Example Sheet #4

Course webpage: `https://www.math.uni-hamburg.de/home/loewe/Lent2019/TST_L19.html`

Example Classes.
#1. Monday 4 February, 3:30–5pm, MR5.
#2. Monday 18 February, 3:30–5pm, MR5.
#3. Monday 4 March, 3:30–5pm, MR5.
#4. Thursday 14 March, 3:30-5pm, MR20.
You hand in your work at the beginning of the Example Class.

In the following, we assume that M is a countable transitive model of ZFC.

(34) Assume that $\mathbb{P} \in M$ and that G is $\mathbb{P}$-generic over M. Show that $M[G] \models$ Replacement.

(35) Assume that $\mathbb{P} \in M$ and that G is $\mathbb{P}$-generic over M. Show that $M[G] \models$ AC.

(36) If $(\mathbb{P}, \leq_\mathbb{P}, \mathbf{1}_\mathbb{P})$ and $(\mathbb{Q}, \leq_\mathbb{Q}, \mathbf{1}_\mathbb{Q})$ are partial orders, then a function $i : \mathbb{P} \to \mathbb{Q}$ is called a *complete embedding* if

   (a) i is order preserving, i.e., if $p \leq_\mathbb{P} p'$, then $i(p) \leq_\mathbb{Q} i(p')$;
   (b) i preserves incompatibility in both directions, i.e., $p \perp_\mathbb{P} p'$ if and only if $i(p) \perp_\mathbb{Q} i(p')$; and
   (c) for all $q \in \mathbb{Q}$ there is a $p \in \mathbb{P}$ such that for all $p' \leq_\mathbb{P}$, we have that $i(p')$ and q are compatible in $\mathbb{Q}$.

   Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding with $i, \mathbb{P}, \mathbb{Q} \in M$ and let H be $\mathbb{Q}$-generic over M. Show that $G := \{p \in \mathbb{P}\,;\, i(p) \in H\}$ is $\mathbb{P}$-generic over M and that $M[G] \subseteq M[H]$.

(37) If $(\mathbb{P}, \leq_\mathbb{P}, \mathbf{1}_\mathbb{P})$ and $(\mathbb{Q}, \leq_\mathbb{Q}, \mathbf{1}_\mathbb{Q})$ are partial orders, then a function $i : \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if

   (a) i is order preserving, i.e., if $p \leq_\mathbb{P} p'$, then $i(p) \leq_\mathbb{Q} i(p')$;
   (b) i preserves incompatibility, i.e., if $p \perp_\mathbb{P} p'$, then $i(p) \perp_\mathbb{Q} i(p')$; and
   (c) the image of $\mathbb{P}$ under i is dense in $\mathbb{Q}$.

   Show that every dense embedding is a complete embedding.

(38) Let $\mathbb{T}$ be the partial order of finite sequences of natural numbers ordered by reverse inclusion. Show that there is a dense embedding from $\mathbb{T}$ to $\mathbb{Q}$.

(39) Let $\mathbb{T}_{\text{bin}}$ be the partial order of finite zero-one sequences. Show that there is no dense embedding from $\mathbb{T}_{\text{bin}}$ to $\mathbb{T}$.

(40) We say that $\mathbb{P}$ *preserves cofinalities* if for every $\mathbb{P}$-generic filter G over M and every limit ordinal $\lambda \in M$, we have that $\text{cf}(\lambda)^M = \text{cf}(\lambda)^{M[G]}$. Prove that if $\mathbb{P}$ preserves cofinalities, then it preserves cardinals.

(41) If κ is a cardinal, we say that $\mathbb{P}$ has the κ-c.c. if every antichain in $\mathbb{P}$ has cardinality smaller than κ. (Thus, the c.c.c. is the $\aleph_1$-c.c.) If κ is a cardinal in M, we say that $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ if for every $\mathbb{P}$-generic filter G over M and every $\lambda \geq \kappa$, we have that $M \models$"λ is a cardinal" if and only if $M[G] \models$"λ is a cardinal". Show that if $M \models$"κ is a regular cardinal" and $M \models$"$\mathbb{P}$ has the κ-c.c.", then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

(42) A partial order $\mathbb{P}$ is called λ-*closed* if whenever $\gamma < \lambda$ and $S := \{p_\xi\,;\, \xi < \gamma\}$ is a decreasing chain of elements in $\mathbb{P}$, then there is a $q \in \mathbb{P}$ such that q is below all elements of S. Suppose that $\mathbb{P}$ is λ-closed, that $\alpha < \lambda$, β is any ordinal, that G is $\mathbb{P}$-generic over M, and that $f \in M[G]$ with $f : \alpha \to \beta$. Show that $f \in M$. Deduce that λ-closed forcing preserves cardinals $\leq \lambda$.

(43) Let $A \subseteq \mathbb{N}$ be infinite. We say that $S \subseteq \mathbb{N}$ *splits* A if both $A \cap S$ and $A \backslash S$ are infinite. We say that S is a *splitting set over* M if for all $A \in M$, S splits A. Let $\mathbb{P} := \text{Fn}(\omega, \omega)$ and G be $\mathbb{P}$-generic over M. Show that there is splitting set over M in $M[G]$.

(44) Let $f : \mathbb{N} \to \mathbb{N}$ a function. We say that f *bounds* M if for every $g \in M$ such that $g : \mathbb{N} \to \mathbb{N}$ there are infinitely many numbers k such that $g(k) < f(k)$.

Let $\mathbb{P} := \{(s, A)\,;\, s \text{ is a partial function with } \text{dom}(s) \in \mathbb{N}, \text{ran}(s) \subseteq \mathbb{N}, \text{ and } A \subseteq \mathbb{N} \text{ is infinite}\}$ with $(s, A) \leq (t, B)$ if $s \supseteq t$, $A \subseteq B$, and $\{s(i)\,;\, i \in \text{dom}(s)\backslash\text{dom}(t)\} \subseteq B$.

Assume that G is $\mathbb{P}$-generic over M and prove that there is a function f that bounds M.

(45) We write $\text{Fn}(I, J, \lambda)$ for the partial order of partial functions p with $\text{dom}(p) \subseteq I$, $\text{ran}(p) \subseteq J$, and $M \models |\text{dom}(p)| < \lambda$, ordered by reverse inclusion. Let $\mathbb{P} := \text{Fn}(\aleph_\omega^M, 2, \aleph_\omega^M)$. Suppose that G is $\mathbb{P}$-generic over M. Show that in $M[G]$, the ordinal $\aleph_\omega^M$ is countable.