Homework Set #5

Capita Selecta: Set Theory
2016/17: 1st Semester; block a
Universiteit van Amsterdam

Homework. Homework has to contain the name and student ID of the student. If your homework is handwritten, make sure that it is legible. You submit your solutions either by e-mail to `h(dot)nobrega(at)uva(dot)nl` or in person before the Tuesday lecture or by placing them in Hugo's mailbox at the ILLC at Science Park 107.

Deadline. The first homework is due on **Tuesday, 11 October 2016** before the lecture.

In all of the exercises, work in a sufficiently strong metatheory and assume that M is a countable transitive model of ZFC.

1. We say that G is $\mathbb{P}$*-antichain generic over* M if for every maximal $\mathbb{P}$-antichain $A \in M$, we have $A \cap G \neq \varnothing$. We call a set B a $\mathbb{P}$*-bar* if for every $p \in \mathbb{P}$ there is a $b \in B$ such that p and b are compatible. We say that G is $\mathbb{P}$*-bar generic over* M if for every $\mathbb{P}$-bar $B \in M$ we have that $B \cap G \neq \varnothing$.

   Let $\mathbb{P} \in M$, and G be a filter over $\mathbb{P}$. Show that the following are equivalent:

   (i) G is $\mathbb{P}$-generic over M,

   (ii) G is $\mathbb{P}$-antichain generic over M, and

   (iii) G is $\mathbb{P}$-bar generic over M.

2. Let $x, y \in M$. Let $\mathbb{P}$ be any partial order. Show:

   (a) If $x \neq y$, then $\mathbf{1} \Vdash^* \neg(\check{x} = \check{y})$.

   (b) If $x \in y$, then $\mathbf{1} \Vdash^* \check{x} \in \check{y}$.

   (c) If $x \notin y$, then $\mathbf{1} \Vdash^* \neg(\check{x} \in \check{y})$.

3. Assume that $\mathbb{P} \in M$. Fix some $p \in \mathbb{P}$ and assume that there is some $q \in \mathbb{P}$ such that $p \perp q$. Show that
$$\{\tau \in M^{\mathbb{P}}\,;\, p \Vdash^* \tau = \check{0}\}$$
   and
$$\{\tau \in M^{\mathbb{P}}\,;\, p \Vdash^* \tau = \check{1}\}$$
   are proper classes in M.

4. Let $\mathbb{P} \in M$ and G be $\mathbb{P}$-generic over M. Assume that $f \in M[G]$ is a function. Show that there is an $x \in M$ such that $\mathrm{ran}(f) \subseteq x$. (*Hint.* Consider the set of b such that it is forced that $\check{b}$ is in the range of the name of the function f.)

5. Suppose that $\mathbb{P} \in M$ such that for every $p \in \mathbb{P}$ there are $q_0, q_1 \leq p$ such that $q_0 \perp q_1$. Define by recursion a sequence $\langle M_i\,;\, i \in \omega\rangle$ of models as follows:

   Let $M_0 := M$. Suppose M_i is defined: find a filter G_i which is $\mathbb{P}$-generic over M_i and let $M_{i+1} := M_i[G_i]$.

   Define $N := \bigcup_{i\in\omega} M_i$ and show that $(N, \in)$ cannot be a model of the power set axiom.