LARGE CARDINALS AND LIGHTFACE DEFINABLE WELL-ORDERS, WITHOUT THE GCH

SY-DAVID FRIEDMAN, PETER HOLY, AND PHILIPP LÜCKE

Abstract. This paper deals with the question whether the assumption that for every inaccessible cardinal κ there is a well-order of $\mathrm{H}(\kappa^+)$ definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a formula without parameters is consistent with the existence of (large) large cardinals and failures of the GCH. We work under the assumption that the SCH holds at every singular fixed point of the $\beth$-function and construct a class forcing that adds such a well-order at every inaccessible cardinal and preserves ZFC, all cofinalities, the continuum function and all supercompact cardinals. Even in the absence of a proper class of inaccessible cardinals, this forcing produces a model of "$\mathrm{V} = \mathrm{HOD}$" and can therefore be used to force this axiom while preserving large cardinals and failures of the GCH. As another application, we show that we can start with a model containing an ω-superstrong cardinal κ and use this forcing to build a model in which κ is still ω-superstrong, the GCH fails at κ and there is a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\mathrm{H}(\kappa^+)$ without parameters. Finally, we can apply the forcing to answer a question about the definable failure of the GCH at a measurable cardinal.

1. Introduction

The *outer model programme* aims to obtain desirable structural features of Gödel's constructible universe in generic extensions of the ground model while preserving large cardinals. A related objective is to obtain desirable structural features of Gödel's constructible universe in generic extensions of the ground model while not only preserving large cardinals, but also other features that are incompatible with the assumption "$\mathrm{V} = \mathrm{L}$". This paper focuses on the existence of well-orders of $\mathrm{H}(\kappa^+)$ definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a formula without parameters, large cardinals and failures of the GCH. Our main result shows that it is possible to add such well-orderings at every inaccessible cardinal κ using a class-forcing that preserves ZFC, all cofinalities, the continuum function and supercompact cardinals. As a corollary, we also obtain a result on the relative consistency of the statement "$\mathrm{V} = \mathrm{HOD}$" together with the existence of large cardinals and possible failures of the GCH. Before we state our results, we fix some definitions and give a brief overview of related existing results.

Definition 1.1. Let κ be an infinite cardinal, $X \subseteq \mathrm{H}(\kappa^+)$, $p \in \mathrm{H}(\kappa^+)$ and $n < \omega$.

2010 *Mathematics Subject Classification.* 03E35, 03E47, 03E55.

Key words and phrases. Class forcing, Definable well-orders, Failures of the GCH, HOD, supercompact cardinals, ω-superstrong cardinals.

The first author wishes to thank the FWF (Austrian Science Fund) for its generous support through Project P 23316 N13. The second author wishes to thank the EPSRC for its generous support through Project EP/J005630/1.

- The set X is *lightface* Σ_n*-definable* (respectively, is Σ_n*-definable in parameter* p) *over* $\mathrm{H}(\kappa^+)$ if X is definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_n-formula without parameters (respectively, with parameter p).
- We say that X is *boldface* Σ_n*-definable over* $\mathrm{H}(\kappa^+)$ if it is Σ_n-definable in some parameter (in $\mathrm{H}(\kappa^+)$) over $\mathrm{H}(\kappa^+)$.
- The set X is lightface Σ_n-definable (respectively, is Σ_n-definable in parameter p) *in* $\mathrm{H}(\kappa^+)$ if X is lightface Σ_n-definable (respectively, Σ_n-definable in parameter p) over $\mathrm{H}(\kappa^+)$ and X is an element of $\mathrm{H}(\kappa^+)$.
- We say that there is a *lightface* Σ_n*-definable well-order* (respectively, a *boldface* Σ_n*-definable well-order*) *of* $\mathrm{H}(\kappa^+)$ if there is a well-order of $\mathrm{H}(\kappa^+)$ which is lightface (respectively, boldface) Σ_n-definable over $\mathrm{H}(\kappa^+)$.
- *Lightface definable* abbreviates lightface Σ_n-definable for some $n \in \omega$, *boldface definable* or simply *definable* abbreviate boldface Σ_n-definable for some $n \in \omega$.

In [2] and [3], David Asperó and the first author showed that assuming the GCH, there is a cofinality-preserving class forcing which introduces a lightface definable well-order of $\mathrm{H}(\kappa^+)$ for every regular uncountable κ while preserving all instances of supercompactness, many other large cardinals and the GCH. In [9], the first and second author independently obtained (as a side result) that assuming GCH, there is a cofinality-preserving forcing which introduces a boldface definable well-order of $\mathrm{H}(\kappa^+)$ for every regular uncountable κ while preserving many large cardinals (including ω-superstrong cardinals) and the GCH. While this result is basically weaker than the aforementioned, enhancements of the techniques developed in the proofs of this statement will be made heavy use of in the following.

The first and third author have shown in [11] that assuming the SCH holds at singular limits of inaccessibles (but not the GCH),[1] there is a cofinality-preserving forcing which introduces a boldface definable well-order of $\mathrm{H}(\kappa^+)$ for every inaccessible κ and preserves all supercompacts, many instances of supercompactness and the continuum function. This construction is based on results of the third author from [13] that allow us to add boldface definable well-orders of $\mathrm{H}(\kappa^+)$ by forcings which preserve the value of 2^κ. These forcings will be another ingredient of our construction.

Our main result improves the result of [11] by obtaining *lightface* instead of *boldface* definable well-orders (starting from a slightly stronger SCH-assumption).

Theorem 1.2. *Assume* SCH *holds at singular fixed points of the* $\beth$*-function. There is a* ZFC*-preserving class forcing* $\mathbb{P}$ *definable without parameters that satisfies the following statements.*

(i) If α *is inaccessible and* G *is* $\mathbb{P}$*-generic over* V *then there is a lightface definable well-order of* $\mathrm{H}(\alpha^+)^{V[\mathsf{G}]}$.
(ii) $\mathbb{P}$ *is cofinality-preserving.*
(iii) $\mathbb{P}$ *preserves the continuum function: If* G *is* $\mathbb{P}$*-generic over* V*, then* $(2^\alpha)^V = (2^\alpha)^{V[\mathsf{G}]}$ *for every* α.
(iv) $\mathbb{P}$ *preserves the supercompactness of supercompact cardinals.*

In [5], Andrew Brooke-Taylor has shown that assuming GCH, there is a cofinality-preserving forcing to obtain "$\mathrm{V} = \mathrm{HOD}$" in the generic extension while preserving

[1]If X is a class of singular strong limit cardinals, SCH at X abbreviates the statement that $2^\kappa = \kappa^+$ for every $\kappa \in X$.

various large cardinals and the GCH. In order to obtain this, he codes a class-function $F : \mathrm{On} \longrightarrow 2$ into the validity of the principle $\Diamond^*_\kappa$ at various regular cardinals κ, making use of the fact that under the GCH, the validity of the principle $\Diamond^*_\kappa$ can be "*switched on and off*" by mild ($<\kappa$-closed and κ^+-c.c.) forcings.

In the proof of the above theorem, we improve this technique and develop a finer coding which does not rely on the GCH. This coding uses the fact that the existence of boldface definable well-orders of $\mathrm{H}(\kappa^+)$ for uncountable κ with $\kappa = \kappa^{<\kappa}$ can be "*switched on and off*" by mild (again $<\kappa$-closed and κ^+-c.c.) forcing that preserves the value of 2^κ. This follows from results contained in [13] that will be restated in Section 2 of this paper. Hence one could define a *Boldface Definable Well-order Oracle Partial Order* similar to the $\Diamond^*$*-Oracle Partial Order* defined in [5, Definition 5] and use it to start with a model where the SCH holds on a proper class of singular strong limit cardinals[2] and force the statement "$\mathrm{V} = \mathrm{HOD}$" while preserving all cofinalities and the continuum function. The class forcing constructed in the proof of Theorem 1.2 contains such a *Boldface Definable Well-order Oracle Partial Order* and uses it (in combination with other coding forcings) to make certain sets locally lightface definable. This construction will allow us to run the argument sketched above and derive the following corollary.[3]

Corollary 1.3. *If $\mathbb{P}$ is the partial order from Theorem 1.2, then in any $\mathbb{P}$-generic forcing extension,* "$\mathrm{V} = \mathrm{HOD}$" *holds.*

In particular, if the SCH holds at singular fixed points of the $\beth$-function, then we can force the statement "$\mathrm{V} = \mathrm{HOD}$" with a class forcing that preserves ZFC, all cofinalities, the continuum function and every supercompact cardinal. This generalizes a result of Arthur Apter and Shoshana Friedman from [1], where the above has been obtained under the assumption of (and preserving) GCH.

Moreover one could replace the coding based on modifying the number of normal measures on measurable cardinals that is used in Section 3 of [1] by the boldface definable wellorder existence coding to obtain, under the assumption of the SCH at singular fixed points of the $\beth$-function, a version of [1, Theorem 3.2] that does not need the assumption of the GCH (and preservation of the GCH is to be replaced by preservation of cofinalities and the continuum function). Since the SCH assumption above is only needed to ensure preservation of cofinalities, this also provides an alternative proof of [6, Theorem 4]. Under the assumption of SCH at singular fixed points of the $\beth$-function, we obtain the following version of that theorem (see [6] for the definition of HOD-supercompactness):

Theorem 1.4. *Assume κ is supercompact and* SCH *holds at all singular fixed points of the $\beth$-function. Then there is a set-forcing extension that preserves both cofinalities and the continuum function, in which κ is still supercompact, but not* HOD-*supercompact.*

We discuss other applications of the forcing construction from Theorem 1.2. In [10], Radek Honzik and the first author show that it is possible to obtain a model

[2]Note that a classical theorem of Solovay (see Theorem 1.9) implies that this assumption becomes void in the presence of a strongly compact cardinal.

[3]Note that if there are unboundedly many inaccessibles, then the corollary follows directly from Theorem 1.2. Only in the case that the inaccessibles are bounded in the universe do we need to argue as above.

containing a measurable cardinal κ with a lightface definable well-order of $\mathrm{H}(\kappa^+)$ and $2^\kappa = \kappa^{++}$, starting with a κ^{++}-strong cardinal κ. The iteration of their paper cannot be used to make 2^κ bigger than κ^{++}. This raises the question whether it is consistent to have a measurable cardinal κ with $2^\kappa > \kappa^{++}$ and a lightface definable well-order of $\mathrm{H}(\kappa^+)$. Starting from a much stronger large cardinal assumption, we use the forcing from Theorem 1.2 to establish this consistency and thus answer [10, Question 6.1].

Corollary 1.5. *Assume that the* SCH *holds at singular fixed points of the $\beth$-function. Let κ be a regular cardinal and δ be the least singular strong limit cardinal of cofinality κ. If κ is δ-supercompact, then there is a cofinality-preserving set forcing which introduces a lightface definable well-order of* $\mathrm{H}(\kappa^+)$ *and preserves the δ-supercompactness (and thus the measurability) of κ and the value of 2^κ.*

With the help of a folklore result (see Lemma 1.8), this immediately gives rise to the following.

Theorem 1.6. *The consistency of* ZFC *plus*

(i) "There is a δ-supercompact cardinal κ such that δ is the least singular strong limit cardinal of cofinality κ"

implies the consistency of ZFC *plus*

(ii) "There is a measurable cardinal κ with $2^\kappa > \kappa^{++}$ and a lightface definable well-order of $\mathrm{H}(\kappa^+)$".

We also show that the ω-superstrength of a given ω-superstrong cardinal (see [9, Definition 5]) may be preserved while forcing with $\mathbb{P}$, which in particular yields the following result (again with the help of Lemma 1.8).

Theorem 1.7. *The following statements are equiconsistent over* ZFC.

(i) There is an ω-superstrong cardinal.
(ii) There is an ω-superstrong cardinal κ with $2^\kappa > \kappa^+$ (or $2^\kappa > \kappa^{++}$) and a lightface definable well-order of $\mathrm{H}(\kappa^+)$.

Throughout this paper, we assume that the SCH holds at singular fixed points of the $\beth$-function. This will imply that the partial order $\mathbb{P}$ constructed in the proof of Theorem 1.2 is cofinality-preserving. The following folklore result says that this assumption can be obtained by mild forcing (i.e., collapsing all "*counterexamples*" while preserving the continuum function and large cardinals).

Lemma 1.8. *There is a class-sized iteration $\mathbb{S}$ which satisfies the following:*

- $\mathbb{S}$ *forces* SCH *at singular fixed points of the $\beth$-function.*
- $\mathbb{S}$ *preserves the cofinality of κ whenever there is no singular fixed point λ of the $\beth$-function with $\lambda^+ < \kappa \leq 2^\lambda$.*
- $\mathbb{S}$ *preserves the value of 2^κ whenever it preserves κ and κ is not a singular fixed point of the $\beth$ function with $2^\kappa > \kappa^+$.*
- $\mathbb{S}$ *preserves the inaccessibility of all inaccessible and the supercompactness of all supercompact cardinals.*
- *If γ is a singular strong limit cardinal but not a fixed point of the $\beth$-function and κ is γ-supercompact with $\gamma^{<\kappa} = \gamma$, then forcing with $\mathbb{S}$ preserves the γ-supercompactness of κ.*
- *If κ is ω-superstrong, then there is a condition in $\mathbb{S}$ which forces that the ω-superstrength of κ is preserved by forcing with $\mathbb{S}$.*

Sketch of the proof. Let $\mathbb{S}$ be the reverse Easton iteration which at stage κ adds a Cohen subset of κ^+ if $2^\kappa > \kappa^+$ and κ is a singular fixed point of the $\beth$-function. That $\mathbb{S}$ forces SCH at singular fixed points of the $\beth$-function, preserves the relevant cofinalities, the relevant parts of the continuum function and all inaccessibles are standard arguments. That it preserves all instances of supercompactness as in the statement of the lemma can be shown by arguments similar to (but easier than) those in the proof of supercompactness preservation provided later in this paper for $\mathbb{P}$. The ω-superstrength preservation can be shown similar to (but easier than) the proof of ω-superstrength preservation provided later in this paper for $\mathbb{P}$. For those reasons and because this lemma is not actually essential to the content of this paper, we will not provide a detailed proof. □

Several computations in the following proofs will rely on the following classical theorem by Robert Solovay (see [12] for the refined statement below).

Theorem 1.9 (Solovay)**.** *If $\gamma > \kappa$, γ is a singular strong limit cardinal and κ is γ-strongly compact, then $2^\gamma = \gamma^+$.*

In particular, this result shows that the assumptions of our main result automatically hold above the first supercompact cardinal.

2. Prerequisites

In this section we will introduce several concepts and results contained in [11] and [13] that we will make use of in later sections. Given an ordinal λ, we let ${}^\lambda\lambda$ denote the set of all functions $f : \lambda \longrightarrow \lambda$ and ${}^{<\lambda}\lambda$ denote the set of all functions g with $\mathrm{dom}(g) \in \lambda$ and $\mathrm{ran}(g) \subseteq \lambda$. We let $\prec\cdot,\cdot\succ$ denote the Gödel pairing function on ordinals.

Definition 2.1. Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. A pair $\langle A, s\rangle$ is a *κ-coding basis* if A is a subset of ${}^\kappa\kappa$ and $s = \langle s_\alpha \mid \alpha < \kappa\rangle$ is an enumeration of ${}^{<\kappa}\kappa$ such that $\mathrm{lh}(s_\alpha) \leq \alpha$ for all $\alpha < \kappa$ and the set $\{\alpha < \kappa \mid s_\alpha = t\}$ is unbounded in κ for each $t \in {}^{<\kappa}\kappa$.

Given an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ and a κ-coding basis $\langle A, s\rangle$, we define $\mathbb{P}_s(A)$ to be the partial order consisting of conditions $p = \langle T_p, g_p, h_p\rangle$ with the following properties.

- (i) T_p is a subtree of ${}^{<\kappa}2$ that satisfies the following statements.
  - (a) T_p has cardinality less than κ.
  - (b) If $t \in T_p$ with $\mathrm{lh}(t) + 1 < \mathrm{ht}(T_p)$, then t has two immediate successors in T_p.
- (ii) $g_p : A \xrightarrow{part} [T_p]$ is a partial function such that $\mathrm{dom}(g_p)$ is of cardinality less than κ.
- (iii) $h_p : A \xrightarrow{part} \kappa$ is a partial function with the following properties.
  - (a) $\mathrm{dom}(h_p) = \mathrm{dom}(g_p)$.
  - (b) For all $x \in \mathrm{dom}(h_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$ with $\alpha = \prec h_p(x), \beta\succ$, we have
$$s_\beta \subseteq x \iff g_p(x)(\alpha) = 1.$$

We define $p \leq_{\mathbb{P}_s(A)} q$ to hold if the following statements are satisfied.

- (a) T_p is an end-extension of T_q.[4]

[4]In the sense that $T_q = \{t \in T_p \mid \mathrm{lh}(t) < \mathrm{ht}(T_q)\}$.

(b) For all $x \in \mathrm{dom}(g_q)$, $x \in \mathrm{dom}(g_p)$ and $g_q(x)$ is an initial segment of $g_p(x)$.
(c) $h_q = h_p \restriction \mathrm{dom}(h_q)$.

Proposition 2.2. *If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $\langle A, s\rangle$ is a κ-coding basis, then $\mathbb{P}_s(A) \subseteq \mathrm{H}(\kappa^+)$, $\mathbb{P}_s(A)$ satisfies the κ^+-chain condition and is $<\kappa$-directed closed with infima*[5]*.*

Proof. The first statement is immediate from the definition of the partial order and the second one is proven in [13, Lemma 3.2]. To show that the third statement holds, let D be a directed subset of $\mathbb{P}_s(A)$ of cardinality less than κ. Define $T = \bigcup\{T_p \mid p \in D\}$,

$$g \;=\; \{\langle x, \bigcup\{g_p(x) \mid p \in D,\ x \in \mathrm{dom}(g_p)\}\rangle \mid \exists p \in D\ x \in \mathrm{dom}(g_p)\}$$

and $h = \bigcup\{h_p \mid p \in D\}$. Then T is a subtree of ${}^{<\kappa}2$ of size less than κ and $\mathrm{dom}(g) = \mathrm{dom}(h)$ is of size less than κ. Given $p_0, p_1 \in D$, $x \in \mathrm{dom}(g)$ and $\alpha < \mathrm{ht}(T)$, there is a $q \in D$ with $q \leq p_0, p_1$, $x \in \mathrm{dom}(g_q)$ and $\mathrm{ht}(T_q) \geq \alpha$. If $x \in \mathrm{dom}(g_{p_i})$, then $g_{p_i}(x) \subseteq g_q(x)$ and $h_{p_i}(x) = h_q(x)$. This shows that g and h are partial functions and $g(x) \in [T]$ for all $x \in \mathrm{dom}(g)$. In particular, the triple $p_D = \langle T, g, h\rangle$ is a condition in $\mathbb{P}_s(A)$ and it is easy to check that p_D is the infimum of D in $\mathbb{P}_s(A)$. □

In the above setting, we let $\dot{T}_\kappa$ and $\dot{F}_\kappa$ denote the canonical $\mathbb{P}_s(A)$-names with the property that $\dot{T}_\kappa^G = \bigcup\{T_p \mid p \in G\}$ and

$$\dot{F}_\kappa^G : A \longrightarrow [\dot{T}_\kappa^G]^{\mathrm{V}[G]};\ x \longmapsto \bigcup\{g_p(x) \mid p \in G,\ x \in \mathrm{dom}(g_p)\}$$

whenever G is $\mathbb{P}_s(A)$-generic over V.

Theorem 2.3 ([13, Corollary 3.7])**.** *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\langle A, s\rangle$ be a κ-coding basis and G be $\mathbb{P}_s(A)$-generic over V. Then $[\dot{T}_\kappa^G]^{\mathrm{V}[G]} = \mathrm{ran}(\dot{F}_\kappa^G)$ and the following statements are equivalent for all $x \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$.*

(i) x is an element of A.
(ii) There is a $z \in [\dot{T}_\kappa^G]^{\mathrm{V}[G]}$ and an $\alpha < \kappa$ such that

$$s_\beta \subseteq x \iff z(\prec\alpha, \beta\succ) = 1$$

holds for all $\beta < \kappa$.

In particular, A is made boldface Σ_1-definable over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. In the proof of Theorem 1.2, we will make use of the following direct consequence of Theorem 2.3.

Corollary 2.4. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\langle A, s\rangle$ be a κ-coding basis and G be $\mathbb{P}_s(A)$-generic over V. If W is an outer model of $\mathrm{V}[G]$ such that κ is a cardinal in W and $[\dot{T}_\kappa^G]^{\mathrm{W}} = [\dot{T}_\kappa^G]^{\mathrm{V}[G]}$, then A is Σ_1-definable in parameters s and $\dot{T}_\kappa^G$ over $\langle \mathrm{H}(\kappa^+)^{\mathrm{W}}, \in\rangle$.* □

Given functions $x, y \in {}^\kappa\kappa$, we let $\prec x, y\succ$ denote the unique function $z \in {}^\kappa\kappa$ with

$$z(\prec\alpha, \beta\succ) = \begin{cases} x(\beta), & \text{if } \alpha = 0, \\ y(\beta), & \text{if } \alpha = 1 \\ 0, & \text{otherwise.} \end{cases}.$$

[5] We say that a subset D of a partial order $\mathbb{P}$ is *directed* if for all $p_0, p_1 \in D$ there is a $p \in D$ with $p \leq p_0, p_1$. Given an infinite regular cardinal κ, a partial order $\mathbb{P}$ is $<\kappa$-directed closed with infima if every directed subset of $\mathbb{P}$ of cardinality less than κ has an infimum in $\mathbb{P}$.

We say that $A \subseteq {}^\kappa\kappa$ *codes a well-order of* ${}^\kappa\kappa$ if there is a well-order $\prec$ of ${}^\kappa\kappa$ such that

$$A = \{\prec x, y\succ \mid x, y \in {}^\kappa\kappa,\ x \prec y\}.$$

The following, which is notationally adapted from the original theorem in [13], provides us with a forcing to introduce a boldface definable well-order of $\mathrm{H}(\kappa^+)$.

Theorem 2.5 ([13, Theorem 6.1])**.** *Assume that κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\langle A, s\rangle$ is a κ-coding basis and A codes a well-order of ${}^\kappa\kappa$. If G is $\mathbb{P}_s(A)$-generic over* V*, then there is a well-ordering of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *that is Δ_2-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ in parameters s and $\dot{T}^G_\kappa$.*

In the other direction, we also need a forcing that allows us make sure that there are no definable well-orders of $\mathrm{H}(\kappa^+)$. The following result is folklore. A proof of this statement can be found in [13, Section 9].

Theorem 2.6. *If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and G is* $\mathrm{Add}(\kappa, \kappa^+)$*-generic over* V*, then the* Axiom of Choice *does not hold in $\langle \mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[G]}), \in\rangle$. In particular, there is no boldface definable well-order of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.

3. Lightface definable well-orders of $\mathrm{H}(\kappa^+)$

In this section, we commence the proof of Theorem 1.2. Let B_s denote the class of all singular fixed points of the $\beth$-function, Γ be the class of all successor cardinals of elements of B_s and $\langle b_\gamma \mid \gamma \in \mathrm{On}\rangle$ be the monotone enumeration of Γ. We will start by introducing a variant $C_f(i)$ of the *canonical function coding*. This coding was first introduced in [2] and [3] and also made use of in [9].

Given an inaccessible cardinal α, $\gamma \in [\alpha, \alpha^+)$, a bijection $f : \alpha \longrightarrow \gamma$ and $i < 2$, a set t is a condition in $C_f(i)$ if the following statements hold.

- t is a closed, bounded subset of α and
- If $\eta \in t$, then $c(otp\, f[\eta]) = i$, where $c : \mathrm{On} \longrightarrow 2$ is defined by

$$c(\delta) = \begin{cases} 1, & \text{if there is a definable well-order of } \mathrm{H}({b_\delta}^+). \\ 0, & \text{otherwise.} \end{cases}$$

Conditions in $C_f(i)$ are ordered by end-extension.

Definition 3.1. Given a set X, the *lottery* of X is the forcing poset consisting of a weakest condition $\mathbf{1}$ and all elements of X, where it is assumed that $\mathbf{1} \notin X$. Elements of X are pairwise incompatible in the lottery. If G is generic for the lottery of X, G will be of the form $\{x, \mathbf{1}\}$ for some $x \in X$; we say that G decides for x in that case.

Before we commence with the details of the proof of Theorem 1.2 by stating the definition of our forcing iteration $\mathbb{P}$, we want to give a rough idea of what the iteration $\mathbb{P}$ will do, as some of the basic ideas seem to be in danger of being buried by technicalities in the actual definition.

Our iteration will basically be Easton supported (but with an additional restriction on the supports) and we only force nontrivially at stage α and $\alpha + 1$ for $\alpha \in \Gamma$ and stages in the interval $[\alpha, \alpha \cdot \alpha)$ for inaccessible α. We perform a lottery of the set $\{0, 1\}$ at stage $\alpha \in \Gamma$. If the lottery decides for 0, then we ensure at stage $\alpha+1$ (by forcing, using Theorem 2.6) that there is no definable well-order of $\mathrm{H}(\alpha^+)$. Otherwise, we ensure at stage $\alpha + 1$ (by forcing, using Theorem 2.5) that there is a definable well-order of $\mathrm{H}(\alpha^+)$. This way we obtain a generic predicate c that is coded by the *boldface definable well-order existence pattern*.

If α is inaccessible, then we will first perform the lottery of all α-coding bases $\langle A, s\rangle$ such that A codes a well-ordering of ${}^{\alpha}\alpha$, to choose such an α-coding basis $\langle A_\alpha, s_\alpha\rangle$. In the next step, we force with $\mathbb{P}_{s_\alpha}(A_\alpha)$ (as defined in Section 2) to introduce a well-order of $\mathrm{H}(\alpha^+)$ that is definable over $H(\alpha^+)$ in some generically added parameter $x_\alpha \subseteq \alpha$. In the next $\alpha \cdot \alpha$-many stages, we perform an iteration $S^\alpha_{<\alpha\cdot\alpha}$, using supports of size less than α, which applies appropriate instances of the forcing $C_f(i)$ to make sure that x_α and the generic for $S^\alpha_{<\alpha\cdot\alpha}$ itself are both definable from $c\cap\alpha$. This will allow us to infer that $\mathrm{H}(\alpha^+)$ has a lightface definable well-order in any $\mathbb{P}$-generic extension.

For the sake of large cardinal preservation, we must in fact not allow for the lottery of $\{0,1\}$ if α is of the form $b_{\kappa+2+\delta}$ for some inaccessible κ and $\delta < \kappa$, but choose either 0 or 1 (non-generically) depending on the generic for the iteration below α. This will be made use of in the proof of Claim 20.

Throughout the following, for every inaccessible κ we fix a uniformly (in parameter κ) definable recursive sequence $\langle f_\gamma \mid \gamma \in [\kappa, \kappa\cdot\kappa]\rangle$ such that every f_γ is a bijection from κ to γ.

Proof of Clauses (i)-(iii) of Theorem 1.2. Our forcing $\mathbb{P}$ will be an "*Easton-like iteration*"[6] of the form $\langle P_{<\alpha}, \dot{P}_\alpha \mid \alpha \in \mathrm{On}\rangle$. $\dot{P}_\alpha$ denotes the trivial forcing unless $\alpha \in \Gamma$, $\alpha = \beta+1$ and $\beta \in \Gamma$ or $\operatorname{card}\alpha = \kappa$ is inaccessible and $\alpha \in [\kappa, \kappa\cdot\kappa)$. If $\alpha \in \Gamma$, let $\dot{P}_\alpha$ and $\dot{P}_{\alpha+1}$ both denote the trivial forcing if $\alpha^{<\alpha} > \alpha$ or α is singular in the $P_{<\alpha}$-generic extension.[7] Similarly, if $\operatorname{card}\alpha = \kappa$ is inaccessible, $\alpha \in [\kappa, \kappa\cdot\kappa)$ and κ is not inaccessible in the $P_{<\alpha}$-generic extension, let $\dot{P}_\alpha$ denote the trivial forcing.[7] Given a generic G for $\mathbb{P}$ or a generic G for some $P_{<\beta}$ and an ordinal $\alpha < \beta$, we let $G_{<\alpha}$ denote the generic on $P_{<\alpha}$ induced by G or G respectively.

Case (α *is inaccessible and remains so in the* $P_{<\alpha}$*-generic extension*)**.** We let $\dot{P}_\alpha$ denote the canonical $P_{<\alpha}$-name for the lottery of (the set of) all α-coding bases $\langle A, s\rangle$, where A codes a well-order of ${}^{\alpha}\alpha$ in the $P_{<\alpha}$-generic extension. If G is $P_{<\alpha}$-generic over V, a generic for $\dot{P}^G_\alpha$ over $\mathrm{V}[G]$ decides for an α-coding basis $\langle A_\alpha, s_\alpha\rangle$, where A_α codes a well-order of $({}^{\alpha}\alpha)^{\mathrm{V}[G]}$. Let $\dot{A}_\alpha$ and $\dot{s}_\alpha$ be canonical $P_{<\alpha+1}$-names for A_α and s_α.

Let $\dot{P}_{\alpha+1}$ be a canonical $P_{<\alpha+1}$-name for $\mathbb{P}_{\dot{s}_\alpha}(\dot{A}_\alpha)$. If G is $P_{<\alpha+2}$-generic over V, $s_\alpha = \dot{s}_\alpha^{G_{<\alpha+1}}$, $A_\alpha = \dot{A}_\alpha^{G_{<\alpha+1}}$ and $\bar{G}$ is the filter in $\mathbb{P}_{s_\alpha}(A_\alpha)$ induced by G, then the well-ordering of $({}^{\alpha}\alpha)^{\mathrm{V}[G_{\alpha+1}]}$ given by A_α is Σ_1-definable in some parameter $x_\alpha \in {}^{\alpha}2$ coding the sets s_α and $\dot{T}^{\bar{G}}_\alpha$, where $\dot{T}_\alpha$ is the $\mathbb{P}_{s_\alpha}(A_\alpha)$-name in $\mathrm{V}[G_{<\alpha+1}]$ constructed in Section 2. Let $\dot{x}_\alpha$ be a canonical $P_{<\alpha+2}$-name for such a parameter x_α. Working in $W = \mathrm{V}[G]$, by Proposition 2.2 we have $\mathrm{H}(\alpha) \subseteq \mathrm{V}[G_{<\alpha+1}]$ and $\alpha^+ = (\alpha^+)^{\mathrm{V}[G_{<\alpha+1}]}$. Let $<^\alpha$ denote the well-ordering of $({}^{\alpha}\alpha)^{\mathrm{V}[G_{<\alpha+1}]}$ determined by the parameter $\dot{x}^G_\alpha$. There is a canonical surjection σ of $({}^{\alpha}2)$ onto $\mathrm{H}(\alpha^+)$ that is Σ_1-definable in parameter α in $\mathrm{H}(\alpha^+)$.[8] Since the definition of this function

[6] This name of course refers to the supports used - those will be specified just before Definition 3.2. A related forcing construction is used in [9].

[7] We will later show by induction that this is never the case (see Claim 7).

[8] Fix $x \in {}^{\alpha}2$. We define a binary relation $\in_x$ on α by setting $\beta \in_x \gamma$ if $x(\prec 0, \prec\beta,\gamma\succ\succ) = 1$. If $\langle\alpha, \in_x\rangle$ is well-founded and extensional and $\pi : \alpha \longrightarrow y$ is the corresponding collapsing map, then we define $\sigma(x) = \{\pi(\beta) \mid x(\prec 1, \beta\succ) = 1\}$. Otherwise, we define $\sigma(x) = \emptyset$. If $z \in \mathrm{H}(\alpha^+)$

is absolute, the set $\mathrm{H}(\alpha^+)^{\mathrm{V}[G_{<\alpha+1}]}$ is equal to the image of $({}^\alpha 2)^{\mathrm{V}[G_{<\alpha+1}]}$ under σ. Hence $<^\alpha$ induces a well-ordering $<_\alpha$ of $\mathrm{H}(\alpha^+)^{\mathrm{V}[G_{<\alpha+1}]}$ in a canonical way.[9] $<_\alpha$ induces a well-ordering of $\mathrm{H}(\alpha)^{\mathrm{V}[G_{<\alpha+1}]}$ of order-type $\xi \in [\alpha, \alpha^+)$ and, by using the $<_\alpha$-least bijection from α to ξ, $<_\alpha$ actually induces a well-ordering $\overline{<_\alpha}$ of $\mathrm{H}(\alpha)^{\mathrm{V}[G_{<\alpha+1}]} = \mathrm{H}(\alpha)$ of order-type α. Let $\overline{<_\alpha}(\delta)$ denote the δ^{th} element of $\mathrm{H}(\alpha)$ in this ordering. We define by induction

- an auxiliary forcing iteration $S^\alpha_{<\alpha\cdot\alpha}$ of the form $\langle S^\alpha_{<\delta}, \dot{S}^\alpha_\delta \mid \delta < \alpha\cdot\alpha\rangle$ with $<\alpha$-support,
- a sequence $\langle R^\alpha_{<\delta} \mid \delta \leq \alpha\cdot\alpha\rangle$ such that $R^\alpha_{<\delta} \subseteq S^\alpha_{<\delta}$ for each $\delta \leq \alpha\cdot\alpha$, and
- a sequence $\langle \dot{a}^\alpha_\delta \mid \delta < \alpha\cdot\alpha\rangle$ such that $\dot{a}^\alpha_\delta$ is an $S^\alpha_{<\delta}$-name for each $\delta < \alpha\cdot\alpha$.

If $S^\alpha_{<\delta}$ is defined, let

$$R^\alpha_{<\delta} \;=\; \{p \in S^\alpha_{<\delta} \mid \exists w_p : \delta \longrightarrow [\alpha]^{<\alpha} \;\forall \gamma < \delta \; p\restriction\gamma \Vdash p(\gamma) = \check{w}_p(\gamma)\}.$$

Note that given $p \in S^\alpha_{<\delta}$, if w_p as desired exists, then it is uniquely determined. This allows us to define an injection $\rho : R^\alpha_{<\delta} \longrightarrow \mathrm{H}(\alpha)$ by setting $\rho(p) = q$ if q is a function with domain $f^{-1}_{\alpha\cdot\alpha}{}''\mathrm{supp}(p)$ and $q(\gamma) = w_p(f_{\alpha\cdot\alpha}(\gamma))$ for all $\gamma \in \mathrm{dom}(q)$.

We define $\dot{a}^\alpha_\delta$ as follows. If δ is of the form $\alpha\cdot\bar{\delta}$, then we let $\dot{a}^\alpha_\delta$ be the canonical $S^\alpha_{<\delta}$-name for $\dot{x}^G_\alpha(\bar{\delta})$. If δ is of the form $\alpha\cdot\bar{\delta}+1+\gamma$ for some $\gamma < \alpha$, then we choose a canonical $S^\alpha_{<\delta}$-name such that

$$(\dot{a}^\alpha_\delta)^K = \begin{cases} 1, & \text{if } \overline{<_\alpha}(\gamma) \in \bar{K}. \\ 0, & \text{otherwise.} \end{cases}$$

holds whenever K is $S^\alpha_{<\delta}$-generic over W and $\bar{K} = \rho''(K \cap R^\alpha_{<\delta})$. Finally, choose $\dot{S}^\alpha_\delta$ such that

$$(\dot{S}^\alpha_\delta)^K \;=\; (C_{f_{\alpha+2+\delta}}((\dot{a}^\alpha_\delta)^K))^{\mathrm{W}[K]}$$

whenever K is $S^\alpha_{<\delta}$-generic over W.

We now want to define the iteration $\mathbb{P}$ in the interval $[\alpha+2, \alpha\cdot\alpha)$ by induction. Assume $\delta < \alpha\cdot\alpha$ and $P_{<\alpha+2+\delta}$ is already defined. Let G be $P_{<\alpha+2+\delta}$-generic over V and let $W = \mathrm{V}[G_{<\alpha+2}]$. Work in W. Define $S^\alpha_{<\delta}$, $R^\alpha_{<\delta}$ and $\dot{a}^\alpha_\delta$ as above. We will sometimes use $\dot{a}^\alpha_\delta$ to also denote a canonical $P_{<\alpha+2}$-name for the actual $S^\alpha_{<\delta}$-name $\dot{a}^\alpha_\delta$ in the $P_{<\alpha+2}$-generic extension. We say that an element q of $R^\alpha_{<\delta}$ is *induced by an element r of G* if

$$r\restriction[\alpha+2, \alpha+2+\gamma) \Vdash r(\alpha+2+\gamma) = \check{w}_q(\gamma).$$

holds in W for all $\gamma < \delta$. Let

$$K \;=\; \{p \in S^\alpha_{<\delta} \mid \exists q \in R^\alpha_{<\delta} \; (q \leq p \wedge q \text{ is induced by an element of } G)\}.$$

---

and $b : \alpha \longrightarrow \mathrm{tc}(\{z\} \cup \alpha)$ is a bijection, then there is an $x \in {}^\alpha 2$ with $\in_x = \{\langle \beta, \gamma\rangle \mid b(\beta) \in b(\gamma)\}$, $x = \{b(\beta) \mid x(\prec 1, \beta\succ) = 1\}$ and hence $\sigma(x) = z$.

[9]Define $x <_\alpha y$ to hold if

$$\begin{aligned} \exists \bar{x}, \bar{y} \in ({}^\alpha 2)^{\mathrm{V}[G_{<\alpha+1}]}[\bar{x} <^\alpha \bar{y} \;\wedge\; \sigma(\bar{x}) = x \;\wedge\; \sigma(\bar{y}) = y \\ \wedge\; \forall \tilde{x} \in ({}^\alpha 2)^{\mathrm{V}[G_{<\alpha+1}]}(\tilde{x} <^\alpha \bar{x} \rightarrow \sigma(\tilde{x}) \neq x) \\ \wedge\; \forall \tilde{y} \in ({}^\alpha 2)^{\mathrm{V}[G_{<\alpha+1}]}(\tilde{y} <^\alpha \bar{y} \rightarrow \sigma(\tilde{y}) \neq y)]. \end{aligned}$$

Then K is a filter in $S^{\alpha}_{<\delta}$.[10] Choose a canonical $P_{<\alpha+2+\delta}$-name $\dot{K}$ for K and define $\dot{P}_{\alpha+2+\delta}$ to be a canonical $P_{<\alpha+2+\delta}$-name such that

$$\dot{P}^{G}_{\alpha+2+\delta} \ = \ (C_{f_{\alpha+2+\delta}}((\dot{a}^{\alpha}_{\delta})^{G_{<\alpha+2}*\dot{K}^{G}}))^{\mathrm{V}[G]}$$

holds whenever G is $P_{<\alpha+2+\delta}$-generic over V. We will later show inductively (see Clause 7 of Claim 7) that $P_{<\alpha+2} * S^{\alpha}_{<\delta}$ can be densely embedded into $P_{<\alpha+2+\delta}$ by a canonical embedding. We will thus sometimes identify $\dot{a}^{\alpha}_{\delta}$ not only with a $P_{<\alpha+2} * S^{\alpha}_{<\delta}$-name, but also with a $P_{<\alpha+2+\delta}$-name.

Case ($\alpha \in \Gamma$, $\alpha^{<\alpha} = \alpha$ *in the* $P_{<\alpha}$*-generic extension*)**.** If there is no inaccessible κ such that α is of the form $b_{\kappa+2+\delta}$ for some $\delta < \kappa$, let $\dot{P}_{\alpha}$ denote the lottery of 0 and all triples $\langle 1, A, s\rangle$ where $\langle A, s\rangle$ is an α-coding basis and A codes a well-order of ${}^{\alpha}\alpha$ in the $P_{<\alpha}$-generic extension. If G is $P_{<\alpha}$-generic over V, a generic for $\dot{P}^{G}_{\alpha}$ over $\mathrm{V}[G]$ decides for either 0 or 1 and if it decides for 1 it also decides for an α-coding basis $\langle A_{\alpha}, s_{\alpha}\rangle$ where A_{α} codes a well-order of $({}^{\alpha}\alpha)^{\mathrm{V}[G]}$. If $\alpha = b_{\kappa+2+\delta}$ for some inaccessible κ and $\delta < \kappa$, we demand that $\dot{P}_{\alpha}$ decides for 1 if and only if $\dot{a}^{\kappa}_{\delta}$ is decided by the filter on $P_{<\kappa+2} * S^{\kappa}_{<\delta}$ induced by G to be 1 (it shall decide for 0 otherwise) and only in that case[11] let $\dot{P}_{\alpha}$ also perform a lottery of all α-coding bases $\langle A, s\rangle$ where A codes a well-order of ${}^{\kappa}\kappa$ in the $P_{<\alpha}$-generic extension, and thus choose an α-coding basis $\langle A_{\alpha}, s_{\alpha}\rangle$ as above. Let $\dot{A}_{\alpha}$ and $\dot{s}_{\alpha}$ be canonical $P_{<\alpha+1}$-names for A_{α} and s_{α}. Let both denote $\emptyset$ if no α-coding basis was chosen.

Assume G is $P_{<\alpha+1}$-generic. Let $A_{\alpha} = \dot{A}^{G}_{\alpha}$ and $s_{\alpha} = \dot{s}^{G}_{\alpha}$. Let $\dot{P}_{\alpha+1}$ be a $P_{<\alpha+1}$-name such that $\dot{P}^{G}_{\alpha+1} = \mathrm{Add}(\alpha, \alpha^{+})$ if G decided for 0 at stage α and such that $\dot{P}^{G}_{\alpha+1} = \mathbb{P}_{s_{\alpha}}(A_{\alpha})$ if G decided for $\langle 1, A_{\alpha}, s_{\alpha}\rangle$ at stage α, with both $\mathrm{Add}(\alpha, \alpha^{+})$ and $\mathbb{P}_{s_{\alpha}}(A_{\alpha})$ defined in the sense of $\mathrm{V}[G]$. Note that by Theorem 2.6, $\mathrm{Add}(\alpha, \alpha^{+})$ ensures that $\mathrm{H}(\alpha^{+})$ has no definable well-order in the generic extension, while $\mathbb{P}_{s_{\alpha}}(A_{\alpha})$ ensures that $\mathrm{H}(\alpha^{+})$ has a definable well-order in the generic extension by Theorem 2.5. Both $\mathrm{Add}(\alpha, \alpha^{+})$ and $\mathbb{P}_{s_{\alpha}}(A_{\alpha})$ are $<\alpha$-closed and satisfy the α^{+}-chain condition.

For $p \in P_{<\alpha}$, λ inaccessible and $\gamma \in \mathrm{supp}(p) \cap [\lambda+2, \lambda\cdot\lambda)$, we write p^{**}_{γ} for $p(\gamma)$. As a notational convention, we say $p^{**}_{\gamma} = \check{\mathbb{1}}$ otherwise. We define the *club support of* p to be the set $\mathrm{C\text{-}supp}(p) = \{\gamma \mid p^{**}_{\gamma} \neq \check{\mathbb{1}}\}$.

It remains to declare the supports used in the construction of our iteration. Assume α is a limit ordinal, $P_{<\gamma}$ is defined for $\gamma < \alpha$, T is the inverse limit of $\langle P_{<\gamma} \mid \gamma < \alpha\rangle$ and $p \in T$. Then $p \in P_{<\alpha}$ if

- if α is regular, $\mathrm{supp}(p)$ is bounded in α and
- for every inaccessible θ, $\mathrm{card}\,(\mathrm{C\text{-}supp}(p) \cap \theta^{+}) < \theta$. [12]

Let $\mathbb{P}$ be the direct limit of $\langle P_{<\alpha} \mid \alpha \in \mathrm{On}\rangle$.

Definition 3.2. If α is inaccessible and $v \subseteq [\alpha, \alpha\cdot\alpha]$ is of size less than α, we say that C_v is the *canonical separating club for* v if

$$C_v = \{\eta < \alpha \mid \forall \gamma_0, \gamma_1 \in v \ \gamma_0 < \gamma_1 \to f_{\gamma_0}[\eta] \textit{ is a proper initial segment of } f_{\gamma_1}[\eta]\}.$$

As defined, C_v is easily seen to be a club subset of α. Note that if $v_1 \supseteq v_0$, we have that $C_{v_1} \subseteq C_{v_0}$. We will tacitly make use of this in the following.

[10] We will prove later that K is in fact generic for $S^{\alpha}_{<\delta}$ (see Clause 7 of Claim 7).

[11] See Claim 20 for a motivation of this extra condition.

[12] The former condition is the reason why we called our iteration "*Easton-like*" earlier on.

Definition 3.3. If $p \in P_{<\alpha}$ and $\eta < \alpha$ is a cardinal, we define $u_\eta(p)$ as follows:[13]

$$u_\eta(p)(\gamma) = \begin{cases} \check{\mathbf{1}} & \text{if } \gamma < \eta \\ \check{\mathbf{1}} & \text{if } \gamma \in (\eta, \eta^+) \\ p(\gamma) & \text{otherwise} \end{cases}.$$

We will usually assume that for every $\gamma \in \operatorname{supp}(p)$, $\mathbf{1}_{P_{<\gamma}} \Vdash p(\gamma) \in \dot{P}_\gamma$; therefore $u_\eta(p) \in P_{<\alpha}$. We let $u_\eta(P_{<\alpha}) = \{u_\eta(p) \mid p \in P_{<\alpha}\}$.

The following claim will often be tacitly used.

Claim 1. *If $p \in P_{<\alpha}$, $\eta < \alpha$ is a cardinal and $q \leq p$, then there is $r \leq q$ such that $q \leq r$ (i.e. q and r are* equivalent*) and $u_\eta(r) \leq u_\eta(p)$.*

Proof. Assume $p \in P_{<\alpha}$, $\eta < \alpha$ is a cardinal and $q \leq p$. We want to construct $r \leq q$ such that $u_\eta(r) \leq u_\eta(p)$. We define r by induction on $i < \alpha$. For $i < \eta$, let $r(i) = q(i)$. If $i \geq \eta$, $r \restriction i$ is defined, $r \restriction i \leq q \restriction i$, $u_\eta(r \restriction i) \leq u_\eta(p \restriction i)$ and $\dot{G}_{<\alpha}$ is the canonical name for the $P_{<\alpha}$-generic, let

$$r(i) = \begin{cases} q(i), & \text{if } r \restriction i \in \dot{G}_{<\alpha}. \\ p(i), & \text{otherwise.} \end{cases}$$

Then $r \restriction i \Vdash r(i) = q(i) \leq q(i)$. Let A be a maximal antichain below $u_\eta(r \restriction i)$ that refines $r \restriction i$, i.e. for every $a \in A$ either $a \leq r \restriction i$ or $a \perp r \restriction i$. If $a \leq r \restriction i$, then $a \Vdash r(i) = q(i) \leq p(i)$. If $a \perp r \restriction i$, then $a \Vdash r(i) = p(i) \leq p(i)$. Hence $u_\eta(r \restriction i) \Vdash r(i) \leq p(i)$. This handles successor stages of our induction. The limit stages are immediate. □

Definition 3.4. If $\eta < \alpha$ is a cardinal and $p \in P_{<\alpha}$, we define $l_\eta(p)$ as follows:

$$l_\eta(p)(\gamma) = \begin{cases} \check{\mathbf{1}}, & \text{if } \gamma \geq \eta^+. \\ \check{\mathbf{1}}, & \text{if } \gamma = \eta. \\ p(\gamma), & \text{otherwise.} \end{cases}$$

We call $l_\eta(p)$ the η-sized or the lower part of p. Note that $l_\eta(p)$ complements $u_\eta(p)$ in the sense that it carries exactly all information about p not contained in $u_\eta(p)$.

Definition 3.5 (stable below η^+)**.** Assume $\langle p^i \mid i < \delta\rangle$ is a decreasing sequence of conditions in $P_{<\alpha}$ of limit length $\delta < \eta^+$ where $\eta < \alpha$ a cardinal. We say that $\langle p^i \mid i < \delta\rangle$ is stable below η^+ iff

- $\langle l_\eta(p^i) \mid i < \delta\rangle$ is eventually constant or
- η is singular and for every cardinal $\mu < \eta$, $\langle l_\mu(p^i) \mid i < \delta\rangle$ is eventually constant.

The following definition will be the key ingredient for Claim 3 below. The basic idea is that if p and q are conditions in $P_{<\alpha}$, for q to be strategically below p means that q is (in a sense that is detailed below) *sufficiently strong* w.r.t. p.

Definition 3.6. Assume $\alpha' \leq \alpha$ and θ is inaccessible. If $p \in P_{<\alpha}$, $q \in P_{<\alpha'}$ and $q \leq p \restriction \alpha'$, we say that q is *strategically below p at θ* if $\text{C-supp}(p) \cap [\theta, \theta^+) = \emptyset$, if $\theta \geq \alpha'$ or all of the following hold, letting $\theta^* = \min(\{\alpha', \theta^+\})$:

[13] We will usually use $\check{\mathbf{1}}$ to denote the canonical name for the weakest condition of a poset in a forcing extension, where both the poset and the forcing extension should always be clear from context.

(i) for all $\gamma \in \text{C-supp}(p) \cap [\theta, \theta^*)$, $q \restriction \gamma$ forces that $\dot{a}^\theta_\xi$ has a $P_{<\sup(\text{supp}(q)\cap\theta)}$-name, where ξ is such that $\gamma = \theta + 2 + \xi$,
(ii) for all $\gamma \in \text{C-supp}(p) \cap [\theta, \theta^*)$, $q \restriction \gamma$ forces that $\max q^{**}_\gamma > \sup(\text{supp}(p) \cap \theta)$ and that $\sup(\text{supp}(q) \cap \theta) > \max p^{**}_\gamma$,
(iii) $\sup(\text{supp}(q) \cap \theta)$ is larger than or equal to some element of $C_{\text{C-supp}(p)\cap[\theta,\theta^+)}$ above $\sup(\text{supp}(p) \cap \theta)$ and
(iv) $\sup(\text{supp}(q) \cap \theta)$ is greater than the cardinality of $\text{C-supp}(p) \cap [\theta, \theta^+)$.
(v) Let $\nu := \text{card} \sup(\text{supp}(p) \cap \theta)$. Then $\sup(\text{supp}(q) \cap \theta) \geq \nu^{+\nu\cdot\nu}$.

If $\eta < \alpha' \leq \alpha$, η is a cardinal and $q \leq p \restriction \alpha'$, we say that q is η^+-strategically below p if for every inaccessible $\theta > \eta$, q is strategically below p at θ. It is immediate that if $\eta_0 < \eta_1$ are both cardinals and q is ${\eta_0}^+$-strategically below p then q is ${\eta_1}^+$-strategically below p.

The common case will be when $\alpha' = \alpha$ in the above. If $p \in P_{<\alpha}$, $q \in P_{<\alpha'}$, $\alpha' < \alpha$ and q is η^+-strategically below p, then q is η^+-strategically below $p \restriction \alpha'$. The reverse direction of this implication will usually not hold, as in general (iii) and (iv) get weaker as α gets smaller. A couple more trivial facts that will be useful later on are the following.

Claim 2.
- *If $\alpha < \alpha^*$, $p, q \in P_{<\alpha^*}$ and q is η^+-strategically below p, then $q \restriction \alpha$ is η^+-strategically below $p \restriction \alpha$.*
- *For $p, q, r \in P_{<\alpha}$ and a cardinal $\eta < \alpha$, if q is η^+-strategically below p and $r \leq q$, then r is η^+-strategically below p.*
- *For $p, q, r \in P_{<\alpha}$ and a cardinal $\eta < \alpha$, if $q \leq p$ and r is η^+-strategically below q, then r is η^+-strategically below p.*

Proof. Straightforward from Definition 3.6. □

Notation. Given a decreasing sequence of conditions $\langle p^i \mid i < \delta \rangle$ in $P_{<\alpha}$ of limit length δ, we say that a sequence $r = \langle r(\gamma) \mid \gamma < \alpha \rangle$ *is the componentwise union of* $\langle p^i \mid i < \delta \rangle$ if the following statements hold.

- If $\gamma < \alpha$ with $\gamma \in [\theta + 2, \theta \cdot \theta)$ for some inaccessible θ, then $r(\gamma)$ is the canonical $P_{<\gamma}$-name for the set $\bigcup_{i<\delta} (p^i)^{**}_\gamma$; we denote $r(\gamma)$ by r^{**}_γ.
- Otherwise, $r(\gamma)$ is the canonical $P_{<\gamma}$-name for $\inf\{p^i(\gamma) \mid i < \delta\}$.

The sequence r is usually not a condition in $P_{<\alpha}$ as the r^{**}_γ are not necessarily names for closed sets, but the supports of r can be calculated as if r were a condition by letting

$$\text{supp}(r) = \{\gamma \mid r(\gamma) \neq \check{\mathbf{1}}\} = \bigcup_{i<\delta} \text{supp}(p^i)$$

and

$$\text{C-supp}(r) = \{\gamma \mid r^{**}_\gamma \neq \check{\mathbf{1}}\} = \bigcup_{i<\delta} \text{C-supp}(p^i).$$

If $\gamma \in \Gamma$, $\dot{x}$ is a $P_{<\gamma}$-name for either 0 or 1 and $p \in P_{<\alpha}$ for some $\alpha > \gamma$, we write $p \restriction \gamma \Vdash p(\gamma) = \dot{x}$ to abbreviate the following: there is a maximal antichain A below $p \restriction \gamma$ which decides $\dot{x}$ such that if $a \in A$ and $a \Vdash \dot{x} = 0$ then $a \Vdash p(\gamma) = 0$, and if $a \in A$ and $a \Vdash \dot{x} = 1$ then there exist $P_{<\gamma}$-names $\dot{s}$ and $\dot{A}$ such that $a \Vdash p(\gamma) = \langle 1, \dot{A}, \dot{s} \rangle$.

Definition 3.7 (Strategic lower bound)**.** Given a cardinal $\eta < \alpha$ and a sequence $\langle p^i \mid i < \delta \rangle$ of conditions in $P_{<\alpha}$ of limit length $\delta < \eta^+$ which is stable below η^+, form their componentwise union r, which exists by our assumptions. $\operatorname{supp}(r)$ is bounded below every regular cardinal and $\text{C-supp}(r) \cap \theta^+$ has size less than θ for every inaccessible θ. We would like to obtain a condition $q \in P_{<\alpha}$ with the following properties for every $\gamma \in \text{C-supp}(r)$, $\gamma \geq \eta^+$, $\gamma = \operatorname{card}\gamma + 2 + \xi$:

(1) $q \restriction \gamma \Vdash q(b_{otp\, f_\gamma[\sup r^{**}_\gamma]}) = \dot{a}^{\operatorname{card}\gamma}_\xi$ and
(2) $q \restriction \gamma \Vdash q^{**}_\gamma := r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}$.

Components of q other than the above should be equal to the respective components of r. If such q exists, we call q the η^+-strategic lower bound for $\langle p^i \mid i < \delta\rangle$. Whenever we want to apply the above, we will be in a situation where each $\sup r^{**}_\gamma$ will have been decided (by any lower bound of $\langle p^i \restriction \gamma \mid i < \delta\rangle$) to equal an actual ordinal value (and is not just a name for an ordinal). Note that (1) implies that $q \restriction \gamma$ forces that $c(otp\, f_\gamma[\sup r^{**}_\gamma]) = \dot{a}^{\operatorname{card}\gamma}_\xi$. It is immediate from the definitions that if our desired q exists as a condition in $P_{<\alpha}$, then q is a lower bound for $\langle p^i \mid i < \delta\rangle$, i.e. $q \leq p^i$ for each $i < \delta$.

Claim 3. *If $\eta < \alpha$ is a cardinal and $\langle p^i \mid i < \delta\rangle$ is a decreasing sequence of conditions in $P_{<\alpha}$ of limit length $\delta < \eta^+$ which is stable below η^+ such that p^{i+1} is η^+-strategically below p^i for every $i < \delta$, then the sequence of conditions $\langle p^i \mid i < \delta\rangle$ has an η^+-strategic lower bound.*

Proof. By induction on $\alpha \geq \eta^+$. If $\alpha = \eta^+$, the claim follows by stability of $\langle p^i \mid i < \delta\rangle$ below η^+. Given that the claim holds below α, we want to show that there exists an η^+-strategic lower bound q^α for $\langle p^i \mid i < \delta\rangle$. Inductively, for $\gamma < \alpha$, let q^γ be the η^+-strategic lower bound of $\langle p^i \restriction \gamma \mid i < \delta\rangle$. We will also use that if $\gamma_0 < \gamma_1 < \alpha$, then $q^{\gamma_1} \restriction \gamma_0 \leq q^{\gamma_0}$. Thus we also have to show that if $\gamma < \alpha$, then $q^\alpha \restriction \gamma \leq q^\gamma$. Let r be the componentwise union of $\langle p^i \mid i < \delta\rangle$. We first show that the sequence $\langle p^i \mid i < \delta\rangle$ has the property that for every inaccessible $\theta \in (\eta, \alpha)$, either $\text{C-supp}(p^i) \cap [\theta, \theta^+) = \emptyset$ for all $i < \delta$ or the following hold:

(i) $\sup(\operatorname{supp}(r) \cap \theta) > \sup(\operatorname{supp}(p^i) \cap \theta)$ for all $i < \delta$,
(ii) for $\gamma \in \text{C-supp}(r) \cap [\theta, \theta^+)$, $q^\gamma \Vdash \sup r^{**}_\gamma = \sup(\operatorname{supp}(r) \cap \theta)$ and
(iii) $f_\gamma[\sup(\operatorname{supp}(r) \cap \theta)] \supseteq \sup(\operatorname{supp}(r) \cap \theta)$,
(iv) for $\gamma_0 < \gamma_1$ both in $\text{C-supp}(r) \cap [\theta, \theta^+)$, $f_{\gamma_0}[\sup(\operatorname{supp}(r) \cap \theta)]$ is a proper initial segment of $f_{\gamma_1}[\sup(\operatorname{supp}(r) \cap \theta)]$,
(v) for $\gamma \in \text{C-supp}(r) \cap [\theta, \theta^+)$, q^γ forces that $\dot{a}^\theta_\xi$ has a $P_{<\sup(\operatorname{supp}(r)\cap\theta)}$-name, where ξ is such that $\theta + 2 + \xi = \gamma$,
(vi) $\sup(\operatorname{supp}(r) \cap \theta) \geq \operatorname{card}(\text{C-supp}(r) \cap [\theta, \theta^+))$ and
(vii) there is no inaccessible $\lambda < \theta$ such that $\sup(\operatorname{supp}(r) \cap \theta) \in [\lambda, \lambda \cdot \lambda)$.

Property (i) immediately follows from property (v) in Definition 3.6. Property (ii) follows from Property (ii) in Definition 3.6, using that q^γ is stronger than $p^i \restriction \gamma$ for every $i < \delta$. Property (iii) follows from Property (v) in Definition 3.6 as the latter implies that $\sup(\operatorname{supp}(r) \cap \theta)$ is a cardinal. Property (iv) follows as Property (iii) in Definition 3.6 implies that $\sup(\operatorname{supp}(r) \cap \theta)$ belongs to $C_{\text{C-supp}(r)\cap[\theta,\theta^+)}$. Property (v) follows from Property (i) in Definition 3.6, Property (vi) follows from Property (iv) in Definition 3.6. We are thus left with proving Property (vii). Note that $\sup(\operatorname{supp}(r) \cap \theta)$ cannot be equal to λ for some inaccessible $\lambda < \theta$ as Property (v) in Definition 3.6 implies that $\sup(\operatorname{supp}(r) \cap \theta)$ has cofinality $\leq \eta$, but stability of $\langle p^i \mid i < \delta\rangle$ implies that $\sup(\operatorname{supp}(r) \cap \theta) \geq \eta$ and equality can only occur if η

is singular. Moreover Property (v) in Definition 3.6 implies that it cannot be in $(\lambda, \lambda \cdot \lambda)$.

Now we show, using (i)-(vii), that we can form the η^+-strategic lower bound q^α of $\langle p^i \mid i < \delta \rangle$. This is trivial (using induction) if $\operatorname{card} \alpha$ is not inaccessible. Note that if we can form $q^\alpha \in P_{<\alpha}$ out of r as in definition 3.7, then $q^\alpha \upharpoonright \gamma \leq q^\gamma$ for every $\gamma < \alpha$. Assume $\theta \in (\eta, \alpha)$ is inaccessible and $\gamma \in [\theta + 2, \theta \cdot \theta)$. Given (i)-(iv), q^γ decides $\sup r^{**}_\gamma$ and $otp\, f_\gamma[\sup r^{**}_\gamma] \geq \sup(\operatorname{supp}(r) \cap \theta)$ is distinct from $otp\, f_\xi[\sup r^{**}_\xi]$ for every $\xi < \gamma$. By (v), if ξ is such that $\theta + 2 + \xi = \gamma$, q^γ forces that $\dot{a}^\theta_\xi$ has a $P_{<\sup(\operatorname{supp}(r)\cap\theta)}$-name, allowing us to satisfy (1) as in definition 3.7, as $b_{otp\, f_\gamma[\sup(\operatorname{supp}(r)\cap\theta)]} \geq \sup(\operatorname{supp}(r) \cap \theta)$ and as by (vii) and (iii), there cannot be some inaccessible $\lambda < \theta$ with $otp\, f_\gamma[\sup(\ suppr \cap \theta)] \in [\lambda, \lambda \cdot \lambda)$. (2) in Definition 3.7 can obviously be satisfied. Finally (vi) implies that $\operatorname{supp}(q^\alpha) \setminus \operatorname{supp}(r)$ (and hence $\operatorname{supp}(q^\alpha)$) is bounded below every regular cardinal and hence q^α actually is a condition in $P_{<\alpha}$. □

Claim 4. *Assume η is a cardinal, $\alpha > \eta$ is a limit ordinal, p and q are conditions in $P_{<\alpha}$, $\langle \alpha_j \mid j < \operatorname{cof}(\alpha) \rangle$ is cofinal in α and increasing with $\alpha_0 > \eta$ s.t. for every $j < \operatorname{cof}(\alpha)$, $q \upharpoonright \alpha_j$ is η^+-strategically below p. Then q is η^+-strategically below p.*

Proof. Immediate from Definition 3.6. □

Claim 5. *Assume $\eta < \alpha$ is a cardinal, α is a limit ordinal, $\langle p^i \mid i < \delta \rangle$ is a decreasing sequence of conditions of limit length $\delta < \eta^+$ in $P_{<\alpha}$ which is stable below η^+, $\langle \alpha_j \mid j < \operatorname{cof}(\alpha) \rangle$ is cofinal in α and increasing such that $\alpha_0 > \eta$ and:*

- *$\forall i < \delta$ there exists $j < \operatorname{cof}(\alpha)$ such that $p^{i+1} \upharpoonright \alpha_j$ is η^+-strategically below p^i and $p^{i+1}[\alpha_j, \alpha) = p^i[\alpha_j, \alpha)$.*
- *$\forall\, j < \operatorname{cof}(\alpha)$ there are unboundedly many $i < \delta$ for which there exists $k \geq j$ such that $p^{i+1} \upharpoonright \alpha_k$ is η^+-strategically below p^i.*

Then the η^+-strategic lower bound for $\langle p^i \mid i < \delta \rangle$ exists.

Proof. By Claim 3, we know that for every $j < \operatorname{cof}(\alpha)$, the η^+-strategic lower bound for $\langle p^i \upharpoonright \alpha_j \mid i < \delta \rangle$ exists and denote it by q^j. It is easily observed that the componentwise union of the q^j is a condition in $P_{<\alpha}$[14] and is the η^+-strategic lower bound for $\langle p^i \mid i < \delta \rangle$. □

Definition 3.8. If D is a dense subset of $P_{<\alpha}$ and $\eta < \alpha$ is a cardinal, we say that q reduces D below η if for every $r \in P_{<\alpha}$ with $u_\eta(r) \leq u_\eta(q)$, there is $s \leq r$ with $u_\eta(s) = u_\eta(r)$ and such that s meets D in the sense that $\exists d \in D\ s \leq d$.

Definition 3.9. If P is a notion of forcing and $D \subseteq P$, we say that D is an *equivalent dense subset* of P if for every $p \in P$, there is $d \in D$ so that $d \leq p$ and $p \leq d$, i.e. p and d are *equivalent*.

Definition 3.10. Let $\gamma(\alpha)$ denote the supremum of the cardinals $\gamma < \alpha$ such that $\dot{P}_\gamma$ does not denote the trivial forcing, i.e. assuming that (as we will show later on inductively) initial segments of our iteration preserve cofinalities, inaccessibles and the continuum function,

$$\gamma(\alpha) = \sup(\{\theta < \alpha \mid \theta \in \Gamma \text{ or } \theta \text{ is inaccessible}\}).$$

[14]As for every $\gamma < \alpha$, $\langle (q^j)^{**}_\gamma \mid j < \operatorname{cof}(\alpha) \rangle$ is eventually constant.

Let $S : \mathrm{On} \longrightarrow \mathrm{Card}$ be defined as follows:

$$S(\alpha) = \begin{cases} 1 & \text{if } \alpha \leq \min(\Gamma), \\ \alpha & \text{if } \alpha \text{ is inaccessible}, \\ \gamma(\alpha)^+ & \text{if } \gamma(\alpha) \text{ is a singular limit point of } \Gamma, \\ 2^{(\gamma(\alpha)^+)} & \text{otherwise.} \end{cases}$$

We will show later that for every α, $P_{<\alpha}$ has a dense subset of size $S(\alpha)$.

Claim 6. *$\forall \beta \in \Gamma\ S(\beta) \leq \beta$.*

Proof. If $\beta \in \Gamma$, $\beta = \lambda^+$ for some $\lambda \in B_s$. If λ is a (singular) limit point of Γ, $S(\beta) = S(\lambda) = \beta$. If λ is not a limit point of Γ, then $\gamma(\beta) < \lambda$ and therefore $S(\beta) \leq 2^{\gamma(\beta)^+} < \lambda < \beta$. □

Claim 7. *Suppose $\omega \leq \eta < \alpha$, $\kappa = \operatorname{card} \alpha$ and η is a cardinal such that either $S(\eta) \leq \eta$ or η is a limit point of $\{\nu \mid S(\nu) \leq \nu\}$.*

1. *[Strategic Successors, Strategic Closure]*
   If $\alpha^* \geq \alpha$, $p \in P_{<\alpha^*}$ and $q \leq p \restriction \alpha$, then there is $r \leq q$ which is η^+-strategically below p such that $l_\eta(r) = l_\eta(q)$. Consequently, $u_\eta(P_{<\alpha})$ is η^+-strategically closed.
2. *[Smallness of the iteration]*
   $P_{<\alpha}$ has a dense subset $E_{<\alpha}$ of size $S(\alpha)$.
3. *[Early Club Information]*
   If $p \in P_{<\alpha}$, then there is $q \leq p$ so that $l_\eta(q) = l_\eta(p)$ and $q \restriction i$ forces that q_i^{**} has a $P_{<\operatorname{card} i}$-name whenever $i \in \mathrm{C}\text{-}\mathrm{supp}(q)$, $i \geq \eta^+$. If q is such, we say that q has early club information above η.
4. *[Chain Condition]*
   Assume η is regular. If J is an antichain of $P_{<\alpha}$ such that $u_\eta(p) \parallel u_\eta(q)$ whenever p and q are in J, then $|J| \leq \eta$.
5. *[Reducing dense sets]*
   - *Assume η is regular and $\langle D_i \mid i < \eta \rangle$ is a collection of dense subsets of $P_{<\alpha}$. Then any condition in $P_{<\alpha}$ can be strengthened to a condition q with the same η-sized part so that for every $i < \eta$, q reduces D_i below η.*
   - *Assume $\eta \leq \alpha$ is singular and $\langle D_i \mid i < \eta \rangle$ is a collection of dense subsets of $P_{<\alpha}$. Then for any cardinal $\zeta < \eta$, any condition in $P_{<\alpha}$ can be strengthened to a condition q with the same ζ-sized part so that for every $i < \eta$ there exists $\eta_i < \eta$ so that q reduces D_i below η_i.*
6. *[Early names]*
   - *Assume η is regular and $\dot{f}$ is a $P_{<\alpha}$-name for an ordinal-valued function with domain η. Then any condition in $P_{<\alpha}$ can be strengthened to a condition q with the same η-sized part forcing that for every $i < \eta$, there is a maximal antichain of size at most η below q deciding $\dot{f}(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. In particular, q forces that $\dot{f}$ has a P_γ-name for some $\gamma < \eta^+$.*
   - *Let $\eta \leq \alpha$ be a singular cardinal. Let $\dot{f}$ be a $P_{<\alpha}$-name for an ordinal-valued function with domain η. Then for any cardinal $\zeta < \eta$, any condition in $P_{<\alpha}$ can be strengthened to a condition q with the same ζ-sized part, forcing that for every $i < \eta$, there is a maximal antichain of size less than η below q deciding $\dot{f}(i)$, where for every element a*

of that antichain, $u_\eta(a) = u_\eta(q)$. In particular, q forces that $\dot f$ has a P_η-name.

7. *[Factoring] If κ is inaccessible, $\alpha \in (\kappa+2, \kappa\cdot\kappa]$ and $\alpha = \kappa + 2 + \delta$, then there is an embedding $\pi\colon P_{<\kappa+2} * S^\kappa_{<\delta} \to P_{<\alpha}$ such that $\pi'' P_{<\kappa+2} * R^\kappa_{<\delta}$ is dense in $P_{<\alpha}$, with $S^\kappa_{<\delta}$ and $R^\kappa_{<\delta}$ as defined at the beginning of this section. Hence $P_{<\alpha} \cong P_{<\kappa+2} * S^\kappa_{<\delta} \cong P_{<\kappa+2} * R^\kappa_{<\delta}$.*
8. *[Covering, Preservation of Cofinalities]*
   For every cardinal θ, for every $p \in P_{<\alpha}$ and every $P_{<\alpha}$-name $\dot x$ for a set of ordinals of size θ there is a set X in V of size θ and an extension q of p such that $q \Vdash \dot x \subseteq X$. Therefore forcing with $P_{<\alpha}$ preserves all cofinalities.
9. *[Preservation of the continuum function]*
   Forcing with $P_{<\alpha}$ preserves the continuum function and thus in particular all inaccessibles.
10. *[Club Extendibility]*
    If $I \subseteq \alpha$ is s.t. $\operatorname{card}(I \cap \theta^+) < \theta$ for every regular θ, $I \subseteq \bigcup\{[\theta+2, \theta\cdot\theta) \mid \theta \text{ inaccessible}\}$ and $\langle \bar\delta^i \mid i \in I\rangle$ is such that $\bar\delta^i < \operatorname{card} i$ for every $i \in I$, then for every $p \in P_{<\alpha}$, there is $q \le p$ such that $\forall i \in I\ q \upharpoonright i \Vdash \max q_i^{**} \ge \bar\delta^i$. Moreover if $\eta < \operatorname{card} \min I$ is regular, there is such q with $l_\eta(q) = l_\eta(p)$.

Proof. We will proceed by induction on α.

Proof of 1. This is trivial if $\eta = \kappa$. Thus assume $\eta < \kappa$. We distinguish several cases for α. Note that (iii), (iv) and (v) in Definition 3.6 can easily be satisfied by choosing r such that $\sup(\operatorname{supp}(r) \cap \theta)$ is sufficiently large whenever $\mathrm{C}\text{-}\operatorname{supp}(p) \cap [\theta, \theta^+) \ne \emptyset$ and $\theta \in (\eta, \alpha)$ is inaccessible. We will thus ignore (iii), (iv) and (v) in the following and concentrate only on (i) and (ii).

Case 1: $\alpha = \beta + 1$ is a successor ordinal

This is trivial by 1 inductively if κ is not inaccessible or $\alpha \le \kappa + 2$. We thus assume that κ is inaccessible and $\beta \ge \kappa + 2$. Assume first that η is regular. Let ξ be such that $\beta = \kappa + 2 + \xi$. Using 6 inductively, we may strengthen q to q^* so that $(q^*)^{**}_\beta = q^{**}_\beta$ and $q^* \upharpoonright \beta$ forces that $\dot a^\kappa_\xi$ and $\sup q^{**}_\beta$ have $P_{<\sup(\operatorname{supp}(q^*)\cap\kappa)}$-names while $l_\eta(q^*) = l_\eta(q)$. Now use 1 inductively to find $r \le q^*$ such that $r \upharpoonright \beta$ is η^+-strategically below p while $l_\eta(r) = l_\eta(q^*)$. Choose a cardinal $\delta < \kappa$ such that

- $\delta > \eta$, $\delta > \sup(\operatorname{supp}(p) \cap \kappa)$, $q^* \upharpoonright \beta$ forces $\delta > \sup q^{**}_\beta$ and
- $otp\, f_\beta[\delta] > \sup(\operatorname{supp}(r) \cap \kappa)$.

The former is possible as by our (inductive) use of 6, there is a maximal antichain of size at most $\eta < \kappa$ of conditions below $q^* \upharpoonright \beta$ deciding $\sup q^{**}_\beta$. Let r^{**}_β be such that $r \upharpoonright \beta \Vdash r^{**}_\beta = q^{**}_\beta \cup \{\delta\}$ and set $r(b_{otp\, f_\beta[\delta]})$ such that $r \upharpoonright \beta \Vdash r(b_{otp\, f_\beta[\delta]}) = \dot a^\kappa_\xi$. Then $r \le q$ is η^+-strategically below p and $l_\eta(r) = l_\eta(q)$, as desired. The case when η is singular is similar.

Case 2: α is a limit ordinal, $\operatorname{cof}(\alpha) = \kappa$

Let $\bar\alpha = \sup(\operatorname{supp}(q) \cap \alpha) < \alpha$. Now use 1 inductively to find $r \le q$ such that $r \upharpoonright \bar\alpha$ is η^+-strategically below p and $r[\bar\alpha, \alpha) = q[\bar\alpha, \alpha)$. Then $r \le q$ is η^+-strategically below p, as desired. The additional lower parts agreement requirements are easy to satisfy using induction.

Case 3: α is a limit ordinal, $\operatorname{cof}(\alpha) < \kappa$

Let $\eta^* = \max\{\eta, \mathrm{cof}(\alpha)\}$. Let $\langle \alpha_i \mid i < \mathrm{cof}(\alpha)\rangle$ be an increasing sequence cofinal in α with $\alpha_0 > \eta^*$. We build a decreasing sequence of conditions $\langle q^i \mid i \leq \mathrm{cof}(\alpha)\rangle$ as follows.

- Let q^0 be so that $q^0 \restriction \alpha_0 \leq q \restriction \alpha_0$ is η^+-strategically below p and $q^0[\alpha_0, \alpha) = q[\alpha_0, \alpha)$.
- Given q^i, let q^{i+1} be so that $q^{i+1} \restriction \alpha_i$ is $(\eta^*)^+$-strategically below q^i and $q^{i+1}[\alpha_i, \alpha) = q^i[\alpha_i, \alpha)$.
- If $\delta \leq \mathrm{cof}(\alpha)$ is a limit ordinal, let q^δ be the $(\eta^*)^+$-strategic lower bound of $\langle q^i \mid i < \delta\rangle$, which exists by Claim 5.

$q^{\mathrm{cof}(\alpha)} \leq q$ is $(\eta^*)^+$-strategically below p by Claim 5, hence by our assumption on q^0 above, $q^{\mathrm{cof}(\alpha)}$ is η^+-strategically below p, as desired. The additional lower parts agreement requirements are easy to satisfy using induction. $\square$

Proof of 2. We will show that $D_{<\alpha} = \{p \in P_{<\alpha} \mid \forall \theta \in \mathrm{Card}\ \mathrm{supp}(p) \cap (\theta, \theta^+) \neq \emptyset \to p \restriction \theta \Vdash p(\theta) \neq \check{\mathbf{1}}\}$ has an equivalent dense subset $E_{<\alpha}$ of size $S(\alpha)$. Note that $D_{<\alpha}$ itself is dense in $P_{<\alpha}$ and that for different α, the $D_{<\alpha}$ cohere. The claim is trivial if $\alpha \leq \min(\Gamma)$, as $P_{<\alpha}$ denotes the trivial iteration of length α in this case.

Assume first that $\alpha = \beta + 1$ and $D_{<\beta}$ has an equivalent dense subset $E_{<\beta}$ of size $S(\beta)$ by 2 inductively. The only nontrivial cases are when $\dot{P}_\beta$ does not denote the trivial forcing, i.e. when either $\beta \in \Gamma$, $\beta = \gamma + 1$ for some $\gamma \in \Gamma$ or $\mathrm{card}\,\beta = \kappa$ is inaccessible and therefore $\gamma(\alpha) = \kappa$.

If $\beta \in \Gamma$, conditions in Q_β can canonically be identified with subsets of β^+, hence a $P_{<\beta}$-name for a condition in $\dot{P}_\beta$ can be identified with a collection of β^+-many antichains of $E_{<\beta}$, which is of size $\leq \beta$ and hence this gives rise to an equivalent dense subset $E_{<\alpha}$ of $D_{<\alpha}$ of size $2^{\beta^+} = S(\alpha)$.

If $\beta = \gamma + 1$ for some $\gamma \in \Gamma$, conditions in Q_β can canonically be identified with subsets of γ^+, hence if $p \in D_{<\alpha}$, we may assume that $p \restriction \beta \in E_{<\beta}$ and $p(\beta)$ can be identified with a collection of γ^+-many antichains of $E_{<\beta}$ below $p \restriction \beta$. Since for any two elements a_0, a_1 of such an antichain $u_\gamma(a_0) = u_\gamma(a_1)$, such an antichain will have size at most γ by 4 inductively, giving rise to an equivalent dense subset $E_{<\alpha}$ of $D_{<\alpha}$ of size $2^{\gamma^+} = S(\alpha)$.

If $\mathrm{card}\,\beta = \kappa$ is inaccessible, then $S(\beta) \leq 2^{\kappa^+}$ and conditions in Q_β can canonically be identified with subsets of κ^+, hence if $p \in D_{<\alpha}$, we may assume that $p \restriction \beta \in E_{<\beta}$ and $p(\beta)$ can be identified with a collection of κ^+-many antichains of $E_{<\beta}$ below $p \restriction \beta$. Since for any two elements a_0, a_1 of such an antichain $u_\kappa(a_0) = u_\kappa(a_1)$, such an antichain will have size at most κ by 4 inductively, giving rise to an equivalent dense subset $E_{<\alpha}$ of $D_{<\alpha}$ of size $2^{\kappa^+} = S(\alpha)$.

Now we consider the case that α is a limit ordinal. Since the $E_{<\beta}$ cohere for $\beta < \alpha$, if α is a limit point of Γ and $p \in D_{<\alpha}$, we may assume that for every $\beta < \alpha$ we have $p \restriction \beta \in E_{<\beta}$. We will thus either obtain an equivalent dense subset $E_{<\alpha}$ of $D_{<\alpha}$ of size α for α inaccessible by the boundedness of the supports or of size α^+ for α singular. If α is singular but not a limit point of Γ or α is regular but not inaccessible, we have that $\gamma(\alpha) < \alpha$ and the result is therefore immediate inductively. $\square$

Proof of 3. This proof is similar to the proof of 1: Let $p \in P_{<\alpha}$ be given. We may assume that κ is inaccessible and $\alpha > \kappa + 2$ as 3 is immediate by 3 inductively otherwise. We distinguish several cases for α.

Case 1: $\alpha = \beta + 1$ is a successor ordinal

Strengthen p to p^* so that $p^* \restriction \beta$ forces that p^{**}_β has a $P_{<\kappa}$-name and so that $l_\eta(p^*) = l_\eta(p)$, using 6 inductively, and let $(p^*)^{**}_\beta = p^{**}_\beta$. Now use 3 inductively to find $q \leq p^*$ with $l_\eta(q) = l_\eta(p^*)$ such that $q \restriction \beta$ has early club information above η and $q^{**}_\beta = (p^*)^{**}_\beta$. Then q has early club information above η, as desired.

Case 2: α is a limit ordinal, $\mathrm{cof}(\alpha) = \kappa$

Let $\bar\alpha = \sup(\mathrm{supp}(p) \cap \alpha) < \alpha$. Now use 3 inductively to find $q \leq p$ such that $q \restriction \bar\alpha$ has early club information above η, $l_\eta(q) = l_\eta(p)$ and $q[\bar\alpha, \alpha) = p[\bar\alpha, \alpha)$. Then q has early club information above η, as desired.

Case 3: α is a limit ordinal, $\mathrm{cof}(\alpha) < \kappa$

Let $\eta^* = \max(\{\eta, \mathrm{cof}(\alpha)\})$. Let $\langle \alpha_i \mid i < \mathrm{cof}(\alpha)\rangle$ be an increasing sequence cofinal in α with $\alpha_0 > \eta^*$. We build a decreasing sequence of conditions $\langle p^i \mid i \leq \mathrm{cof}(\alpha)\rangle$ as follows.

- Let $p^0 \leq p$ be so that $p^0 \restriction \alpha_0$ has early club information above η and $l_\eta(p^0) = l_\eta(p)$.
- Given p^i, let p^{i+1} be so that p^{i+1} is $(\eta^*)^+$-strategically below p^i, $p^{i+1} \restriction \alpha_i$ has early club information above η^* and $l_{\eta^*}(p^{i+1}) = l_{\eta^*}(p^i)$.
- If $\delta \leq \mathrm{cof}(\alpha)$ is a limit ordinal, let p^δ be the $(\eta^*)^+$-strategic lower bound of $\langle p^i \mid i < \delta\rangle$, which exists by Claim 5.

$q := p^{\mathrm{cof}(\alpha)} \leq p$ has early club information above η, as desired. □

Proof of 4. We may assume that η is inaccessible or $\eta \in \Gamma$, as otherwise 4 is immediate from our assumption that $S(\eta) \leq \eta$ and the fact that $\mathbb{P}$ is trivial in $[\eta, \eta^+)$. Assume J is an antichain of $P_{<\alpha}$ such that whenever p and q are in J, $u_\eta(p) \parallel u_\eta(q)$. We may assume that all conditions in J are from $E_{<\alpha}$ and have early club information. Assume for a contradiction that J has size at least η^+. As $E_{<\eta}$ has size $\leq \eta$, $p \restriction \eta$ is the same for η^+-many conditions in J and thus we may assume it is the same for all conditions in J. As $\eta^{<\eta} = \eta$ in V by the SCH at singular fixed points of the $\beth$-function, we can apply a Δ-system argument and obtain that there is $W \subseteq J$ of size η^+ and a subset A of η^+ of size less than η such that $\mathrm{C\text{-}supp}(p) \cap \mathrm{C\text{-}supp}(q) \cap [\eta, \eta^+) = A$ whenever $p \neq q$ are both in W. Using $\eta^{<\eta} = \eta$ in the $P_{<\eta}$-generic extension by 9 inductively, it follows that for η^+-many conditions p in W, $\langle p^{**}_i \mid i \in A\rangle$ is the same (modulo equivalence). Using the assumption that $u_\eta(p) \parallel u_\eta(q)$ it follows that W induces an antichain of $\mathbb{P}_{s_\kappa}(A_\kappa)$ of size κ^+, contradicting Proposition 2.2. □

Proof of 5. We will first show the following:

Claim 8. *Assume $p \in P_{<\alpha}$, D is a dense subset of $P_{<\alpha}$ and $\nu < \alpha$ is regular with $S(\nu) \leq \nu$. Then there is $q \leq p$ such that $l_\nu(q) = l_\nu(p)$ and q reduces D below ν.*

Proof. Build a decreasing sequence of conditions in $P_{<\alpha}$ below p as follows: Let $p^0 = p$. Choose q^0 so that $q^0 \leq p^0$ and $q^0 \in D$. By possibly passing to an equivalent condition, we may also ensure that $u_\nu(q^0) \leq u_\nu(p^0)$. At stage $j + 1$, let $p^{j+1} \leq p^0$ be any condition incompatible with all q^k, $k \leq j$, such that $u_\nu(p^{j+1}) = u_\nu(q^j)$ if such exists and choose q^{j+1} such that:

- $q^{j+1} \leq p^{j+1}$,
- $q^{j+1} \in D$ and
- $u_\nu(q^{j+1})$ is chosen according to the strategy for ν^+-strategic closure below $\langle u_\nu(q^k) \mid k \leq j\rangle$.

At limit stages $j < \nu^+$, let $p^j \leq p^0$ be a condition which is incompatible with all q^k, $k < j$ so that for all $k < j$, $u_\nu(p^j) \leq u_\nu(q^k)$ if such exists. Note that a p^j satisfying the latter condition can always be found by the strategic choice of the $u_\nu(q^k)$. Choose $q^j \leq p^j$ so that $q^j \in D$ and $u_\nu(q^j) \leq u_\nu(p^j)$. Proceed until at some stage j no condition p^j as above can be chosen. By 4, this will be the case for some $j < \nu^+$. We can then find $q \in P_{<\alpha}$ so that $u_\nu(q) \leq u_\nu(q^k)$ for every $k < j$ and $l_\nu(q) = l_\nu(p)$. By our construction, q reduces D below ν. □

Using 1 and the claim for $\nu = \eta$, the case of regular η follows immediately. For the case of $\eta \leq \alpha$ singular, choose a continuous, cofinal in η, increasing sequence $\langle \eta_i \mid i < \mathrm{cof}(\eta)\rangle$ of cardinals where η_0 and each η_{i+1} is regular with $S(\eta_{i+1}) \leq \eta_{i+1}$, $S(\eta_0) \leq \eta_0$ and $\eta_0 \geq \zeta, \mathrm{cof}(\eta)$. This is possible by our requirement that η is a limit point of $\{\nu \mid S(\nu) \leq \nu\}$. Build a sequence of conditions $\langle q^i \mid i < \mathrm{cof}(\eta)\rangle$ so that $q^{i+1} = q^i$ for limit ordinals i and otherwise q^{i+1} reduces the first η_i-many given dense sets below η_i, $l_{\eta_i}(q^{i+1}) = l_{\eta_i}(q^i)$ and $u_{\eta_i}(q^{i+1})$ is chosen according to the strategy for $(\eta_i)^+$-strategic closure of $u_{\eta_i}(P_{<\alpha})$ for each $i < \mathrm{cof}(\eta)$. At limit stages $i \leq \mathrm{cof}(\eta)$, we may take lower bounds of the conditions obtained so far using stability of the obtained sequence of conditions below η_i and Claim reflowerbound. □

Proof of 6. Apply 5 to reduce the dense sets D_i of conditions which decide $\dot f(i)$, $i < \eta$. □

Proof of 7. Using 8 for $\kappa+2$ inductively, we let π be the canonical embedding from $P_{<\kappa+2} * S^\kappa_{<\delta}$ to $P_{<\alpha}$ - given $(p,t) \in P_{<\kappa+2} * S^\kappa_{<\delta}$, let $q \in P_{<\alpha}$ be so that $q \restriction (\kappa+2) = p$ and for $\xi < \delta$ such that $\gamma = \kappa+2+\xi$, $q(\gamma)$ is a $P_{<\gamma}$-name for $t(\xi)$.

To show that $\pi''(P_{<\kappa+2} * R^\kappa_{<\delta})$ is dense in $P_{<\alpha}$, given $p \in P_{<\alpha}$ let $q \leq p$ be so that for all $\gamma > \kappa$ in C-supp(q), $q \restriction \gamma$ forces that q^{**}_γ has a $P_{<\kappa}$-name. This is possible by 3. We can now find $s \in P_{<\kappa+2} * R^\kappa_{<\delta}$ such that $\pi(s)$ is equivalent to q. □

Proof of 8. Assume θ is a cardinal, $p \in P_{<\alpha}$ and $\dot x$ is a $P_{<\alpha}$-name for a set of ordinals of size θ. If $S(\theta) \leq \theta$, we may use 6 to reduce $\dot x$ below θ and obtain q as desired. Otherwise, let $\nu \geq \theta$ be least such that $S(\nu) \leq \nu$. If $\nu < \alpha$, we may use 6 to find $p' \leq p$ wich reduces $\dot x$ below η. We let $p' = p$ otherwise. Note that our iteration is trivial on $[\theta, \eta]$ by the case assumption. But this means that p' forces that $\dot x$ has a $P_{<\gamma(\theta)^+}$-name. If $\gamma(\theta) \in \Gamma$, we can find $p'' \leq p$ and a $P_{<\gamma(\theta)}$-name $\dot y$ for a set of ordinals of size θ such that $p'' \Vdash \dot x \subseteq \dot y$ using that $\dot P_{\gamma(\theta)+1}$ is forced to be $<\gamma(\theta)$-closed and $\gamma(\theta)^+$-cc and $\dot P_{\gamma(\theta)}$ does not add any new sets. The desired result then follows using 8 inductively for $P_{<\gamma(\theta)}$.

Now assume $\gamma(\theta)$ is inaccessible. For every $\delta \leq \gamma(\theta)\cdot\gamma(\theta)$, $P_\delta \cong P_{<\gamma(\theta)+2} * R_{\bar\delta}$ with $\delta = \gamma(\theta)+2+\bar\delta$ by 7. $R_{\bar\delta}$ has size $\gamma(\theta) \leq \theta$ in any $P_{<\gamma(\theta)+2}$-generic extension and therefore has the desired covering property for $\dot x$. As $\dot P_{\gamma(\theta)+1}$ is forced to be $<\gamma(\theta)$-closed and $\gamma(\theta)^+$-cc and $\dot P_{\gamma(\theta)}$ does not add any new sets, the desired result follows using 8 inductively for $P_{<\gamma(\theta)}$ as above.

The remaining case is when $\gamma(\theta) \in \mathrm{Lim}(\Gamma)$ is singular. Note that in this case $S(\theta) = S(\gamma(\theta)) = \gamma(\theta)^+$. Therefore the only relevant case is when $\theta = \gamma(\theta) \in \mathrm{Lim}(\Gamma)$. But we may now use the singular case of 6 to reduce $\dot{x}$ below θ and obtain q as desired. $\square$

Proof of 9. Let θ be any infinite cardinal and let $\theta^* = \min(\{\theta^+, \alpha\})$. All subsets of θ which are added by $P_{<\alpha}$ are in fact added by $P_{<\theta^*}$ by 6. If $S(\theta^*) \leq \theta$, our desired result follows immediately. Otherwise, let $\nu = \gamma(\theta^*) \leq \theta$. If $\nu \in \Gamma$, our desired result follows as $\dot{P}_{\nu+1}$ is forced to be ν^+-cc, $\dot{P}_\nu$ does not add any new sets and thus both preserve the value of 2^θ and $P_{<\nu}$ preserves the value of 2^θ as $S(\nu) \leq \nu$. If ν is inaccessible, our desired result follows using 7 as $R^\nu_{<\nu\cdot\nu}$ has size ν and thus preserves the value of 2^θ, $\dot{P}_{\nu+1}$ is forced to be ν^+-cc, $\dot{P}_\nu$ does not add any new sets and thus both preserve the value of 2^θ and $P_{<\nu}$ preserves the value of 2^θ as $S(\nu) \leq \nu$. Finally if $\nu \in \mathrm{Lim}(\Gamma)$ is singular, then $S(\theta^+) = S(\nu) = \nu^+$ and thus the only relevant case is $\nu = \theta$. But then any subset of θ added by $P_{<\alpha}$ is in fact added by $P_{<\theta}$ as $P_{<\alpha}$ is trivial in $[\theta, \theta^+)$. The singular case of 6 shows that every $P_{<\theta}$-name for a subset of θ can densely be reduced below θ and the claim follows as there are only θ^+-many inequivalent such reduced names by an easy cardinalities argument using 3. $\square$

Proof of 10. Given $p \in P_{<\alpha}$, $I \subseteq \alpha$ and $\langle \bar{\delta}^i \mid i \in I \rangle$ as in the statement of the claim, let $p' \leq p$ be such that for every θ with $I \cap [\theta, \theta^+) \neq \emptyset$, we have that $\sup(\mathrm{supp}(p') \cap \theta) \geq \sup(\{\bar{\delta}^i \mid i \in I \cap [\theta, \theta^+)\})$. Let $q \leq p'$ be η^+-strategically below p' (or ω_1-strategically below p' if no $\eta < \operatorname{card} \min I$ is specified). It follows that q is as desired. If $\eta < \operatorname{card} \min I$ is regular, we may easily ensure that $l_\eta(q) = l_\eta(p)$ in the above. $\square$

This completes the proof of Claim 7. $\square$

Note: For every $i \in [\kappa + 2, \kappa \cdot \kappa)$ with κ inaccessible, $\bigcup_{p \in \mathsf{G}} p_i^{**}$ is club in κ for any $\mathbb{P}$-generic G. This is immediate from Clause 10 of Claim 7.

Corollary 3.11. *$\mathbb{P}$ preserves* ZFC*, cofinalities and the value of 2^κ for every κ.*

Proof. For every cardinal κ and $P_{<\kappa}$-generic G, $\mathbb{P}/G$ is $<\kappa$-distributive by Claim 7, Clause 6 and the observations that if $\nu = \min\{\theta \geq \kappa \mid S(\theta) \leq \theta\} > \kappa$, then $P[\kappa, \kappa^+)$ is $<\kappa$-closed (or even trivial) and $\mathbb{P}$ is trivial in the interval $[\kappa^+, \nu)$. By [7, Lemma 2.31], this implies that $\mathbb{P}$ is tame below κ. Thus $\mathbb{P}$ is tame, which implies that $\mathbb{P}$ preserves ZFC (see [7, pp. 32]). Preservation of cofinalities is immediate from Claim 7, Clauses 8 and 6. Preservation of the continuum function is immediate from Claim 7, Clauses 9 and 6. $\square$

Claim 9. *Assume $\alpha > \gamma$ are ordinals, let $G_{<\gamma}$ be generic for $P_{<\gamma}$ over* V *and let $P[\gamma, \alpha)$ denote the iteration $\mathbb{P}$ from stage γ to stage α, as defined in* $\mathrm{V}[G_{<\gamma}]$*. Let $P[\gamma, \alpha)$ denote the canonical $P_{<\gamma}$-name for this iteration. Then*

$$P_{<\alpha} \;\cong\; P_{<\gamma} * P[\gamma, \alpha).$$

Proof. The proof is a standard argument using the covering properties of $P_{<\gamma}$ provided by Clause 8 of Claim 7. $\square$

Corollary 3.12. *Suppose $\omega \leq \eta < \eta^+ < \alpha, \kappa = \operatorname{card} \alpha$ and η is a cardinal. If η is not inaccessible, then $P[\eta, \alpha)$ is $<\eta$-strategically closed in every $P_{<\eta}$-generic* *extension. If moreover $\eta \notin \Gamma$, then $P[\eta,\alpha)$ is $<\eta^+$-strategically closed in every $P_{<\eta}$-generic extension.*

Proof. The second statement follows from the first one since if η is not inaccessible and not in Γ, then $P[\eta,\eta^+)$ is trivial. Work in a $P_{<\eta}$-generic extension. The partial order $P[\eta,\eta^+)$ is either trivial or $<\eta$-closed by Proposition 2.2. It is seen as in the proof of Claim 9 that

$$P[\eta,\alpha) \;\cong\; P[\eta,\eta^+) * P[\eta^+,\alpha)$$

and it thus suffices to show that $P[\eta^+,\alpha)$ is $<\eta$-strategically closed, but this follows directly from Clause 1 of Claim 7. □

Claim 10. *If κ is inaccessible, G is $P_{<\kappa^+}$-generic, $K_{<\gamma}$ is the induced $S^\kappa_{<\gamma}$-generic and $a^\kappa_\gamma = (\dot{a}^\kappa_\gamma)^{G_{<\kappa+2}*K_{<\gamma}}$ for every $\gamma < \kappa\cdot\kappa$, then the sequence $\langle a^\kappa_\gamma \mid \gamma < \kappa\cdot\kappa\rangle$ is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.*

Proof. Let c denote the class function defined at the beginning of this section. Then Clause 6 of Claim 7 implies $c^{\mathrm{V}[G]}\restriction\kappa = c^{\mathrm{V}[G_{<\kappa+2}]}\restriction\kappa : \kappa \longrightarrow 2$ and this function is Δ_0-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Since the sequence $\langle f_\gamma : \kappa \longrightarrow \gamma \mid \gamma \in [\kappa,\kappa\cdot\kappa]\rangle$ is Δ_0-definable in parameter κ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and the equivalence

$$a^\kappa_\gamma = 1 \iff \{\delta<\kappa \mid c(otp\, f_{\kappa+2+\gamma}[\delta]) = 1\} \textit{ contains a club}$$

holds in $\mathrm{V}[G]$, we can conclude the statement of the claim. □

Claim 11. *Assume κ is inaccessible and G is $P_{<\kappa^+}$-generic. Let $\rho : R^\kappa_{<\kappa\cdot\kappa} \longrightarrow \mathrm{H}(\kappa)^{\mathrm{V}[G_{<\kappa+2}]}$ denote the injection in $\mathrm{V}[G_{<\kappa+2}]$ constructed at the beginning of this section. Then the set $\mathcal{R}_\kappa = \rho'' R^\kappa_{<\kappa\cdot\kappa}$ is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.*

Proof. In $\mathrm{V}[G]$, a function $w : \kappa\cdot\kappa \longrightarrow [\kappa]^{<\kappa}$ in $\mathrm{V}[G_{<\kappa+2}]$ witnesses that some condition in $S^\kappa_{<\kappa\cdot\kappa}$ is an element of $R^\kappa_{<\kappa\cdot\kappa}$ if the following statements hold.

(i) $\mathrm{card}\,(\{\gamma<\kappa\cdot\kappa \mid w(\gamma)\neq\emptyset\}) < \kappa$.
(ii) $w(\gamma)$ is a closed, bounded subset of κ for all $\gamma < \kappa\cdot\kappa$.
(iii) $c(otp\, f_{\kappa+2+\gamma}[\eta]) = a^\kappa_\gamma$ for all $\gamma<\kappa\cdot\kappa$ and $\eta \in w(\gamma)$.

Clause 6 of Claim 7 shows that all bounded subsets of κ added by forcing with $\mathbb{P}$ already appear after forcing with $P_{<\kappa}$. This implies that $\mathcal{R}_\kappa$ is the set of all partial functions $q : \kappa \xrightarrow{part} [\kappa]^{<\kappa}$ in $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ such that $q(\alpha)$ is a closed, bounded subset of κ and $c(otp\, f_{\kappa+2+\gamma}[\eta]) = a^\kappa_\gamma$ for all $\alpha\in\mathrm{dom}(q)$, $\eta\in q(\alpha)$ and $\gamma = f_{\kappa\cdot\kappa}(\alpha)$. By Claim 10 and its proof, $\mathcal{R}_\kappa$ is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. □

Claim 12. *If κ is inaccessible, G is $P_{<\kappa^+}$-generic over V and $<^\kappa$ is the corresponding well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ in $\mathrm{V}[G]$, then the sets $({}^\kappa\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ and $<^\kappa$ are Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ over $\mathrm{H}(\kappa^+)^{V[G]}$.*

Proof. We will use the notation from Section 2. Let $\langle A_\kappa, s_\kappa\rangle$ be the κ-coding basis chosen by $G_{<\kappa+1}$ and let $\bar{G}$ be the filter in $\mathbb{P}_{s_\kappa}(A_\kappa)$ induced by G. By the definition of the a^κ_δ's and Claim 10, $\dot{x}_\kappa^{G_{<\kappa+2}}$ is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Let $\dot{F}_\kappa$ and $\dot{T}_\kappa$ denote the $\mathbb{P}_{s_\kappa}(A_\kappa)$-names in $\mathrm{V}[G_{<\kappa+1}]$ constructed in Section 2. In this situation, Corollary 2.4 shows that the equality $[\dot{T}_\kappa^{\bar{G}}]^{\mathrm{V}[G]} = [\dot{T}_\kappa^{\bar{G}}]^{\mathrm{V}[G_{<\kappa+2}]}$ would imply the statement of the claim.

Therefore, let us asssume for a contradiction that there is a condition $p_0 \in G$ and a $P_{<\kappa^+}$-name $\dot{z}$ with

$$p_0 \ \Vdash \ \dot{z} \in [\dot{T}_\kappa] \ \wedge \ \dot{z} \notin \operatorname{ran}(\dot{F}_\kappa).$$

We combine this assumption with Clauses 1 and 6 of Claim 7 and the fact that κ is a regular cardinal in every $P_{<\kappa^+}$-generic extension to construct

- a descending sequence $\langle p_n \mid n<\omega\rangle$ of conditions in $P_{<\kappa^+}$,
- strictly increasing sequences $\langle \alpha_n \mid n<\omega\rangle$ and $\langle \beta_n \mid n<\omega\rangle$ of ordinals smaller than κ, and
- a sequence $\langle \dot{s}_n \mid n<\omega\rangle$ of $P_{<\kappa}$-names

such that the following statements hold for all $n<\omega$.

(i) p_{n+1} is ω_1-strategically below p_n,
(ii) $\alpha_{n+1} > \beta_n$,
(iii) $p_{n+1} \restriction (\kappa+1) \Vdash \operatorname{ht}(T_{p_n(\kappa+1)}) = \check{\alpha}_n$,
(iv) $p_{n+1} \Vdash \dot{z} \restriction \check{\alpha}_n = \dot{s}_n$, and
(v) $p_{n+1} \Vdash \forall x \in \operatorname{dom}(g_{p_n(\kappa+1)}) \ \dot{F}_\kappa(x) \restriction \check{\beta}_n \neq \dot{z} \restriction \check{\beta}_n$.

By Claim 3, there is a condition q in $P_{<\kappa^+}$ with $q \leq p_n$ for all $n<\omega$ and

$$q \restriction (\kappa+1) \ \Vdash \ q(\kappa+1) = \inf_{\dot{P}_{\kappa+1}} \{p_n(\kappa+1) \mid n<\omega\}.$$

Let H be $P_{<\kappa+1}$-generic over V with $q \restriction (\kappa+1) \in H$, $\alpha = \sup_{n<\omega} \alpha_n \in \kappa \cap \mathrm{Lim}$ and $s = \bigcup\{\dot{s}_n^H \mid n<\omega\}$. Our construction ensures that $\operatorname{ht}(T_{q(\kappa+1)^H}) = \alpha$ and $s \neq g_{q(\kappa+1)^H}(x)$ for all $x \in \operatorname{dom}(g_{q(\kappa+1)^H})$. This allows us to construct a condition $r_{\kappa+1}$ in $\dot{P}^H_{\kappa+1}$ with $\operatorname{ht}(T_{r_{\kappa+1}}) = \alpha+1$, $r_{\kappa+1} \leq q(\kappa+1)$ and $s \notin T_{r_{\kappa+1}}$. If r is the condition in $P[\kappa{+}1,\kappa^+)^{\mathrm{V}[H]}$ with $r(\kappa{+}1) = r_{\kappa+1}$ and $r \restriction [\kappa{+}2,\kappa^+) = q \restriction [\kappa{+}2,\kappa^+)$, then the above construction ensures

$$r \restriction [\kappa+1,\kappa^+) \ \Vdash \ \dot{z} \in [\dot{T}_\kappa] \ \wedge \ \dot{z} \restriction \check{\alpha} = \check{s} \ \wedge \ \check{s} \notin \dot{T}_\kappa,$$

a contradiction. □

Claim 13. *If κ is inaccessible, G is $P_{<\kappa^+}$-generic over* V*, $\langle A_\kappa, s_\kappa\rangle$ is the κ-coding basis chosen by $G_{<\kappa+1}$ and $\bar{G}$ is the filter in $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ induced by G, then the sets $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ and $\bar{G}$ are Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ over $\mathrm{H}(\kappa^+)^{V[G]}$.*

Proof. $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G]} = \mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ as $\mathrm{V}[G]$ and $\mathrm{V}[G_{<\kappa+1}]$ contain the same bounded subsets of κ by Clause 6 of Claim 7. The proof of Claim 12 shows that A_κ is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ over $\mathrm{H}(\kappa^+)^{V[G]}$ and we can conclude that $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G]}$ is definable in the same way. Next, if $\dot{T}_\kappa$ denotes the $\mathbb{P}_{s_\kappa}(A_\kappa)$-name in $\mathrm{V}[G_{<\kappa+1}]$ defined in Section 2, then Claim 10 and the proof of Claim 12 show that the sets $\dot{T}_\kappa^{\bar{G}}$ and s_κ are also definable in this way and $[\dot{T}_\kappa^{\bar{G}}]^{\mathrm{V}[G]} = [\dot{T}_\kappa^{\bar{G}}]^{\mathrm{V}[G_{<\kappa+2}]}$ holds. In this situation, the computations of [13, Proof of Proposition 6.3] show that $\bar{G}$ is also definable in the desired way. □

Claim 14. *If κ is inaccessible, then every $\mathbb{P}$-generic extension of the ground model contains a well-ordering of $\mathrm{H}(\kappa^+)$ that is Δ_2-definable in parameter κ over $\mathrm{H}(\kappa^+)$.*

Proof. Let G be $\mathbb{P}$-generic over V, G be the filter in $P_{<\kappa^+}$ induced by G, $\langle A_\kappa, s_\kappa\rangle$ be the κ-coding basis chosen by $G_{<\kappa+1}$, $\bar{G}$ be the filter in $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ induced by G and $<^\kappa$ be the corresponding well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ in $\mathrm{V}[G_{<\kappa+2}]$. By the distributivity of the tails of $\mathbb{P}$, we have $\mathrm{H}(\kappa^+)^{\mathrm{V}[\mathbb{G}]} \subseteq \mathrm{V}[G]$.

By the remarks at the beginning of this section and Claim 12, there is a surjection $\sigma : ({}^\kappa 2)^{\mathrm{V}[G_{<\kappa+1}]} \longrightarrow \mathrm{H}(\kappa^+)^{\mathrm{V}[G_{<\kappa+1}]}$ that is contained in $\mathrm{V}[G_{<\kappa+1}]$ and Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Together with Claim 12, this shows that $\mathrm{H}(\kappa^+)^{\mathrm{V}[G_{<\kappa+1}]}$ is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. We define

$$e_0 : ({}^\kappa 2)^{\mathrm{V}[G_{<\kappa+1}]} \longrightarrow \mathcal{P}(\kappa)^{\mathrm{V}[G_{<\kappa+2}]}$$

to be the map with

$$e_0(x) \ = \ \{\alpha < \kappa \mid \exists y \in {}^\kappa\kappa \ \exists \beta < \kappa \ [\sigma(y) \in \bar{G} \wedge \forall \gamma < \kappa \ y(\gamma) = x(\prec\alpha, \prec\beta, \gamma\succ\succ)]\}.$$

Claim 12 and Claim 13 show that e_0 is Σ_1-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Since $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$ satisfies the κ^+-chain condition in $\mathrm{V}[G_{<\kappa+1}]$ and every element of $\mathcal{P}(\kappa)^{\mathrm{V}[G_{<\kappa+2}]}$ is represented by a $\mathbb{P}_{s_\kappa}(A_\kappa)^{\mathrm{V}[G_{<\kappa+1}]}$-nice name of cardinality at most κ, we have $\mathrm{ran}(e_0) = \mathcal{P}(\kappa)^{\mathrm{V}[G_{<\kappa+2}]}$ and this set is definable in the above way.

Let $<_\kappa$ be the canonical well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G_{<\kappa+1}]}$ induced by $<^\kappa$, ξ be the order-type of the restriction of $<_\kappa$ to $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$, f be the $<_\kappa$-least bijection from κ to ξ, $\overline{<_\kappa}$ be the corresponding well-ordering of $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ of order-type K be the filter in $S^\kappa_{<\kappa\cdot\kappa}$ induced by G. We define

$$e_1 : \mathcal{P}(\kappa)^{\mathrm{V}[G_{<\kappa+2}]} \longrightarrow \mathcal{P}(\kappa)^{\mathrm{V}[G]}$$

by setting

$$e_1(x) \ = \ \{\alpha < \kappa \mid \exists \beta < \kappa \ (\prec\alpha, \beta\succ \in x \wedge \overline{<_\kappa}(\beta) \in \rho''(K \cap R^\kappa_{<\kappa\cdot\kappa}))\}.$$

Our construction ensures

$$\overline{<_\kappa}(\beta) \in \rho''(K \cap R^\kappa_{<\kappa\cdot\kappa}) \iff a^\kappa_{\kappa\cdot\delta+1+\beta} = 1 \textit{ for some } \delta < \kappa.$$

Now Claim 10 shows that e_1 is Σ_1-definable in parameters $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ and $\overline{<_\kappa}$ over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. By Clause 7 of Claim 7, every element in $\mathcal{P}(\kappa)^{\mathrm{V}[G]}$ has an $R^\kappa_{<\kappa\cdot\kappa}$-nice name in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G_{<\kappa+2}]}$ and this implies that e_1 is surjective.

By using the surjection σ constructed at the beginning of this section, we can find a surjection $\tau : \mathcal{P}(\kappa)^{\mathrm{V}[G]} \longrightarrow \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is Σ_1-definable in parameter κ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Given $x, y \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, we define $x <^*_\kappa y$ to hold if

$$\begin{aligned} \exists \bar{x}, \bar{y} \in ({}^\kappa 2)^{\mathrm{V}[G_{<\kappa+1}]} \ \big[& \bar{x} <^\kappa \bar{y} \ \wedge \ (\tau \circ e_1 \circ e_0)(\bar{x}) = x \ \wedge \ (\tau \circ e_1 \circ e_0)(\bar{y}) = y \\ & \wedge \ \forall \tilde{x} \in ({}^\kappa 2)^{\mathrm{V}[G_{<\kappa+1}]} (\tilde{x} <^\kappa \bar{x} \longrightarrow (\tau \circ e_1 \circ e_0)(\tilde{x}) \neq x) \\ & \wedge \ \forall \tilde{y} \in ({}^\kappa 2)^{\mathrm{V}[G_{<\kappa+1}]} (\tilde{y} <^\kappa \bar{y} \longrightarrow (\tau \circ e_1 \circ e_0)(\tilde{y}) \neq y) \big]. \end{aligned}$$

Then $<^*_\kappa$ is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is Σ_2-definable (and hence also Π_2-definable) in parameters $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ and $\overline{<_\kappa}$ over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. By its definition and Claim 12, the well-ordering $<_\kappa$ of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G_{<\kappa+1}]}$ is Δ_2-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. This implies that the ordering $<_\kappa \restriction (\mathrm{H}(\kappa)^{\mathrm{V}[G]} \times \mathrm{H}(\kappa)^{\mathrm{V}[G]})$ is Δ_2-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. We can conclude that the ordinal ξ, the function f and the ordering $\overline{<_\kappa}$ are all Σ_2-definable in parameter $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Since $\mathrm{H}(\kappa)^{\mathrm{V}[G]}$ is Π_1-definable in parameter κ in this model, the above shows that $<^*_\kappa$ is Δ_2-definable in parameter κ in $\mathrm{V}[G]$. □

Claim 15. *If κ is inaccessible, then every $\mathbb{P}$-generic extension of the ground model contains a well-ordering of $\mathrm{H}(\kappa^+)$ that is lightface Δ_3-definable over $\mathrm{H}(\kappa^+)$.*

Proof. This follows directly from Claim 14 and the fact that the ordinal κ is lightface Π_2-definable in $\mathrm{H}(\kappa^+)$. $\square$

This completes the proof of Clauses (i)-(iii) of Theorem 1.2. $\square$

All that remains to prove Theorem 1.2 is to show that $\mathbb{P}$ preserves supercompact cardinals. We will do this in the next section.

4. Supercompactness preservation

This section will be devoted to the proof of the remaining clause of our main theorem.

Proof of Clause (iv) of Theorem 1.2. It will suffices to prove the following claim that directly implies the last part of Theorem 1.2.

Claim 16. *If γ is a singular strong limit cardinal but not a fixed point of the $\beth$-function and κ is γ-supercompact such that $\gamma^{<\kappa} = \gamma$, then forcing with $\mathbb{P}$ preserves the γ-supercompactness of κ.*

In the following, fix κ and γ as in the above claim, let

$$\nu \ = \ \sup\{\alpha < \gamma \mid \alpha \textit{ is inaccessible or } \alpha \in \Gamma\} \ < \ \gamma$$

and let $j : \mathrm{V} \longrightarrow \mathrm{M}$ denote an elementary embedding witnessing the γ-supercompactness of κ. We may assume that $\mathrm{M} = \mathrm{Ult}(\mathrm{V}, U)$, where U is a normal ultrafilter on $\mathcal{P}_\kappa(\gamma)$. Note that we have $2^\gamma = \gamma^+$ by Theorem 1.9 and our assumptions imply that $S(\gamma) \leq 2^{(\nu^+)} < \gamma$.

Claim 17. *If $\alpha \leq \gamma$, then $P^{\mathrm{M}}_{<\alpha} = P_{<\alpha} \subseteq {}^\alpha\mathrm{V}_\gamma$.*

Proof. Since γ is a strong limit cardinal, the definition of the forcing iteration directly implies $P_{<\alpha} \subseteq \mathrm{V}_\gamma$ for all $\alpha < \gamma$ and hence $P_{<\gamma} \subseteq {}^\gamma\mathrm{V}_\gamma$. The clauses defining $P_{<\gamma}$ are absolute between transitive ZFC-models with the same γ-sequences and we can conclude $P_{<\alpha} = P^{\mathrm{M}}_{<\alpha}$ holds for all $\alpha \leq \gamma$. $\square$

Claim 18. *If $G_{<\gamma}$ is $P_{<\gamma}$-generic over* V*, then $G_{<\gamma}$ is $P^{\mathrm{M}}_{<\gamma}$-generic over* M *and $({}^\gamma\mathrm{M}[G_{<\gamma}])^{\mathrm{V}[G_{<\gamma}]} \subseteq \mathrm{M}[G_{<\gamma}]$.*

Proof. Let $x \in \mathcal{P}(\gamma)^{\mathrm{V}[G_{<\gamma}]}$. By Clause 2 of Claim 7, $P_{<\gamma}$ has a dense subset of cardinality $S(\gamma) < \gamma$ and therefore satisfies the γ-chain condition. Let

$$\dot{x} \ = \ \bigcup_{\alpha<\gamma} A_\alpha \ \times \ \{\check{\alpha}\},$$

be a $P_{<\gamma}$-nice name with $x = \dot{x}^{G_{<\gamma}}$. Then the sequence $\langle A_\alpha \mid \alpha < \gamma\rangle$ is an element of M, $\dot{x} \in \mathrm{M}$ and $x = \dot{x}^{G_{<\gamma}} \in \mathrm{M}[G_{<\gamma}]$.

Next, let $X \in \mathrm{V}[G_{<\gamma}]$ be a set of ordinals with $(\mathrm{card}\, X)^{\mathrm{V}[G_{<\gamma}]} \leq \gamma$. Clause 8 of Claim 7 shows that there is $\bar{X}$ in V with $(\mathrm{card}\, \bar{X})^{\mathrm{V}} = \gamma$ and $X \subseteq \bar{X}$. Then $\bar{X} \in \mathrm{M}$ and $(\mathrm{card}\, \bar{X})^{\mathrm{M}} = \gamma$. Let $\langle a_\alpha \mid \alpha < \gamma\rangle$ be an enumeration of $\bar{X}$ in M. By the above computations, the set $\{\alpha < \gamma \mid a_\alpha \in X\}$ is an element of $\mathrm{M}[G_{<\gamma}]$ and this shows that $X \in \mathrm{M}[G_{<\gamma}]$. We can conclude $({}^\gamma\mathrm{On})^{\mathrm{V}[G_{<\gamma}]} \subseteq \mathrm{M}[G_{<\gamma}]$ and this implies the statement of the claim. $\square$

Claim 19. *If $G_{<\gamma}$ is $P_{<\gamma}$-generic over* V*, then there is a set $\mathcal{D} \in \mathrm{M}[G_{<\gamma}]$ such that the following statements hold.*

(i) *If* $D \in \mathcal{D}$, *then* D *is a dense subset of* $P[\gamma, j(\nu^+))^{\mathrm{M}[G_{<\gamma}]}$.
(ii) *Every* $\mathcal{D}$*-generic filter for* $P[\gamma, j(\nu^+))^{\mathrm{M}[G_{<\gamma}]}$ *is* $P[\gamma, j(\nu^+))^{\mathrm{M}[G_{<\gamma}]}$*-generic over* $\mathrm{M}[G_{<\gamma}]$.
(iii) *The set* $\mathcal{D}$ *has cardinality at most* γ^+ *in* $\mathrm{V}[G_{<\gamma}]$.

Proof. Clause 2 of Claim 7 implies that $P^{\mathrm{M}}_{<j(\nu^+)}$ contains a dense subset E of cardinality less than $j(\gamma)$ in M. Clauses 8 and 9 of the same claim show that $j(\gamma)$ is still a strong limit cardinal in $\mathrm{M}[G_{<\gamma}]$. In $\mathrm{M}[G_{<\gamma}]$, we define

$$E[\gamma, j(\nu^+)) \;=\; \{q \in P[\gamma, j(\nu^+))^{\mathrm{M}[G_{<\gamma}]} \mid \exists p \in G_{<\gamma}\; p \cup q \in E\}.$$

Then $E[\gamma, j(\nu^+))$ is a dense subset of $P[\gamma, j(\nu^+))^{\mathrm{M}[G_{<\gamma}]}$ of cardinality less than $j(\gamma)$ in $\mathrm{M}[G_{<\gamma}]$. In $\mathrm{M}[G_{<\gamma}]$, let $\mathcal{D}$ denote the set of all subsets of $E[\gamma, j(\nu^+))$ that are dense in $P[\gamma, j(\nu^+))^{\mathrm{M}[G_{<\gamma}]}$. By the above remarks, $\mathcal{D}$ has cardinality less than $j(\gamma)$ in $\mathrm{M}[G_{<\gamma}]$ and it clearly satisfies the first two statements of the claim.

To complete the proof of the claim, it suffices to show that $j(\gamma)$ has cardinality at most γ^+ in V. If $\alpha < j(\gamma)$, then α is represented in $\mathrm{M} = \mathrm{Ult}(\mathrm{V}, U)$ by a function $f : \mathcal{P}_\kappa(\gamma) \longrightarrow \gamma$ in V. In V, the set $\mathcal{P}_\kappa(\gamma)$ has cardinality γ and the set of all such functions has cardinality $2^\gamma = \gamma^+$ by Theorem 1.9. □

We will need a strengthening of Claim 3 in order to handle directed sets in suitable tails of the iteration $\mathbb{P}$ instead of decreasing sequences of conditions in $\mathbb{P}$.

Notation. Work in a $P_{<\gamma}$-generic extension of the universe. Given $\alpha > \gamma$ and a directed set D of conditions in $P[\gamma, \alpha)$ of size less than γ, we say that a sequence $r = \langle r(\delta) \mid \delta \in [\gamma, \alpha)\rangle$ is the *componentwise union of* D if the following statements hold.

- If $\delta \in [\gamma, \alpha)$ with $\delta \in [\theta + 2, \theta^+)$ for some inaccessible θ, then $r(\delta)$ is the canonical $P[\gamma, \delta)$-name for the set $\bigcup_{p \in D} p^{**}_\delta$; we denote $r(\delta)$ by r^{**}_δ.
- Otherwise, $r(\delta)$ is the canonical $P_{<\delta}$-name for $\inf(\{p(\delta) \mid p \in D\})$.[15]

The sequence r is usually not a condition in $P[\gamma, \alpha)$ as the r^{**}_δ are not necessarily names for closed sets, but the supports of r can be calculated as if r were a condition by letting

$$\mathrm{supp}(r) = \{\delta \mid r(\delta) \neq \check{\mathbb{1}}\} = \bigcup_{p \in D} \mathrm{supp}(p)$$

and

$$\mathrm{C\text{-}supp}(r) = \{\delta \mid r^{**}_\delta \neq \check{\mathbb{1}}\} = \bigcup_{p \in D} \mathrm{C\text{-}supp}(p).$$

Definition 4.1 (Strategic lower bound)**.** Assume that $G_{<\nu^+}$ is $P_{<\nu^+}$-generic over V (and hence over M by Claim 17). Note that $\mathbb{P}$ is trivial in the interval $[\nu^+, \gamma)$ and therefore we can identify $G_{<\gamma}$ and $G_{<\nu^+}$. For every $\alpha \in [\gamma, j(\nu^+)]$, let

$$D_\alpha \;=\; \{j(p) \restriction [\gamma, \alpha) \mid p \in G_{<\nu^+}\}.$$

Fix some particular $\alpha \in [\gamma, j(\nu^+)]$. Let r be the componentwise union of D_α. $\mathrm{supp}(r)$ is bounded below every regular cardinal, $\mathrm{C\text{-}supp}(r) \cap \theta^+$ has size less than θ for every inaccessible θ. If for some ordinal ξ, h is the bijection from $\mathrm{card}\,\xi$ to ξ chosen by some condition or generic in M, we denote this bijection by f^{M}_ξ, in order to be able to distinguish it from its V-counterpart f_ξ.

[15]This infimum exists by Proposition 2.2.

Working in $\mathrm{M}[G_{<\gamma}] = \mathrm{M}[G_{<\nu^+}]$, we would like to obtain a condition $q \in P[\gamma,\alpha)^{\mathrm{M}}$ that satisfies the following properties for every $\delta \in \text{C-supp}(r)$ with $\delta = \operatorname{card}\delta + 2 + \xi$.

(1) If $\delta > j(\kappa)^+$, then
$$q \restriction \delta \Vdash q(b_{otp\, f_\delta^{\mathrm{M}}[\sup r_\delta^{**}]}) = \dot{a}_\xi^{\operatorname{card}\delta}.$$
(2) $q \restriction \delta \Vdash q_\delta^{**} := r_\delta^{**} \cup \{\sup r_\delta^{**}\}$.
(3) Components of q other than the above should be equal to the respective components of r.

If such q exists, we call q the *strategic lower bound of D_α*.

Claim 20. *In $\mathrm{M}[G_{<\gamma}]$, if our desired q exists as a condition in $P[\gamma,\alpha)^{\mathrm{M}}$, then q is a lower bound for D_α, i.e. $q \le p$ for every $p \in D_\alpha$.*

Proof. Work in $\mathrm{M}[G_{<\gamma}]$. The only thing to check is that if $\delta \in \text{C-supp}(r) \cap [j(\kappa), j(\kappa)^+)$ with $\delta = j(\kappa) + 2 + \xi$, then there is $p \in G_{<\gamma}$ such that
$$q \restriction \delta \ \Vdash \ p(b_{otp\, f_\delta^{\mathrm{M}}[\sup r_\delta^{**}]}) = (\dot{a}_\xi^{j(\kappa)})^{\mathrm{M}}.$$

Note that
$$\text{C-supp}(r) \cap [j(\kappa), j(\kappa)^+) = \bigcup\{j(A) \mid A \subseteq [\kappa+2, \kappa\cdot\kappa) \wedge \operatorname{card} A < \kappa\} = j''[\kappa+2, \kappa\cdot\kappa)$$
and hence such δ will be of the form $j(\bar\delta)$ for some $\bar\delta \in [\kappa+2, \kappa\cdot\kappa)$ and $f_\delta^{\mathrm{M}} = j(f_{\bar\delta})$. Similarly, $(\dot{a}_\xi^{j(\kappa)})^{\mathrm{M}} = j(\dot{a}_{\bar\xi}^{\kappa})$ where $\xi = j(\bar\xi)$ and $\bar\delta = \kappa+2+\bar\xi$. Note that $\sup r_\delta^{**} = \kappa$, and hence
$$otp\, f_\delta^{\mathrm{M}}[\sup r_\delta^{**}] \ = \ otp\, j(f_{\bar\delta})[\kappa] \ = \ otp\, j'' f_{\bar\delta}[\kappa] \ = \ otp\, j''\bar\delta = \bar\delta$$
as j is order-preserving. Now the claim follows as $q(b_{\bar\delta}) = \dot{a}_{\bar\xi}^{\kappa}$ by the definition of our iteration. $\square$

Claim 21. *Assume $G_{<\nu^+}$ is $P_{<\nu^+}$-generic over V. Then for every $\alpha \in [\gamma, j(\nu^+)]$, the set $D_\alpha = \{j(p) \restriction [\gamma,\alpha) \mid p \in G_{<\nu^+}\}$ has a strategic lower bound in $\mathrm{M}[G_{<\gamma}]$.*

Proof. By induction on $\alpha \ge \gamma$ in $\mathrm{M}[G_{<\gamma}]$. If $\alpha = \gamma$, the claim is trivial. Given that the claim holds below α, we want to show that there exists a strategic lower bound q^α for D_α, which is a directed set. Inductively, for $\beta < \alpha$, let q^β be the strategic lower bound for D_β. We also assume inductively that if $\beta_0 < \beta_1 < \alpha$, then $q^{\beta_1} \restriction \beta_0 \le q^{\beta_0}$. Thus we also have to show that if $\beta < \alpha$, then $q^\alpha \restriction \beta \le q^\beta$. Let r be the componentwise union of D_α. We first show that either for every inaccessible $\theta \in (j(\kappa), \alpha)$, $\text{C-supp}(p) \cap [\theta, \theta^+) = \emptyset$ for all $p \in D_\alpha$ or the following hold:

(o) $\sup(\operatorname{supp}(r) \cap \theta) > \gamma$,
(i) $\sup(\operatorname{supp}(r) \cap \theta) > \sup(\operatorname{supp}(p) \cap \theta)$ for all $p \in D_\alpha$,
(ii) for $\delta \in \text{C-supp}(r) \cap [\theta, \theta^+)$, we have
$$q^\delta \Vdash \sup r_\delta^{**} = \sup(\operatorname{supp}(r) \cap \theta)$$
(iii) for $\delta \in \text{C-supp}(r) \cap [\theta, \theta^+)$, we have
$$f_\delta^{\mathrm{M}}[\sup(\operatorname{supp}(r) \cap \theta)] \supseteq \sup(\operatorname{supp}(r) \cap \theta)$$
(iv) for $\delta_0 < \delta_1$ both in $\text{C-supp}(r) \cap [\theta, \theta^+)$, the set $f_{\delta_0}^{\mathrm{M}}[\sup(\operatorname{supp}(r) \cap \theta)]$ is a proper initial segment of $f_{\delta_1}^{\mathrm{M}}[\sup(\operatorname{supp}(r) \cap \theta)]$,
(v) for $\delta = \theta + 2 + \xi \in \text{C-supp}(r) \cap [\theta, \theta^+)$, we have
$$q^\delta \Vdash (\dot{a}_\xi^\theta)^{\mathrm{M}} \text{ has a } P[\gamma, \sup(\operatorname{supp}(r) \cap \theta))^{\mathrm{M}[G_{<\gamma}]}\text{-name},$$

(vi) $\sup(\operatorname{supp}(r) \cap \theta) \geq \operatorname{card}(\text{C-supp}(r) \cap [\theta, \theta^+))$,

(vii) there is no inaccessible $\lambda < \theta$ such that $\sup(\operatorname{supp}(r) \cap \theta) \in [\lambda, \lambda \cdot \lambda)$.

Proof of (o)*:* Let $p \in D_\alpha$ such that $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$ and $p = j(t) \restriction [\gamma, \alpha)$ for some $t \in G_{<\nu^+}$. Define X to be the set consisting of all $s \in P_{<\nu^+}$ such that

$$\text{C-supp}(t) \cap [\eta, \eta^+) \neq \emptyset \;\longrightarrow\; \sup(\operatorname{supp}(s) \cap \eta) > \kappa$$

holds for all inaccessible $\eta > \kappa$. X is dense below t. Pick $s \in X \cap G_{<\nu^+}$. Then

$$\sup(\operatorname{supp}(j(s)) \cap \theta) \;>\; j(\kappa) \;>\; \gamma$$

and hence $\sup(\operatorname{supp}(r)) \cap \theta > \gamma$. □

Proof of (i)*:* Assume $p \in D_\alpha$. Then $p = j(t) \restriction [\gamma, \alpha)$ for some $t \in G_{<\nu^+}$. Define X to be the set consisting of all $s \in P_{<\nu^+}$ such that

$$\text{C-supp}(t) \cap [\eta, \eta^+) \neq \emptyset \;\longrightarrow\; \sup(\operatorname{supp}(s) \cap \eta) > \sup(\operatorname{supp}(t) \cap \eta)$$

holds for every inaccessible η. X is dense below t. Then

$$\sup(\operatorname{supp}(r) \cap \theta) \;\geq\; \sup(\operatorname{supp}(j(s)) \cap \theta) \;>\; \sup(\operatorname{supp}(p) \cap \theta)$$

for all $s \in X \cap G_{<\nu^+}$. □

Proof of (ii)*:* Assume $\xi < \sup(\operatorname{supp}(r) \cap \theta)$. Then we can find a $p \in D_\alpha$ such that $\xi < \sup(\operatorname{supp}(p) \cap \theta)$, $\delta \in \text{C-supp}(p)$ and $p = j(t) \restriction [\gamma, \alpha)$ for some $t \in G_{<\nu^+}$. Define X to be the set consisting of all $s \in P_{<\nu^+}$ such that

$$s \restriction \zeta \;\Vdash\; \sup(s^{**}_\zeta) \geq \sup(\operatorname{supp}(t) \cap \operatorname{card} \zeta)$$

holds for all $\zeta \in \text{C-supp}(t)$. X is dense below t. Let $s \in X \cap G_{<\nu^+}$. Then

$$q^\delta \leq j(s) \restriction [\gamma, \delta) \;\Vdash\; \sup j(s)^{**}_\delta \geq \sup(\operatorname{supp}(p) \cap \theta)$$

and hence $q^\delta \Vdash \sup r^{**}_\delta \geq \sup(\operatorname{supp}(r) \cap \theta)$. That also $q^\delta \Vdash \sup r^{**}_\delta \leq \sup(\operatorname{supp}(r) \cap \theta)$ is shown similarly. □

Proof of (iii)*:* We show that $\sup(\operatorname{supp}(r) \cap \theta)$ is a cardinal in M, which clearly implies (iii). But the former follows as for every $t \in G_{<\nu^+}$, the set consisting of all $s \in P_{<\nu^+}$ such that

$$\text{C-supp}(t) \cap [\eta, \eta^+) \neq \emptyset \;\longrightarrow\; \sup(\operatorname{supp}(s) \cap \eta) > (\sup(\operatorname{supp}(t) \cap \eta))^+$$

holds for every inaccessible η is dense below t. □

Proof of (iv)*:* Assume $\xi < \sup(\operatorname{supp}(r) \cap \theta)$. Then we can find a $p \in D_\alpha$ such that $\xi < \sup(\operatorname{supp}(p) \cap \theta)$, $\delta_0, \delta_1 \in \text{C-supp}(p)$ and $p = j(t) \restriction [\gamma, \alpha)$ for some $t \in G_{<\nu^+}$. Define X to be the set consisting of all $s \in P_{<\nu^+}$ such that

$$f_{\zeta_0}[\sup(\operatorname{supp}(t) \cap \eta)] \;\subseteq\; f_{\zeta_1}[\sup(\operatorname{supp}(s) \cap \eta)]$$

and

$$f_{\zeta_1}[\sup(\operatorname{supp}(t) \cap \eta)] \;\cap\; \zeta_0 \;\subseteq\; f_{\zeta_1}[\sup(\operatorname{supp}(s) \cap \eta)]$$

hold for every inaccessible η and all $\zeta_0 < \zeta_1$ both of cardinality η and both in $\text{C-supp}(t)$. X is dense below t. Let $s \in X \cap G_{<\nu^+}$. Then $f^{\mathrm{M}}_{\delta_0}(\xi)$ is an element of $f^{\mathrm{M}}_{\delta_1}[\sup(\operatorname{supp}(r) \cap \theta)]$ and $f^{\mathrm{M}}_{\delta_1}(\xi)$ is an element of $f^{\mathrm{M}}_{\delta_0}[\sup(\operatorname{supp}(r) \cap \theta)]$ if the former is less than δ_0. But this means that $f^{\mathrm{M}}_{\delta_0}[\sup(\operatorname{supp}(r) \cap \theta)]$ is a proper initial segment of $f^{\mathrm{M}}_{\delta_1}[\sup(\operatorname{supp}(r) \cap \theta)]$, as desired. □

Proof of (v)*:* Pick $p \in D_\alpha$ with $\delta \in \text{C-supp}(p)$ and $t \in G_{<\nu^+}$ with $p = j(t) \upharpoonright [\gamma, \alpha)$. Define X to be the set consisting of all $s \in P_{<\nu^+}$ such that

$$s \upharpoonright \zeta \ \Vdash \ \dot{a}^{\operatorname{card}\zeta}_{\bar{\xi}} \text{ has a } P_{<\sup(\operatorname{supp}(s)\cap \operatorname{card}\zeta)}\text{-name}$$

holds for all $\zeta \in \text{C-supp}(t)$ with $\zeta = \operatorname{card}\zeta + 2 + \bar{\xi}$. X is dense below t. Pick $s \in X \cap G_{<\nu^+}$. Then

$$j(s) \upharpoonright \delta \ \Vdash \ (\dot{a}^{\theta}_{\xi})^{\mathrm{M}} \text{ has a } P^{\mathrm{M}}_{<\sup(\operatorname{supp}(j(s))\cap\theta)}\text{-name},$$

hence in $M[G_{<\gamma}]$, $q^\delta \leq j(s) \upharpoonright [\gamma, \delta) \Vdash (\dot{a}^{\theta}_{\xi})^{\mathrm{M}}$ has a $P[\gamma, \sup(\operatorname{supp}(r) \cap \theta))^{\mathrm{M}}$-name, as desired. □

Proof of (vi)*:* We have

$$\operatorname{card}(\text{C-supp}(r) \cap [\theta, \theta^+)) \ = \ \operatorname{card}\big(\bigcup_{t \in G_{<\nu^+}} \text{C-supp}(j(t)) \cap [\theta, \theta^+)\big).$$

There are only $2^\nu < \gamma < j(\kappa) < \theta$-many possiblities for $\text{C-supp}(p)$ for $p \in G_{<\nu^+}$. This implies that

$$\operatorname{card}(\text{C-supp}(r) \cap [\theta, \theta^+)) \ = \ \sup(\{\operatorname{card}(\text{C-supp}(j(t)) \cap [\theta, \theta^+)) \mid t \in G_{<\nu^+}\}).$$

Pick some $t \in G_{<\nu^+}$ and define X to be the set consisting of all $s \in P_{<\nu^+}$ such that

$$\sup(\operatorname{supp}(s) \cap \eta) \ \geq \ \operatorname{card}(\text{C-supp}(t) \cap [\eta, \eta^+))$$

holds for every inaccessible η. X is dense below t. Let $s \in X \cap G_{<\nu^+}$. Then

$$\sup(\operatorname{supp}(j(s)) \cap \theta) \ \geq \ \operatorname{card}(\text{C-supp}(j(t)) \cap [\theta, \theta^+))$$

and (vi) follows by the above. □

Proof of (vii)*:* Let us say that $E \subseteq \mathrm{On}$ is Easton iff E is bounded below every regular cardinal. In M, we have

$$\sup(\operatorname{supp}(r) \cap \theta) \ = \ \sup(\bigcup\{j(E) \cap \theta \mid E \subseteq \nu \text{ Easton}\}).$$

It follows that $\sup(\operatorname{supp}(r) \cap \theta)$ has cofinality $\leq 2^\nu < \gamma$ while $\sup(\operatorname{supp}(r) \cap \theta) > \gamma$ in M as $j(\kappa) \in \operatorname{supp}(r)$ and therefore $\sup(\operatorname{supp}(r) \cap \theta)$ cannot be inaccessible.

$\sup(\operatorname{supp}(r) \cap \theta) \notin (\lambda, \lambda \cdot \lambda)$ for any inaccessible λ as for every $t \in G_{<\nu^+}$, letting $\xi_\eta = \sup(\operatorname{supp}(t) \cap \eta)$, the set

$$\{s \in P_{<\nu^+} \mid \forall \eta \text{ inaccessible } (\text{C-supp}(t) \cap [\eta, \eta^+) \neq \emptyset \rightarrow \sup(\operatorname{supp}(s) \cap \eta) > \xi_\eta^{+\xi_\eta \cdot \xi_\eta})\}$$

is dense in $P_{<\nu^+}$. □

Now we show, using (o)-(vii), that we can form the strategic lower bound q^α of D_α. This is trivial (using induction) if $\operatorname{card}\alpha$ is not inaccessible. Note that if we can form $q^\alpha \in P[\gamma, \alpha)^{\mathrm{M}}$ out of r as in Definition 4.1, then $q^\alpha \leq q^\alpha \upharpoonright \gamma \leq q^\gamma$ for every $\gamma < \alpha$. Assume $\theta \in (j(\kappa), \alpha)$ is inaccessible and $\delta \in [\theta + 2, \theta \cdot \theta)$. Given (i)-(iv), q^δ decides $\sup r^{**}_\delta$ and forces that

$$\mathit{otp}\, f^{\mathrm{M}}_\delta[\sup r^{**}_\delta] \ \geq \ \sup(\operatorname{supp}(r) \cap \theta)$$

is distinct from $\mathit{otp}\, f^{\mathrm{M}}_\xi[\sup r^{**}_\xi]$ for every $\xi < \delta$. By (v), if ξ is such that $\theta + 2 + \xi = \delta$, q^δ forces that $(\dot{a}^{\theta}_{\xi})^{\mathrm{M}}$ has a $P^{\mathrm{M}}[\gamma, \sup(\operatorname{supp}(r) \cap \theta))$-name, allowing us to satisfy (1) as in Definition 4.1, as

$$b_{\mathit{otp}\, f^{\mathrm{M}}_\delta[\sup(\operatorname{supp}(r)\cap\theta)]} \ \geq \ \sup(\operatorname{supp}(r) \cap \theta) \ > \ \gamma$$

by (o) and as by (vii) and (iii), there cannot be some inaccessible $\lambda < \theta$ with

$$otp\ f^{\mathrm{M}}_{\delta}[\sup(\operatorname{supp}(r) \cap \theta)] \in [\lambda, \lambda \cdot \lambda).$$

The statement (2) in Definition 4.1 can obviously be satisfied. Finally (vi) implies that $\operatorname{supp}(q^{\alpha}) \setminus \operatorname{supp}(r)$ (and hence $\operatorname{supp}(q^{\alpha})$) is bounded below every regular cardinal and hence q^{α} actually is a condition in $P[\gamma, \alpha)^{\mathrm{M}}$. □

Proof of Claim 16. Let $G_{<\gamma}$ be $P_{<\gamma}$-generic over V. Corollary 3.12 shows that the partial order $P[\gamma, j(\nu^{+}))^{\mathrm{M}[G_{<\gamma}]}$ is γ^{+}-strategically closed. Let q be the strategic lower bound of

$$D_{j(\nu^{+})} \ = \ \{j(p) \restriction [\gamma, j(\nu^{+})) \mid p \in G_{<\gamma}\}$$

in $\mathrm{M}[G_{<\gamma}]$ provided by Claim 21 and let $\mathcal{D}$ denote the collection of dense subsets of $P[\gamma, j(\nu^{+})]^{\mathrm{M}[G_{<\gamma}]}$ provided by Claim 19. Work in $\mathrm{V}[G_{<\gamma}]$ and fix an enumeration $\langle D_{\alpha} \mid 1 \leq \alpha < \gamma^{+}\rangle$ of $\mathcal{D}$. By the γ^{+}-strategic closure of $P[\gamma, j(\nu^{+}))^{\mathrm{M}[G_{<\gamma}]}$, we can construct a decreasing sequence $\langle p^{\alpha} \mid \alpha < \gamma^{+}\rangle$ of conditions in $P[\gamma, j(\nu^{+}))^{\mathrm{M}[G_{<\gamma}]}$ such that $p^{0} = q$ and $\alpha > 0$ implies that $p^{\alpha} \leq d_{\alpha}$ for some $d_{\alpha} \in D_{\alpha}$. Define

$$H \ = \ \{p \in P[\gamma, j(\nu^{+}))^{\mathrm{M}[G_{<\gamma}]} \mid \exists \alpha < \gamma^{+}\ p^{\alpha} \leq p\} \ \in \ \mathrm{V}[G_{<\gamma}].$$

By Claim 19, H is $P[\gamma, j(\nu^{+}))^{\mathrm{M}[G_{<\gamma}]}$-generic over $\mathrm{M}[G_{<\gamma}]$. Since the partial order $P[j(\nu^{+}), j(\gamma))$ is trivial in every $P_{<j(\nu^{+})}$-generic extension of M, $G * H$ induces a filter $H_{<j(\gamma)}$ in $P^{\mathrm{M}}_{<j(\gamma)}$ that is generic over M. Our constructions ensure that $j[G_{<\gamma}] \subseteq H_{<j(\gamma)}$ and we can extend j to an elementary embedding $j_{*} : \mathrm{V}[G_{<\gamma}] \longrightarrow \mathrm{M}[H_{<j(\gamma)}]$. By Claim 18, we have

$$({}^{\gamma}\mathrm{On})^{\mathrm{V}[G_{<\gamma}]} \ \subseteq \ \mathrm{M}[G_{<\gamma}] \ \subseteq \ \mathrm{M}[H_{<j(\gamma)}]$$

and this implies $({}^{\gamma}\mathrm{M}[H_{<j(\gamma)}])^{\mathrm{V}[G_{<\gamma}]} \subseteq \mathrm{M}[H_{<j(\gamma)}]$. Since $H_{<j(\gamma)}$ is contained in $\mathrm{V}[G_{<\gamma}]$, this shows that κ is still γ-supercompact in $\mathrm{V}[G_{<\gamma}]$.

Now, let G be $\mathbb{P}$-generic over V and let $G_{<\gamma}$ be the filter in $P_{<\gamma}$ induced by G. By our assumptions, $P[\gamma, \gamma^{+})$ is the trivial forcing in $\mathrm{V}[G_{<\gamma}]$ and Corollary 3.12 implies that $\mathcal{P}(\mathcal{P}_{\kappa}(\gamma))^{\mathrm{V}[\mathsf{G}]} \in \mathrm{V}[G_{<\gamma}]$. The above computations now show that there is a normal ultrafilter on $\mathcal{P}_{\kappa}(\gamma)$ in $\mathrm{V}[\mathsf{G}]$ and hence κ remains γ-supercompact in $\mathrm{V}[\mathsf{G}]$. □

We will now derive Clause (iv) of Theorem 1.2 from Claim 16. Let κ be supercompact, $\alpha \geq \kappa$ and let ν denote the least fixed point of the $\beth$-function that is $\geq \alpha$. We let $\gamma = \beth_{\nu+\kappa}$ denote the smallest strong limit cardinal of cofinality κ above ν. Then $\gamma^{<\kappa} = \gamma$, γ is singular and $2^{\gamma} = \gamma^{+}$ by Theorem 1.9. Moreover γ is not a fixed point of the $\beth$-function. Hence by Claim 16, κ remains α-supercompact in every $\mathbb{P}$-generic extension of the ground model. □

We have thus finished the proof of Theorem 1.2.

5. Omega-Superstrongs

This section is devoted to the proof of Theorem 1.7. We will make use of the following folklore theorem, which we provide a sketch of the proof for sake of completeness.

Theorem 5.1. *If κ is ω-superstrong and the* GCH *holds, $2^{\kappa} = \kappa^{++}$ holds in a set forcing extension which preserves the ω-superstrength of κ and preserves* SCH.

Proof. Let P be the reverse Easton iteration that adds λ^{++}-many Cohen subsets of λ for every inaccessible $\lambda < j^\omega(\kappa)$. This obviously forces $2^\kappa = \kappa^{++}$and preserves SCH by standard reduction arguments. Hence it suffices to prove the following statement.

Claim 22. *Some condition in P forces that P preserves the ω-superstrength of κ.*

Proof. Let $j\colon \mathrm{V} \to \mathrm{M}$ witness the ω-superstrength of κ in V. We may assume that j is given by an extender ultrapower embedding.[16] For every $n < \omega$, $P_{<j^n(\kappa)} = {P_{<j^n(\kappa)}}^{\mathrm{M}}$, latter denoting the M-version of $P_{<j^n(\kappa)}$. Let G be the canonical name for the P-generic. By the closure properties of our iterands, $(j''G)\restriction[\kappa, j^\omega(\kappa))$ gives rise to a condition in P (a "master condition") which forces that $j''G \subseteq G$. Furthermore $G \cap P^{\mathrm{M}}$ is P^{M}- generic over M, using that j is given by an extender ultrapower embedding (the argument is given in the proof of 1 in the proof of Theorem 23 below). This enables us to lift j to $j^*\colon \mathrm{V}[G] \to \mathrm{M}[G \cap P^{\mathrm{M}}]$. $\mathrm{V}[G]_{j^\omega(\kappa)} \subseteq \mathrm{M}[G \cap P^{\mathrm{M}}]$ follows as every element of $\mathrm{V}[G]_{j^\omega(\kappa)}$ has a P-name in $\mathrm{V}_{j^n(\kappa)}$ for some $n < \omega$ by the closure properties of tails of P. □

This completes the proof of the theorem. □

Proof of Theorem 1.7. Assume κ is ω-superstrong in V. By [8], we can force the GCH to hold while preserving the ω-superstrength of κ. Using Theorem 5.1, we may force further to obtain $2^\kappa = \kappa^{++}$ while preserving SCH and the ω-superstrength of κ. In this situation, the assumptions of Theorem 1.2 hold.

Claim 23. *There is a condition in $\mathbb{P}$ which forces that the ω-superstrength of κ is preserved.*

Proof. Let $j : \mathrm{V} \longrightarrow \mathrm{M}$ witness the ω-superstrength of κ in V. Let $\mathbb{P}$ and $P_{<\alpha}$ denote the iteration and its restriction to α as defined in the proof of Theorem 1.2, let $\mathbb{P}^M$ and $P^M_{<\alpha}$ denote their respective M-versions. For every $n < \omega$, $P_{<j^n(\kappa)} = P^{\mathrm{M}}_{<j^n(\kappa)}$. Moreover if $\delta < j^\omega(\kappa)$ and $\delta = \operatorname{card}\delta + 2 + \xi$, $f_\delta = f^{\mathrm{M}}_\delta$, $\dot{a}^{\operatorname{card}\delta}_\xi = (\dot{a}^{\operatorname{card}\delta}_\xi)^{\mathrm{M}}$, $P^{\mathrm{M}}_{<j^\omega(\kappa)} = P_{<j^\omega(\kappa)} \cap \mathrm{M}$ and also the extension relations of those two forcings agree. We will use those facts tacitly in the following. We want to find a V-generic $G \subseteq \mathbb{P}$ and an M-generic $G^* \subseteq \mathbb{P}^{\mathrm{M}}$ such that $j''G \subseteq G^*$ and $\mathrm{V}[G]_{j^\omega(\kappa)} \subseteq \mathrm{M}[G^*]$. After finding a suitable $P_{<j^\omega(\kappa)}$-generic $G_{<j^\omega(\kappa)}$, we will let $G^*_{<j^\omega(\kappa)}$ be $G_{<j^\omega(\kappa)} \cap P_{<j^\omega(\kappa)}$. We will let G^* be the filter generated by $G^*_{<j^\omega(\kappa)}$ together with the image of G under j. The inclusion $\mathrm{V}[G]_{j^\omega(\kappa)} \subseteq \mathrm{M}[G^*]$ follows as every element of $\mathrm{V}[G]_{j^\omega(\kappa)}$ has a P-name in $\mathrm{V}_{j^n(\kappa)}$ for some $n < \omega$ by Clause 6 of Claim 7.

It remains to show the following statements.

1. $G^*_{<j^\omega(\kappa)}$ is $P^{\mathrm{M}}_{<j^\omega(\kappa)}$-generic over M.
2. G^* is P^*-generic over M.
3. We can choose $G_{<j^\omega(\kappa)}$ in such a way that $j''G_{<j^\omega(\kappa)} \subseteq G^*_{<j^\omega(\kappa)}$.

We will assume 3 for the moment and prove 1 and 2 using 3. We will then prove 3 without using either 1 or 2. Assume that j is given by an extender ultrapower embedding.

[16] This means that each element of M is of the form $j(f)(a)$, where a belongs to $V_{j^\omega(\kappa)}$ and f is a function in V with domain $V_{j^\omega(\kappa)}$. This assumption is harmless as if the initial $j\colon \mathrm{V} \to \mathrm{M}$ does not satisfy it, we can replace M by the transitive collapse $\bar{M}$ of H, the elementary submodel of M consisting of all $j(f)(a)$ of the above form and replace j by $k \circ j$, where $k\colon H \cong \bar{M}$.

Proof of 1. Suppose $D \in \mathrm{M}$ is dense on $P^{\mathrm{M}}_{<j^\omega(\kappa)}$ and write D as $j(f)(a)$ where $\mathrm{dom}(f) = V_{j^\omega(\kappa)}$ and $a \in V_{j^{n+1}(\kappa)}$ for some $n \in \omega$. Choose $p \in G_{<j^\omega(\kappa)}$ such that p reduces $f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is dense on $P_{<j^\omega(\kappa)}$. The existence of p follows from Clause 5 of Theorem 7, using that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$. Then $j(p)$ belongs to $j''G_{<j^\omega(\kappa)} \subseteq G^*_{<j^\omega(\kappa)}$ by 3 and reduces D below $j^{n+1}(\kappa)$. Hence $E = \{q \in P_{<j^{n+2}(\kappa)} \mid q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D\}$ is dense below $j(p) \restriction j^{n+2}(\kappa)$ in $P_{<j^{n+2}(\kappa)}$. Since $G_{<j^{n+2}(\kappa)}$ contains $j(p) \restriction j^{n+2}(\kappa)$ and is $P_{<j^{n+2}(\kappa)}$-generic over M, $G_{<j^{n+2}(\kappa)} \cap E \neq \emptyset$. Choose q in that intersection. Then $q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D \cap G^*_{<j^\omega(\kappa)}$. □

Proof of 2. Like 1, using that $j''G \subseteq G^*$ as an immediate consequence of 3. □

Proof of 3. We will specify a *master condition* $q \in P_{<j^\omega(\kappa)}$ so that $q \in G_{<j^\omega(\kappa)}$ ensures $j''G_{<j^\omega(\kappa)} \subseteq G^*_{<j^\omega(\kappa)}$. Let G be the canonical name in V for the $\mathbb{P}$-generic. We define r as the componentwise union of $(j''G)[j(\kappa), j^\omega(\kappa))$ and define the strategic lower bound q for this (directed) set to be - similar to Definition 4.1 - the condition obtained by letting, for every $\delta \in \text{C-supp}(r)$ with $\delta = \mathrm{card}\,\delta + 2 + \xi$:

(1) If $\delta > j(\kappa)^+$, then
$$q \restriction \delta \Vdash q(b_{otp\, f_\delta[\sup r^{**}_\delta]}) = \dot{a}^{\mathrm{card}\,\delta}_\xi.$$
(2) $q \restriction \delta \Vdash q^{**}_\delta := r^{**}_\delta \cup \{\sup r^{**}_\delta\}$.
(3) Components of q other than the above should be equal to the respective components of r.

If such q exists as a condition in $P_{<j^\omega(\kappa)}$,[17] we call q the *strategic lower bound of* $(j''G)[j(\kappa), j^\omega(\kappa))$. In this case, it is seen as in the proof of Claim 20 that q is a lower bound for $(j''G)[j(\kappa), j^\omega(\kappa))$. Furthermore it is seen similar to the proof of Claim 21 that such q exists as a condition in $P_{<j^\omega(\kappa)}$.[18] We will finish the proof of 3 by showing the following claim.

Claim 24. *Whenever $p \leq q$, $p \in G$, then $p \leq j(p)$; hence if $q \in G_{<j^\omega(\kappa)}$ and p is any condition in $G_{<j^\omega(\kappa)}$, then $j(p) \in G_{<j^\omega(\kappa)}$, i.e. $j''G_{<j^\omega(\kappa)} \subseteq G^{\mathrm{M}}_{j^\omega(\kappa)}$.*

Proof. Assume $p \leq q$. Then $p \leq j(p)$ as $p \restriction \kappa = j(p) \restriction \kappa$, $j(p) \restriction [\kappa, j(\kappa)) = \mathbf{1}$ and $p \restriction j(\kappa) \Vdash p[j(\kappa), j^\omega(\kappa)) \leq q[j(\kappa), j^\omega(\kappa)) \leq j(p)[j(\kappa), j^\omega(\kappa))$ by our choice of q. Now if p is any condition in $G_{<j^\omega(\kappa)} \ni q$, then there is $p' \in G_{<j^\omega(\kappa)}$ which is stronger than both p and q. By the above, $p' \leq j(p')$, but by elementarity $j(p') \leq j(p)$ and therefore $j(p) \in G_{<j^\omega(\kappa)}$. Since $j(p) \in \mathrm{M}$, the last statement of the claim follows. □

If we now choose $G_{<j^\omega(\kappa)}$ containing q, the above implies that 3 holds. □

This completes the proof of Claim 23. □

In the situation assembled above, we can now combine Theorem 1.2, Theorem 5.1 and Claim 23 to force the existence of a lightface definable well-order of H_{κ^+} together with a failure of the GCH at κ. □

Note that we could have similarly obtained $2^\kappa > \kappa^{++}$ in the above proof.

[17] In contrast to Section 4, we do not claim here that either r or q are elements of M.

[18] The only difference is that for every $n < \omega$, the intervals $[j^n(\kappa), j^{n+1}(\kappa))$ have to be treated separately.

6. Corollaries

This short section contains the proofs of the two corollaries mentioned in the Introduction.

Proof of Corollary 1.3. We show that in any $\mathbb{P}$-generic extension, for any ordinal α and $x \subseteq \alpha$, there is an ordinal ξ so that for all $\gamma < \alpha$,

$$\gamma \in x \iff c_{\xi+\gamma} = 1.$$

Let α, $x \subseteq \alpha$ and $p \in \mathbb{P}$ be given. By the distributivity properties of $\mathbb{P}$, it follows that there is a condition $p' \leq p$ which forces that x has a $P_{<\alpha^+}$-name $\dot{x}$. Let ξ be greater than both $\sup(\operatorname{supp}(p'))$ and α. We may extend p' to q so that for every $\gamma < \alpha$, q forces that $c(\xi + \gamma) = 1$ iff $\gamma \in \dot{x}$. But this means that q forces that $\dot{x}$ is ordinal-definable. □

Proof of Corollary 1.5. As in the proof of Clause *(iv)* of Theorem 1.2 at the end of Section 4, letting $\alpha = \kappa$. □

7. Open Questions

One could ask whether the results of [2] and [3] can be generalized to a non-GCH context without any restrictions (like in the case of Theorem 1.2 the restriction to inaccessible κ). It seems like completely new techniques would be neccessary to provide an answer.

Question 7.1. *Does every model of set theory (which satisfies* SCH*?) have a cofinality-preserving forcing extension in which for every regular uncountable κ there is a lightface definable well-order of* $\mathrm{H}(\kappa^+)$*? Can this be done at least for boldface definable well-orders?*

Easier questions are the following.

Question 7.2. *Can one force to add a boldface definable well-order of* $\mathrm{H}(\kappa^+)$ *when $\kappa^{<\kappa} > \kappa$ and κ is regular while preserving cofinalities?*

Question 7.3. *Can one force to add a lightface definable well-order of* $\mathrm{H}(\kappa^+)$ *when $\kappa^{<\kappa} = \kappa$ but κ is not inaccessible while preserving cofinalities?*

The latter question will be answered positively in the forthcoming [4].

While answering [10, Question 6.1], Corollary 1.5 still left open the following question, which again seems cannot be answered using the techniques introduced in our paper.

Question 7.4. *Starting from a large cardinal assumption weaker than a strong cardinal, is it possible to obtain a model with a measurable cardinal κ with $2^\kappa > \kappa^{++}$ and a definable well-order of* $\mathrm{H}(\kappa^+)$*?*

References

[1] Arthur W. Apter and Shoshana Friedman. Coding into HOD via normal measures with some applications. *MLQ Math. Log. Q.*, 57(4):366–372, 2011.
[2] David Asperó and Sy-David Friedman. Large cardinals and locally defined well-orders of the universe. *Ann. Pure Appl. Logic*, 157(1):1–15, 2009.
[3] David Asperó and Sy-David Friedman. Definable well-orders of $H(\omega_2)$ and GCH. *J. Symbolic Logic*, 77(4):1101–1121, 2012.

[4] David Asperó, Peter Holy, and Philipp Lücke. Forcing lightface definable well-orders without GCH. In preparation.
[5] Andrew D. Brooke-Taylor. Large cardinals and definable well-orders on the universe. *J. Symbolic Logic*, 74(2):641–654, 2009.
[6] Shoshana Friedman. Accessing the switchboard via set forcing. *MLQ Math. Log. Q.*, 58(4-5):303–306, 2012.
[7] Sy-David Friedman. *Fine structure and class forcing*, volume 3 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2000.
[8] Sy-David Friedman. Large cardinals and L-like universes. In *Set theory: recent trends and applications*, volume 17 of *Quad. Mat.*, pages 93–110. Dept. Math., Seconda Univ. Napoli, Caserta, 2006.
[9] Sy-David Friedman and Peter Holy. Condensation and large cardinals. *Fund. Math.*, 215(2):133–166, 2011.
[10] Sy-David Friedman and Radek Honzik. A definable failure of the singular cardinal hypothesis. *Israel J. Math*, 2012.
[11] Sy-David Friedman and Philipp Lücke. Large cardinals and definable well-orders, without the GCH. Accepted for publication in the *Annals of Pure and Applied Logic*.
[12] A. Kanamori and M. Magidor. The evolution of large cardinal axioms in set theory. In *Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977)*, volume 669 of *Lecture Notes in Math.*, pages 99–275. Springer, Berlin, 1978.
[13] Philipp Lücke. $\mathbf{\Sigma}^1_1$-definability at uncountable regular cardinals. *J. Symbolic Logic*, 77(3):1011–1046, 2012.

Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, 1090 Wien, Austria

E-mail address: sdf@logic.univie.ac.at

University of Bristol, Department of Mathematics, University Walk, Bristol BS8 1TW, United Kingdom

E-mail address: maxph@bristol.ac.uk

Rheinische Friedrich-Wilhelms-Universität Bonn, Mathematisches Institut, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: pluecke@math.uni-bonn.de