Infinite Games
Lent Term 2020
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #2

Examples Classes.
#1. Monday 3 February 2020, 3:30-5pm, MR14.
#2: Monday 17 February 2020, 5-6:30pm, MR14.
#3: Monday 2 March 2020, 3:30-5pm, MR14.
#4: Monday 16 March 2020, 3:30-5pm, MR14.
You hand in your work at the beginning of the Examples Class.

15. A function $c : \omega^{<\omega} \to \omega^{<\omega}$ is called *coherent* if for $p \subseteq q$, we have $c(p) \subseteq c(q)$, and for $x \in \omega^\omega$, we have $|c(x{\upharpoonright}n)| \to \infty$. If c is coherent, we define $f_c : \omega^\omega \to \omega^\omega$ by $f_c(x) := \bigcup_{n\in\omega} c(x{\upharpoonright}n)$. Prove that the following are equivalent:

    (i) $f : \omega^\omega \to \omega^\omega$ is continuous and

    (ii) there is a coherent c such that $f = f_c$.

16. Let $f : \omega^\omega \to \omega^\omega$ be any function and consider the following game $\mathrm{G}(f)$ on the move set $\omega \cup \{\mathsf{pass}\}$: player I may play only elements of ω, but player II may play pass; suppose player I produces x and player II produces a sequence $y \in (\omega \cup \{\mathsf{pass}\})^\omega$; remove all of the pass moves from y and obtain y^*; if $y^* \notin \omega^\omega$, then player II loses; otherwise player II wins if and only if $y^* = f(x)$. Show that the following are equivalent:

    (i) $f : \omega^\omega \to \omega^\omega$ is continuous and

    (ii) player II has a winning strategy in the game $\mathrm{G}(f)$.

    What happens if you do not require the extra possibility of pass moves?

17. Consider the real numbers $\mathbb{R}$ with their usual topology and their subspace $\mathbb{Q}$. Show that $\boldsymbol{\Delta}^0_2(\mathbb{Q}) \neq \{A \cap \mathbb{Q}\,;\, A \in \Delta^0_2(\mathbb{R})\}$.

18. Again, consider the real numbers $\mathbb{R}$ with their usual topology and let $F \subseteq \mathbb{R}$ be closed. Consider any continuous map $f : \mathbb{R} \to X$ and show that

    $$f[F] := \{x \in X\,;\, \exists r \in F(x = f(r))\}$$

    is F_σ.

19. Let $\mathbf{\Gamma}$ be a boldface pointclass. We say that $A \subseteq Y$ is *$\mathbf{\Gamma}$-hard for X* if for all $B \in \mathbf{\Gamma}(X)$, there is a continuous function $f : X \to Y$ such that $f^{-1}[A] = B$. If in addition, $A \in \mathbf{\Gamma}(Y)$, we call A *$\mathbf{\Gamma}$-complete for X*.

    Show that universal sets for $\mathbf{\Gamma}$ are $\mathbf{\Gamma}$-complete.

20. In the lectures, our construction of an ω^ω-universal set for $\mathbf{\Sigma}^0_\alpha$ from ω^ω-universal sets for all $\mathbf{\Pi}^0_\beta$ for $\beta < \alpha$ used a surjection $\pi : \omega \to \alpha$ that hits each element of α infinitely many times.

    Suppose $\alpha = \xi + 1$ is a successor ordinal and show that a bijection $\pi : \omega \to \xi + 1$ is not enough for the proof to work.

21. Let A and B be disjoint subsets of ω^ω. We say that A and B are *Borel separable* if there is a Borel set C such that $A \subseteq C$ and $B \cap C = \varnothing$.

    Consider sets $\{A_n\,;\, n \in \omega\}$ and $\{B_n\,;\, n \in \omega\}$. Suppose that for each $n, m \in \omega$, A_n and B_m are Borel-separable. Then $\bigcup_{n\in\omega} A_n$ and $\bigcup_{n\in\omega} B_n$ are Borel separable.

22. Show the *Luzin Separation Theorem*: any two disjoint analytic sets are Borel separable.

23. Show that a set $B \subseteq \omega^\omega$ is Borel if and only if it is $\mathbf{\Delta}^1_1(\omega^\omega)$, i.e., both analytic and co-analytic.

24. Let κ be the smallest cardinality such that there is some $A \subseteq \mathbb{R}$ with $|A| = \kappa$ which is not Lebesgue-null. Assume that A is such a set of cardinality κ and that R is a wellorder of A of order type κ. Show that R cannot be Lebesgue-measurable.

    [*Hint.* Fubini's theorem in the following form may help: if $B \subseteq \mathbb{R} \times \mathbb{R}$ is Lebesgue-measurable, then it is a null set if and only if the set of all vertical (or horizontal) sections which are not null is null.]

25. Let $A \subseteq \omega^\omega$ and consider the following game $\mathrm{G}^{**}(A)$: players I and II play nonempty finite sequences $p_i \in \omega^{<\omega}$; consider $x := p_0p_1p_2\ldots$; player I wins if $x \in A$, otherwise player II wins. Show that

    (i) Player I has a winning strategy in $\mathrm{G}^{**}(A)$ if and only if there is a position $p \in \omega^{<\omega}$ such that $[p]\backslash A$ is meagre.

    (ii) Player II has a winning strategy in $\mathrm{G}^{**}(A)$ if and only if A is meagre.

26. Show that $A \subseteq \omega^\omega$ has the Baire property if and only if for all open sets P, the game $\mathrm{G}^{**}(A\backslash P)$ is determined.

27. Let $\mathbf{\Gamma}$ be a boldface pointclass closed under finite intersections and containing the open sets. Show that the determinacy of all $\mathbf{\Gamma}$ sets implies that all $\mathbf{\Gamma}$ sets have the Baire property.