Topics in Set Theory
Lent Term 2018
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #3

Example Classes.
#1. Thursday 1 February, 3–4pm, MR13.
#2. Thursday 15 February, 3–4pm, MR20.
#3. Thursday 1 March, 3–4pm, MR21.
#4. Wednesday 14 March, 3–4pm, MR21.
You hand in your work at the beginning of the Example Class.

(23) Assume that $\mathsf{ZFC} + \mathsf{IC}$ is consistent and show that the following theory is consistent: $\mathsf{ZFC}+$ "there are ordinals $\alpha < \beta < \aleph_1$ such that $\mathbf{L}_\alpha \models \mathsf{ZFC}$, $\mathbf{L}_\beta \models \mathsf{ZFC}$ and $\mathbf{L}_\beta \models$ 'α is countable' ".

(24) Let x be any transitive set and define by transfinite recursion:

$$\begin{aligned} \mathbf{L}_0(x) &:= x, \\ \mathbf{L}_{\alpha+1}(x) &:= \mathcal{D}(\mathbf{L}_\alpha(x)), \\ \mathbf{L}_\lambda(x) &:= \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha(x) \text{ (for } \lambda \text{ limit).} \end{aligned}$$

As usual, we define $\mathbf{L}(x) := \bigcup_{\alpha \in \mathrm{Ord}} \mathbf{L}_\alpha(x)$ and write $\mathbf{V}{=}\mathbf{L}(x)$ for the formula $\forall x \exists \alpha (x \in \mathbf{L}_\alpha(x))$ (note that this is a formula with the parameter x). Show that

(a) for each α, $\mathbf{L}_\alpha$ is transitive,

(b) if x is countable and $\alpha \geq \omega$, then $|\mathbf{L}_\alpha(x)| = |\alpha|$,

(25) Let x be any transitive set. Show that there is a formula Φ such that the following holds: if M is transitive, $x \in M$, and $(M, \in) \models \Phi$, then there is a limit ordinal λ such that $M = \mathbf{L}_\lambda(x)$.

(26) Suppose that $x \subseteq \mathbb{N}$ and prove the *Condensation Lemma for* $\mathbf{L}(x)$:

Suppose $\mathbf{V}{=}\mathbf{L}(x)$ and that $A \subseteq \mathbb{N}$. Then there is $\lambda < \omega_1$ such that $A \in \mathbf{L}_\lambda(x)$.

Conclude that $\mathbf{L}(x)$ satisfies CH (and also GCH). If $x \subseteq \kappa$ for some uncountable cardinal κ, what bound does the proof of the condensation lemma give and what can you say about the size of $2^{\aleph_0}$ in $\mathbf{L}(x)$?

(27) Assume that for all $x \subseteq \mathbb{N}$, we have that $\aleph_1^{\mathbf{L}(x)}$ is countable. Show that $\mathbf{L} \models$ "$\aleph_1^{\mathbf{V}}$ is inaccessible".

(28) Let $(\mathbb{P}, \leq)$ be a partial order and $p \in \mathbb{P}$. Show that

(a) if D is dense below p and $r \leq p$, then D is dense below r;

(b) if $\{r\,;\, D \text{ is dense below } r\}$ is dense below p, then D is dense below p.

(29) We say that G is $\mathbb{P}$-*antichain generic over* M if for every maximal $\mathbb{P}$-antichain $A \in M$, we have $A \cap G \neq \varnothing$. We call a set B a $\mathbb{P}$-*bar* if for every $p \in \mathbb{P}$ there is a $b \in B$ such that p and b are compatible. We say that G is $\mathbb{P}$-*bar generic over* M if for every $\mathbb{P}$-bar $B \in M$ we have that $B \cap G \neq \varnothing$.

Let $\mathbb{P} \in M$, and G be a filter over $\mathbb{P}$. Show that the following are equivalent:

(i) G is $\mathbb{P}$-generic over M,

(ii) G is $\mathbb{P}$-antichain generic over M, and

(iii) G is $\mathbb{P}$-bar generic over M.

(30) Let M be a transitive model of set theory, and $x, y \in M$. Define (in M) a partial order by $\mathbb{P}_{x,y} := \{p\,;\, p : \text{dom}(p) \to 2 \text{ and dom}(p) \text{ is a finite subset of } x \times y\}$ and $p \leq q$ if $p \supseteq q$. Show that if there is a $\mathbb{P}_{x,y}$-generic filter over M, then there is an injection from x to $\wp(y)$.

(31) Let μ be Lebesgue measure on the real line $\mathbb{R}$. Consider $\mathbb{P}_\varepsilon := \{G \subseteq \mathbb{R}\,;\, G \text{ is open and } \mu(G) < \varepsilon\}$. Suppose that $\{N_\alpha\,;\, \alpha < \kappa\}$ is any family of Lebesgue null sets. Find a family $\mathcal{D}$ of κ-many dense sets in $\mathbb{P}_\varepsilon$ such that the following holds:

> If G is $\mathbb{P}_\varepsilon$-generic for $\mathcal{D}$, then there is an open set G with $\mu(G) < \varepsilon$ such that $\bigcup_{\alpha<\kappa} N_\alpha \subseteq G$.

(32) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Suppose $\sigma, \tau \in M^{\mathbb{P}}$ and that D is a filter on $\mathbb{P}$. Show that $\text{val}(\sigma \cup \tau, D) = \text{val}(\sigma, D) \cup \text{val}(\tau, D)$.

(33) Let M be a transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order, and $x \in M$. Define $\text{can}(x) := \{(\text{can}(y), p)\,;\, y \in x \wedge p \in \mathbb{P}\}$. Show that if D is a filter, then $\text{val}(\text{can}(x), D) = x$.

(34) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Assume that $\tau \in M^{\mathbb{P}}$ and that D is a filter on $\mathbb{P}$. Define

$$\pi := \{(\varrho, p)\,;\, \exists(\sigma, q) \in \tau \exists r(\varrho, r) \in \sigma \wedge p \leq r \wedge p \leq q\}$$

and show that $\text{val}(\pi, D) = \bigcup \text{val}(\tau, D)$. Conclude that $M[D]$ satisfies the Union axiom.