Infinite Games
Lent Term 2020
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #3

Examples Classes.
#1. Monday 3 February 2020, 3:30-5pm, MR14.
#2: Monday 17 February 2020, 5-6:30pm, MR14.
#3: Monday 2 March 2020, 3:30-5pm, MR14.
#4: Monday 16 March 2020, 3:30-5pm, MR14.
Revision: Friday 22 May 2020, time TBA.
You hand in your work at the beginning of the Examples Class.

28. We proved that if there is a projective wellordering of ω^ω, then there is a projective set that is not determined. Assume that the projective wellordering is $\mathbf{\Delta}^1_n$. What is (an upper bound of) the complexity of the non-determined set?

29. Show that the following statement is inconsistent with ZFC: "$2^{\aleph_0} = \aleph_3$ and there is a $\mathbf{\Pi}^1_1$ set of cardinality $\aleph_2$".

    [*Hint.* You are allowed to use Borel Determinacy.]

30. Show that if T is a tree on X, then T is well-founded if and only if there is an order preserving map from $(T, \supseteq)$ to an ordinal. What can you say about the size of that ordinal?

31. Suppose that S and T are trees on X. Show that there is an order preserving map from $(S, \supseteq)$ to $(T, \supseteq)$ if and only if either T is illfounded or $\mathrm{ht}(S) \leq \mathrm{ht}(T)$.

32. Let s and t be finite sequences of ordinals. We say $s <_{\mathrm{KB}} t$ if either $t \subseteq s$ (*sic*!) or if i if the least number such that $s(i) \neq t(i)$, we have $s(i) < t(i)$. This order is called the *Kleene-Brouwer order*. Show that it is a total order on the class of finite sequences of ordinals and that if T is a tree on κ, then T is well-founded if and only if $(T, <_{\mathrm{KB}})$ is a wellorder.

33. Let $A \subseteq \mathrm{WO} \times \mathrm{WO}$. The following game $\mathrm{G_S}(A)$ is called the *Solovay game* on A: players I and II produce a play z in the usual way, $x := z_{\mathrm{I}}$ and $y := z_{\mathrm{II}}$. Player I loses if $x \notin \mathrm{WO}$. Otherwise, player II loses if $y \notin \mathrm{WO}$. If both of them play in WO, then player I wins if $(x, y) \in A$.

    Let $A := \{(x, y)\,;\, \|x\| \geq \|y\|\}$ and show that player I cannot have a winning strategy in $\mathrm{G_S}(A)$.

34. Let σ be a winning strategy for player I in some Solovay game $\mathrm{G_S}(A)$. Show that there is a function $f : \aleph_1 \to \aleph_1$ such that if $\alpha = \|y\| < \xi$, then there is some x such that $\|x\| < f(\xi)$ and $(x, y) \in A$.

35. Recall that for a limit ordinal λ, $\mathrm{cf}(\lambda) := \min\{|C|\,;\, C \text{ is a cofinal subset of } \lambda\}$. Show that for each λ, $\mathrm{cf}(\lambda)$ is a regular cardinal and that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$.

36. Let κ be a cardinal. If $X \subseteq \kappa$, we write $[X]^n$ for the set of n-element subsets of X and $[X]^{<\omega}$ for the set of finite subsets of X.

    If $\chi : [\kappa]^2 \to \gamma$ is a colouring of $[\kappa]^2$ with γ many colours, we say that $H \subseteq \kappa$ is *χ-homogeneous* if $[H]^2$ is monochromatic under the map χ (i.e., all sets $\{x, y\} \in [H]^2$ get the same colour under χ). We call a cardinal κ *weakly compact* if every colouring χ with two colours has a homogeneous set of size κ (in Erdős-Rado notation: $\kappa \to (\kappa)^2_2$). Show that every weakly compact cardinal is strongly inaccessible.

    [*Hint.* Note that for any cardinal λ the set 2^λ with the lexicographic ordering cannot have any strictly increasing or decreasing sequences of length $> \lambda$.]

37. If $\chi : [\kappa]^{<\omega} \to \gamma$ is a colouring of the finite subsets of κ, we say that $H \subseteq \kappa$ is *χ-homogeneous* if for each $n \in \omega$, $[H]^n$ is monochromatic under the map χ (i.e., all sets in $[H]^n$ get the same colour under χ). What happens if you replace n in this definition with "$<\omega$"?

38. An ultrafilter U on κ is called *normal* if for any family $\{X_\alpha\,;\, \alpha < \kappa\} \subseteq U$, the *diagonal intersection*

$$\triangle_{\alpha<\kappa} X_\alpha := \{\xi < \kappa\,;\, \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$$

    lies in U. If $S \subseteq \kappa$, a function $f : S \to \kappa$ is called *regressive* if for all $\xi \neq 0$, $f(\xi) < \xi$. A set S is called *U-stationary* if for all $X \in U$, we have that $X \cap S \in U$.

    Let U be a κ-complete ultrafilter on κ. Show that the following are equivalent:

    (i) U is normal,
    (ii) for every U-stationary set S and every regressive function $f : S \to \kappa$ there is an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is U-stationary, and
    (iii) for every function $f : \kappa \to \kappa$, if $\{\xi < \kappa\,;\, f(\xi) < \xi\} \in U$, then there is some $\alpha < \kappa$ such that $\{\xi < \kappa\,;\, f(\xi) = \alpha\} \in U$.

39. Prove *Rowbottom's Theorem*: If κ is measurable, U is a normal ultrafilter on κ, $\gamma < \kappa$, and $\chi : [\kappa]^{<\omega} \to \gamma$ is a colouring, then there is a χ-homogeneous set in U.

40. A set $A \subseteq (\omega^\omega)^n$ is called *κ-Suslin* if there is a tree T on $\kappa \times \omega^n$ such that $A = \mathrm{p}[T]$. Prove that every set A is $2^{\aleph_0}$-Suslin and that a set is $\aleph_0$-Suslin if and only if it is $\mathbf{\Sigma}^1_1$.

41. Prove that the class of κ-Suslin sets (cf. Example 40) is closed under projections, continuous preimages and unions of size κ.