some B is not an A"	$b(\mathbf{1} - a) \neq \mathbf{0}$.

Celarent.

- *We assume:* $ba = \mathbf{0}$ and $c(\mathbf{1} - b) = \mathbf{0}$.
- *We have to show:* $ca = \mathbf{0}$.
- $ba = \mathbf{0}$ implies that $cba = c\mathbf{0} = \mathbf{0}$.
- $ca = ca - \mathbf{0} = ca - cba = a(c - bc) = a(c(\mathbf{1} - b)) = ac\mathbf{0} = \mathbf{0}$.

Venn.

John Venn (1834-1923).



- Lecturer in Moral Science at Cambridge (1862).
- Area of interest: logic and probability theory.
- *Symbolic Logic* (1881).
- *The Principles of Empirical Logic* (1889).
- *Alumni Cantabrigienses*.

Venn diagrams.

Boolean Algebras (1).

A structure $\mathrm{B} = \langle B, 0, 1, +, \cdot, - \rangle$ is a Boolean algebra if

- B is a set with $0, 1 \in B$.
- $+$ and $\cdot$ are binary operations on B satisfying the commutative and associative laws.
- $-$ is a unary operation on B.
- $+$ distributes over $\cdot$ and *vice versa*: $x + (y \cdot z) = (x + y) \cdot (x + z)$ and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
- $x \cdot x = x + x = x$ (idempotence), $- - x = x$.
- $-(x \cdot y) = (-x) + (-y)$, $-(x + y) = (-x) \cdot (-y)$ (de Morgan's laws).
- $x \cdot (-x) = 0$, $x + (-x) = 1$, $x \cdot 1 = x$, $x + 0 = x$, $x \cdot 0 = 0$, $x + 1 = 1$.
- $-1 = 0$, $-0 = 1$.

Example. $B = \{0, 1\}$.

$\cdot$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	1

Boolean Algebras (2).

$X := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}, \text{Themistokles}\}$

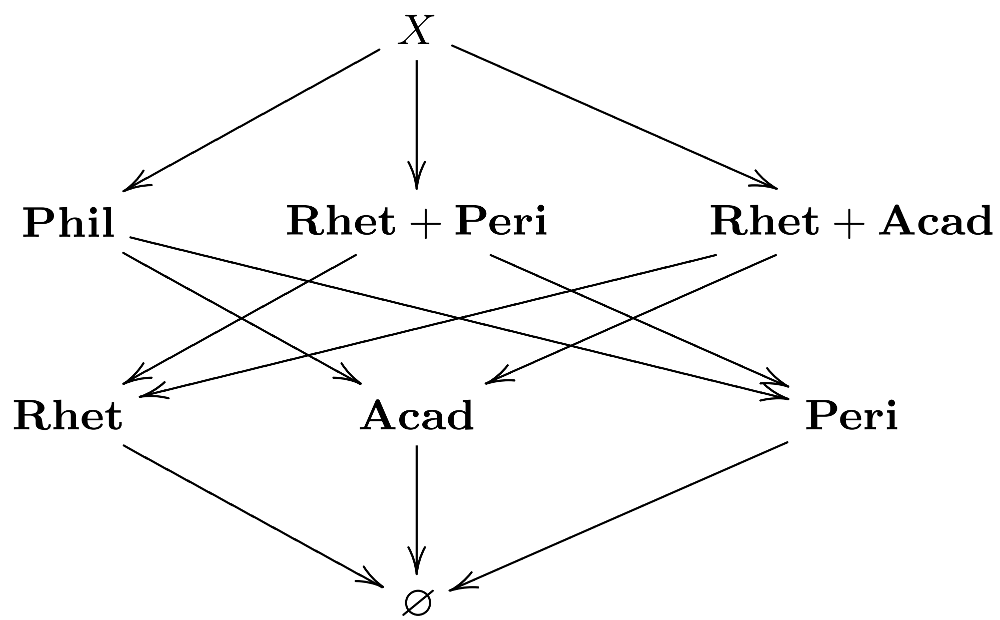
$\mathbf{Phil} := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}\}$

$\mathbf{Rhet} := \{\text{Themistokles}\}$

$\mathbf{Acad} := \{\text{Platon}, \text{Speusippus}\}$

$\mathbf{Peri} := \{\text{Aristotle}\}$

$B := \{\varnothing, X, \mathbf{Phil}, \mathbf{Rhet}, \mathbf{Acad}, \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Acad}\}.$



Boolean Algebras (3).

If X is a set, let $\wp(X)$ be the power set of X, *i.e.*, the set of all subsets of X.
For $A, B \in \wp(X)$, we can define

- $A \cdot B := A \cap B$,
- $A + B := A \cup B$,
- $0 := \varnothing$,
- $1 := X$,
- $-A := X \backslash A$.

Then $\langle \wp(X), 0, 1, +, \cdot, - \rangle$ is a Boolean algebra, denoted by $\mathbf{Pow}(X)$.

Boolean Algebras (4).

Define the notion of isomorphism of Boolean algebras: Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ and $\mathbf{C} = \langle C, \bot, \top, \oplus, \otimes, \ominus \rangle$ be Boolean algebras. A function $f : B \to C$ is a Boolean isomorphism if

- f is a bijection,
- for all $x, y \in B$, we have $f(x + y) = f(x) \oplus f(y)$, $f(x \cdot y) = f(x) \otimes f(y)$, $f(-x) = \ominus f(x)$, $f(0) = \bot$, $f(1) = \top$.

Stone Representation Theorem. If $\mathbf{B}$ is a Boolean algebra, then there is some set X such that $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{Pow}(X)$.

Circuits.

- $+$ corresponds to having two switches in parallel: if either (or both) of the switches are **ON**, then the current can flow.
- $\cdot$ corresponds to having two switches in series: if either (or both) of the switches are **OFF**, then the current is blocked.

Mathematics and real content.

Mathematics getting more abstract...

Imaginary numbers.

Nicolo Tartaglia (1499-1557)

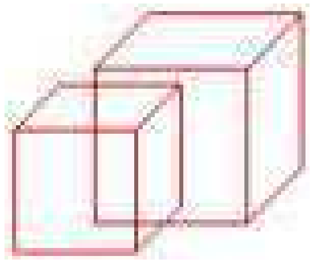
Girolamo Cardano (1501-1576)

Carl Friedrich Gauss (1777-1855)

Ideal elements in number theory.

Richard Dedekind (1831-1916)

The Delic problem (1).



If a cube has height, width and depth 1, then its volume is $1 \times 1 \times 1 = 1^3 = 1$.
If a cube has height, width and depth 2, then its volume is $2 \times 2 \times 2 = 2^3 = 8$.
In order to have volume 2, the height, width and depth of the cube must be $\sqrt[3]{2}$:

$$\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = (\sqrt[3]{2})^3 = 2.$$

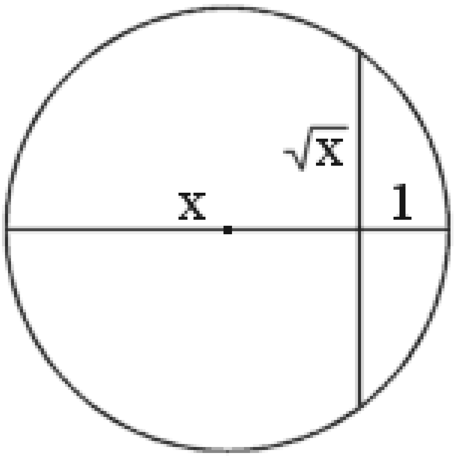
The Delic problem (2).

Question. Given a compass and a ruler that has only integer values on it, can you give a geometric construction of $\sqrt[3]{2}$?

Example. If x is a number that is constructible with ruler and compass, then $\sqrt{x}$ is constructible.

Proof.

If x is the sum of two squares (*i.e.*, $x = n^2 + m^2$), then this is easy by Pythagoras. In general:



The Delic problem (3).

It is easy to see what a positive solution to the Delic problem would be. But a negative solution would require reasoning about all possible geometric constructions.

Geometries (1).

- We call a structure $\langle P, L, I\rangle$ a plane geometry if $I \subseteq P \times L$ is a relation.
- We call the elements of P "points", the elements of L "lines" and we read $pI\ell$ as "p lies on ℓ".
- If ℓ and ℓ^* are lines, we say that ℓ and ℓ^* are parallel if there is no point p such that $pI\ell$ and $pI\ell^*$.
- **Example.** If $P = \mathbb{R}^2$, then we call $\ell \subseteq P$ a line if

$$\ell = \{\langle x, y\rangle \,;\, y = a \cdot x + b\}$$

  for some $a, b \in \mathbb{R}$. Let $\mathcal{L}$ be the set of lines. We write $pI\ell$ if $p \in \ell$. Then $\langle P, \mathcal{L}, I\rangle$ is a plane geometry.

Geometries (2).

- (A1) For every $p \neq q \in P$ there is exactly one $\ell \in L$ such that $pI\ell$ and $qI\ell$.
- (A2) For every $\ell \neq \ell^* \in L$, either ℓ and ℓ^* are parallel, or there is exactly one $p \in P$ such that $pI\ell$ and $pI\ell^*$.
- (N) For every $p \in P$ there is an $\ell \in L$ such that p doesn't lie on ℓ and for every $\ell \in L$ there is an $p \in P$ such that p doesn't lie on ℓ.
- (P2) For every $\ell \neq \ell^* \in L$, there is exactly one $p \in P$ such that $pI\ell$ and $pI\ell^*$.

A plane geometry that satisfies (A1), (A2) and (N) is called a plane. A plane geometry that satisfies (A1), (P2) and (N) is called a projective plane.

Geometries (3).

- (A1) For every $p \neq q \in P$ there is exactly one $\ell \in L$ such that $pI\ell$ and $qI\ell$.
- (A2) For every $\ell \neq \ell^* \in L$, either ℓ and ℓ^* are parallel, or there is exactly one $p \in P$ such that $pI\ell$ and $pI\ell^*$.
- (N) For every $p \in P$ there is an $\ell \in L$ such that p doesn't lie on ℓ and for every $\ell \in L$ there is an $p \in P$ such that p doesn't lie on ℓ.

Let $\mathbf{P} := \langle \mathbb{R}^2, \mathcal{L}, \in \rangle$. Then $\mathbf{P}$ is a plane.

- (WE) ("the weak Euclidean postulate") For every $\ell \in L$ and every $p \in P$ such that p doesn't lie on ℓ, there is an $\ell^* \in L$ such that $pI\ell^*$ and ℓ and ℓ^* are parallel.
- (SE) ("the strong Euclidean postulate") For every $\ell \in L$ and every $p \in P$ such that p doesn't lie on ℓ, there is exactly one $\ell^* \in L$ such that $pI\ell^*$ and ℓ and ℓ^* are parallel.

$\mathbf{P}$ is a strongly Euclidean plane.

Geometries (4).

Question. Do (A1), (A2), (N), and (WE) imply (SE)?

It is easy to see what a positive solution would be, but a negative solution would require reasoning over all possible proofs.

Semantic version of the question. Is every weakly Euclidean plane strongly Euclidean?

Syntactic versus semantic.

	Does Φ imply ψ?	**Does every Φ-structure satisfy ψ?**
Positive	Give a proof	Check all structures
	$\exists$	$\forall$
Negative	Check all proofs	Give a counterexample
	$\forall$	$\exists$

Euclid's Fifth Postulate (1).

- Ptolemy (c.85-c.165)
- Proclus (411-485)
- Omar Khayyam (1048-1131)
- Nasir ad-Din at-Tusi (1201-1274)
- Girard Desargues (1591-1661)
- Blaise Pascal (1623-1662)
- Gerolamo Saccheri (1667-1733): Hypothesis of the acute angle
- Heinrich Lambert (1728-1777)
- John Playfair (1748-1819)
- Adrien-Marie Legendre (1752-1833): (SE) is equivalent to "the sum of angles of a triangle is equal to $180°$".

Euclid's Fifth Postulate (2).

"the scandal of elementary geometry" (D'Alembert 1767)
"In the theory of parallels we are even now not further than Euclid. This is a shameful part of mathematics..." (Gauss 1817)

Johann Carl Friedrich Gauss

(1777-1855)

1817

Nikolai Ivanovich Lobachevsky

(1792-1856)

1829

János Bolyai

(1802-1860)

1823

A non-Euclidean geometry.

Take the usual geometry $\mathbf{P} = \langle \mathbb{R}^2, \mathcal{L}, \in \rangle$ on the Euclidean plane.

Consider $\mathbb{U} := \{x \in \mathbb{R}^2\,;\, \|x\| < 1\}$. We define the restriction of $\mathcal{L}$ to $\mathbb{U}$ by $\mathcal{L}^{\mathbb{U}} := \{\ell \cap \mathbb{U}\,;\, \ell \in \mathcal{L}\}$.

$\mathbf{U} := \langle \mathbb{U}, \mathcal{L}^{\mathbb{U}}, \in \rangle$.

Theorem. $\mathbf{U}$ is a weakly Euclidean plane which is not strongly Euclidean.

Cantor (1).

Georg Cantor
(1845-1918)
studied in Zürich, Berlin, Göttingen
Professor in Halle

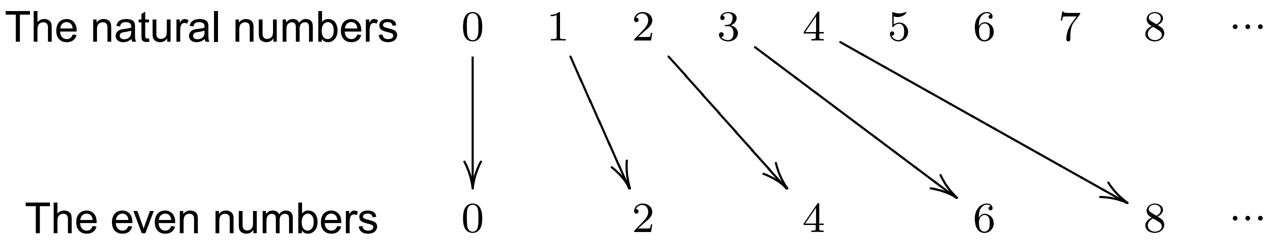
- Work in analysis leads to the notion of cardinality (1874): most real numbers are transcendental.
- Correspondence with Dedekind (1831-1916): bijection between the line and the plane.
- Perfect sets and iterations of operations lead to a notion of ordinal number (1880).

Cantor (2).

Georg Cantor (1845-1918)

- 1877. Leopold Kronecker (1823-1891) tried to prevent publication of Cantor's work.
- Cantor is supported by Dedekind and Felix Klein.
- 1884: Cantor suffers from a severe depression.
- 1888-1891: Cantor is the leading force in the foundation of the *Deutsche Mathematiker-Vereinigung*.
- Development of the foundations of set theory: 1895-1899.

Cardinality (1).



- There is a 1-1 correspondence (bijection) between $\mathbb{N}$ and the even numbers.
- There is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.
- There is a bijection between $\mathbb{Q}$ and $\mathbb{N}$.
- There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.
- There is no bijection between $\mathbb{R}$ and $\mathbb{N}$.

Cardinality (2).

Theorem (Cantor). There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.

Theorem (Cantor). There is a bijection between the real line and the real plane.

Proof. Let's just do it for the set of infinite 0-1 sequences and the set of pairs of infinite 0-1 sequences:
If x is an infinite 0-1 sequence, then let

$$x_0(n) := x(2n), \text{ and}$$

$$x_1(n) := x(2n+1).$$

Let $F(x) := \langle x_0, x_1 \rangle$. F is a bijection. q.e.d.

Cantor to Dedekind (1877): "*Ich sehe es, aber ich glaube es nicht!*"