Structural Reflection and the HOD Conjecture

4. Lecture: A consistency proof from large cardinals beyond choice

Philipp Moritz Lücke
Universität Hamburg

Contemporary Set Theory Workshop – Gdansk
27. March 2025

Introduction

Theorem (The HOD Dichotomy, Woodin)

If δ is an extendible cardinal, then one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds.
- Every regular cardinal greater than or equal to δ is ω-strongly measurable in HOD.

The Weak HOD Conjecture (Woodin)

The theory

$$\mathrm{ZFC} + \text{“}\textit{There is a huge cardinal above an extendible cardinal}\text{”}$$

proves that a proper class of regular cardinals is not ω-strongly measurable in HOD.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem (Aguilera–Bagaria–L.)

If λ is exacting, then λ is a singular cardinal that is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Corollary

If ZFC is consistent with the existence of an exacting cardinal above an extendible cardinal, then the Weak HOD Conjecture fails.

Note that, in both ZFC-models with exacting cardinals constructed in the second lecture, there are no extendible cardinals below the given exacting cardinals.

In the following, we will start with models of ZF containing *large cardinals beyond choice* and use them to construct models of ZFC with extendible cardinals below ultraexacting cardinals.

Forcing Choice

Recall that, given an infinite cardinal λ, the *Dependent Choice principle* λ-DC states that for every non-empty set D and every binary relation R with the property that for all $s \in {}^{<\lambda}D \setminus \{\emptyset\}$, there exists $d \in D$ with $s \mathrel{R} d$, there exists a function $f : \lambda \longrightarrow D$ with the property that

$$(f \restriction \alpha) \mathrel{R} f(\alpha)$$

holds for all $\alpha < \lambda$.

It is easy to see that the Axiom of Choice is equivalent to the statement that λ-DC holds for every cardinal λ.

Theorem (Woodin, ZF)

If δ is a supercompact cardinal, then there is a partial order $\mathbb{Q} \subseteq \mathrm{V}_\delta$ such that the following hold:

- $\mathbb{Q}$ is homogeneous.
- $\mathbb{Q}$ is Σ_3-definable without parameters in V_δ.
- If G is $\mathbb{Q}$-generic over V, then $\mathrm{V}[G]_\delta$ is a model of ZFC and every extendible cardinal smaller than δ in V is extendible in $\mathrm{V}[G]_\delta$.

...

. . .

- If $\lambda < \delta$ is a cardinal in $C^{(3)}$ and $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ is a non-trivial elementary embedding with first non-trivial fixed point λ, then there is a complete suborder $\mathbb{P}$ of $\mathbb{Q}$ with $\mathbb{P} \subseteq V_{\lambda+1}$ and a $\mathbb{P}$-name $\dot{\mathbb{R}}$ for a partial order such that the following statements hold:
    - $\mathbb{P}$ is homogeneous and Σ_3-definable without parameters in $V_{\lambda+1}$.
    - There is a dense embedding of $\mathbb{Q}$ into $\mathbb{P} * \dot{\mathbb{R}}$ that maps every condition p in $\mathbb{P}$ to $(p, \mathbb{1}_{\dot{\mathbb{R}}})$.
    - $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{R}}$ *is homogeneous and* $<\check{\lambda}^+$*-closed*"
    - $\mathbb{1}_{\mathbb{P}} \Vdash \check{\lambda}$-DC
    - There is a condition p in $\mathbb{P}$ with the property that whenever G_0 is $\mathbb{P}$-generic over V with $p \in G_0$, then $j[G_0] \subseteq G_0$ holds.

A consistency proof from large cardinals beyond choice

Theorem (Aguilera–Bagaria–L., BG)

Let $j : V \longrightarrow V$ be a non-trivial elementary embedding with first non-trivial fixed point $\lambda \in C^{(3)}$ and let $\delta > \lambda$ be a supercompact cardinal with $j(\delta) = \delta$. If $\mathbb{Q}$ is the partial order given by Woodin's theorem and G is $\mathbb{Q}$-generic over V, then λ is an ultraexacting cardinal in $\mathrm{V}[G]_\delta$.

Let $j : V \longrightarrow V$ be a non-trivial elementary embedding with first non-trivial fixed point $\lambda \in C^{(3)}$, let $\delta > \lambda$ be a supercompact cardinal with $j(\delta) = \delta$ and let $\mathbb{Q}$ be the partial order given by Woodin's theorem.

Pick a complete suborder $\mathbb{P}$ of $\mathbb{Q}$ and a $\mathbb{P}$-name $\dot{\mathbb{R}}$ for a partial order as in the statement of Woodin's theorem.

Fix a condition p in $\mathbb{P}$ with the property that whenever G_0 is $\mathbb{P}$-generic over V with $p \in G_0$, then $j[G_0] \subseteq G_0$ holds.

Let G be $\mathbb{Q}$-generic over V with $p \in G$ and let $G_0 * G_1$ denote the filter on $\mathbb{P} * \dot{\mathbb{R}}$ induced by G.

Then $p \in G_0$ and $j \restriction V_{\lambda+1}$ lifts to an elementary embedding

$$j_* : V[G_0]_{\lambda+1} \longrightarrow V[G_0]_{\lambda+1}$$

in $V[G_0]$.

We then know that λ is a limit of strongly inaccessible cardinals in $V[G_0]$, and it follows that $V[G]_\lambda$ has cardinality λ in $V[G]_\delta$.

Thus, in $V[G]_\delta$, we can find $\lambda < \eta \in C^{(2)}$ and an elementary submodel X of $V[G]_\eta$ of cardinality λ with $V[G]_\lambda \cup \{j_* \restriction V[G]_\lambda\} \subseteq X$.

Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse.

Then $M \in \mathrm{H}(\lambda^+)^{V[G]_\delta}$ and it follows that M is an element of $V[G_0]$.

The homogeneity of $\dot{\mathbb{R}}^{G_0}$ in $V[G_0]$ then implies that whenever F is $\dot{\mathbb{R}}^{G_0}$-generic over $V[G_0]$, then, in $V[G_0, F]_\delta$, we can find a cardinal $\lambda < \zeta \in C^{(2)}$ and an elementary submodel Y of $V[G_0, F]_\zeta$ such that $V[G_0, F]_\lambda \cup \{\lambda\} \subseteq Y$ and the transitive collapse of Y is equal to M.

Pick a $\mathbb{P}$-name $\dot{M}$ in V with $\dot{M}^{G_0} = M$.

Then, there is a condition p_0 in G_0 with the property that whenever $H_0 * H_1$ is $(\mathbb{P} * \dot{\mathbb{R}})$-generic over V with $p_0 \in H_0$, then, in $V[H_0, H_1]_\delta$, we can find $\lambda < \zeta \in C^{(2)}$ and an elementary submodel Y of $V[H_0, H_1]_\zeta$ such that $V[H_0, H_1]_\lambda \cup \{\lambda\} \subseteq Y$ and the transitive collapse of Y is equal to $\dot{M}^{H_0}$.

Since $j(\mathbb{P}) = \mathbb{P}$, we then have that whenever $H_0 * H_1$ is $(\mathbb{P} * j(\dot{\mathbb{R}}))$-generic over V with $j(p_0) \in H_0$, then, in $V[H_0, H_1]_\delta$, we can find $\lambda < \zeta \in C^{(2)}$ and an elementary submodel Y of $V[H_0, H_1]_\zeta$ such that $V[H_0, H_1]_\lambda \cup \{\lambda\} \subseteq Y$ and the transitive collapse of Y is equal to $j(\dot{M})^{H_0}$.

Note also that since $\mathbb{Q}$ is definable in V_δ by a formula without parameters, and since $j(\delta) = \delta$, we know that $j(\mathbb{Q}) = \mathbb{Q}$.

By elementarity, this implies that there is a dense embedding of $\mathbb{Q}$ into $\mathbb{P} * j(\dot{\mathbb{R}})$ in V that sends every condition q in $\mathbb{P}$ to $(q, \mathbb{1}_{j(\dot{\mathbb{R}})})$.

Hence, there is $F \in V[G]$ that is $j(\dot{\mathbb{R}})^{G_0}$-generic over $V[G_0]$ with $V[G] = V[G_0, F]$.

Since $p_0 \in G_0$ and $j[G_0] \subseteq G_0$, we may now conclude that, in $V[G]_\delta$, there exists $\lambda < \zeta \in C^{(2)}$ and an elementary submodel Y of $V[G]_\zeta$ with $V[G]_\lambda \cup \{\lambda\} \subseteq Y$ and the transitive collapse τ of Y is an isomorphism onto $j(\dot{M})^{G_0}$.

The elementary embedding j_*, being a lifting of $j \restriction V_{\lambda+1}$ to $V[G_0]_{\lambda+1}$, now yields that $j_*(M) = j(\dot{M})^{G_0}$.

This shows that

$$j_* \restriction M : M \longrightarrow j(\dot{M})^{G_0}$$

is an elementary embedding in $V[G]_\delta$.

Moreover, the composition

$$i \ = \ \tau^{-1} \circ (j_* \restriction M) \circ \pi : X \longrightarrow Y$$

is an elementary elementary embedding from X to $V[G]_\zeta$ in $V[G]_\delta$.

Since $\pi \restriction V[G]_\lambda = \mathrm{id}_{V_\lambda}$ and $\tau^{-1} \restriction V[G]_\lambda = \mathrm{id}_{V[G]_\lambda}$, we can conclude that

$$i \restriction V[G]_\lambda \ = \ j_* \restriction V[G]_\lambda \ \in \ X.$$

Definition (GB)

A cardinal κ is *super Reinhardt* if for every ordinal α, there is an elementary embedding $j : \mathrm{V} \longrightarrow \mathrm{V}$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

Proposition (BG)

If κ is super Reinhardt cardinal, then $\mathrm{V}_\kappa \prec \mathrm{V}$ and there is a proper class of supercompact cardinals.

Theorem (Aguilera–Bagaria–L., BG)

If there is a super Reinhardt cardinal, then there is a model of ZFC with an exacting cardinal above an extendible cardinal.

Open questions

Question

Is it possible to derive the consistency of ZFC with the existence of an extendible cardinal below an exacting cardinal from the consistency of ZFC with some well-studied large cardinal axiom?

Question

Does the consistency of ZFC with the existence of an extendible cardinal below an exacting cardinal imply the consistency of ZFC with an I2-embedding?

Question

Is it possible to derive the consistency of ZFC with the existence of an ultraexacting cardinal below an extendible cardinal from the consistency of ZFC with some well-studied large cardinal axiom?

Question

Is it possible to derive the consistency of ZFC with the existence of a proper class of exacting cardinals from the consistency of ZFC with an I0-embedding?

Question

Is there a canonical strengthening of the notion of ultraexacting cardinals that is compatible with the Axiom of Choice and implies that the given cardinal is HOD-Berkeley?

Question

Are ultraexacting cardinals measurable in HOD?

Thank you for listening!