Definable clubs

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Joint work with Omer Ben-Neria (Jerusalem).

Introduction

The work presented in this talk contributes to a programme that aims to study strong combinatorial properties of uncountable cardinals through the interaction of these properties with set-theoretic definability.

Analogous to the study of analytical real numbers, our results deal with the definability of subsets of uncountable cardinals by formulas using only simple parameters, focussing on the definability of closed unbounded subsets of singular cardinals of countable cofinality.

We investigate the corresponding notions of *stationary sets*, i.e. subsets of the given cardinal that intersect all closed unbounded sets definable in a given way, and the structural properties of the collections of these sets.

Definition

A class X is *definable* by a formula $\varphi(v_0, \ldots, v_n)$ and parameters $z_0, \ldots, z_{n-1}$ if

$$X = \{y \mid \varphi(y, z_0, \ldots, z_{n-1})\}.$$

Definition

- A formula in the language $\mathcal{L}_\in$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under negation, disjunction and bounded quantification.
- Given $n < \omega$, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \, \neg\varphi(x)$ for some Σ_n-formula φ.

Definition

Let κ be a uncountable cardinal, let $n < \omega$ and let A be a class.

- A subset S of κ is $\Sigma_n(A)$-*stationary in* κ if $C \cap S \neq \emptyset$ holds for every closed unbounded subset C of κ with the property that $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$.
- A subset S of κ is $\mathbf{\Sigma}_n(A)$-*stationary in* κ if it is $\Sigma_n(A \cup \mathrm{H}(\kappa))$-stationary in κ.
- A subset S of κ is $\mathbf{\Sigma}_n$-*stationary in* κ if it is $\mathbf{\Sigma}_n(\emptyset)$-stationary in κ.

We focus on the following two questions:

- How much can the collection of $\Sigma_1(A)$-stationary subsets of an uncountable cardinal κ differ from the collection of all stationary subsets of κ? What is the situation at cardinals of countable cofinality, where stationarity coincides with coboundedness?

- For which cardinals is it possible to develop a non-trivial structure theory for $\Sigma_1(A)$-stationary subsets?

Proposition

Assume that $\mathrm{V} = \mathrm{L}$ *holds and* κ *is an uncountable cardinal. Then a subset of* κ *is* $\Sigma_1(\kappa^+)$*-stationary in* κ *if and only if it is stationary in* κ.

Proof.

If A is a subset of κ, then there is an ordinal $\gamma < \kappa^+$ with the property that the set $\{A\}$ is definable by a Σ_1-formula with parameter γ. □

Proposition

Assume that Martin's Maximum holds. Then a subset of ω_1 *is* $\mathbf{\Sigma}_1$*-stationary in* ω_1 *if and only if it is stationary in* ω_1*.*

Proof.

Woodin proved that *Martin's Maximum* implies *admissible club guessing*, i.e., every closed unbounded subset of ω_1 contains a closed unbounded subset of the form

$$\{\alpha < \omega_1 \mid \mathrm{L}_\alpha[x] \models \mathrm{KP}\}$$

for some $x \in \mathbb{R}$. □

Jónsson cardinals

Definition

A cardinal κ is *Jónsson* if for every function $f : [\kappa]^{<\omega} \longrightarrow \kappa$, there is a proper subset H of κ of cardinality κ with $f[[H]^{<\omega}] \subseteq H$.

Question

Does $\mathbf{ZFC}$ prove that ω_ω is not Jónsson?

Theorem

If ω_ω is Jónsson, then every infinite subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1$-stationary in ω_ω.

Definition

Given uncountable cardinals $\mu < \kappa$, we say that the cardinal κ has the $\Sigma_1(\mu)$-*undefinability property* if no ordinal α in the interval $[\mu, \kappa)$ has the property that the set $\{\alpha\}$ is definable by a Σ_1-formula with parameters in the set $\mathrm{H}(\mu) \cup \{\kappa\}$.

Lemma

Given uncountable cardinals $\mu < \kappa$, if the cardinal κ has the $\Sigma_1(\mu)$-undefinability property, then $\{\mu\}$ is $\Sigma_1(\mathrm{H}(\mu))$-stationary in κ.

Lemma

Given uncountable cardinals $\mu < \kappa$*, if the cardinal* κ *has the* $\Sigma_1(\mu)$*-undefinability property, then* $\{\mu\}$ *is* $\Sigma_1(\mathrm{H}(\mu))$*-stationary in* κ*.*

Proof.

Let C be a closed unbounded subset of κ with the property that the set $\{C\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\mu) \cup \{\kappa\}$.

Assume, towards a contradiction, that μ is not an element of C. Set

$$\rho \ = \ \max((C \cup \{0\}) \cap \mu) \ < \ \mu$$

and

$$\nu \ = \ \min(C \setminus (\rho + 1)) \ \in \ (\mu, \kappa).$$

Then the set $\{\nu\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\mu) \cup \{\kappa\}$, a contradiction. □

Definition

Given uncountable cardinals $\mu < \kappa$, we say that the cardinal κ has the $\Sigma_1(\mu)$-*undefinability property* if no ordinal α in the interval $[\mu, \kappa)$ has the property that the set $\{\alpha\}$ is definable by a Σ_1-formula with parameters in the set $\mathrm{H}(\mu) \cup \{\kappa\}$.

Lemma

Given uncountable cardinals $\mu < \kappa$, if the cardinal κ has the $\Sigma_1(\mu)$-undefinability property, then $\{\mu\}$ is $\Sigma_1(\mathrm{H}(\mu))$-stationary in κ.

Corollary

Let κ be a limit cardinal and $E \subseteq \kappa$ be a set of uncountable cardinals that is unbounded in κ. If κ has the $\Sigma_1(\mu)$-undefinability property for all $\mu \in E$, then E is $\boldsymbol{\Sigma}_1$-stationary in κ.

Definition

Given uncountable cardinals $\nu < \kappa$, the cardinal κ is ν-Rowbottom if and only if

$$\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \kappa, {<}\nu \rangle$$

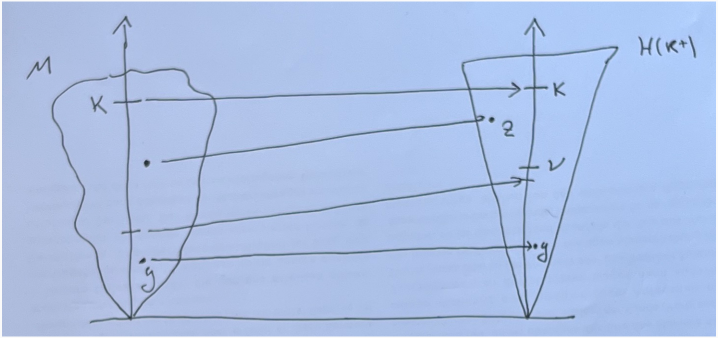
holds for all $\lambda < \kappa$, i.e., given a countable first-order language $\mathcal{L}$ with a unary predicate symbol $\dot{R}$, every $\mathcal{L}$-structure A with domain κ and $|\dot{R}^A| = \lambda$ has an elementary substructure B of size κ with $|\dot{R}^B| < \nu$.

Lemma

- *If κ is ν-Rowbottom for some $\nu < \kappa$, then κ is Jónsson.*
- *If κ is the least Jónsson cardinal, then κ is ν-Rowbottom for some uncountable $\nu < \kappa$.*

Lemma

Let κ be a ν-Rowbottom cardinal with ν regular, let $y \in \mathrm{H}(\nu)$ and let $z \in \mathrm{H}(\kappa^+)$. Then there exists a transitive set M with $\kappa, y \in M$ and a non-trivial elementary embedding $j : M \longrightarrow \mathrm{H}(\kappa^+)$ satisfying $\mathrm{crit}(j) < \nu$, $j(y) = y$, $j(\kappa) = \kappa$ *and* $z \in \mathrm{ran}(j)$.



Lemma

If ω_ω is ω_n-Rowbottom for some $0 < n < \omega$, then ω_ω has the $\Sigma_1(\omega_n)$-undefinability property.

Proof.

Assume that there is an ordinal $\alpha \in [\omega_n, \omega_{n+1})$ and $y \in \mathrm{H}(\aleph_n)$ such that the set $\{\alpha\}$ is definable by a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and the parameters κ and y.

Pick a transitive set M with $\omega_\omega, y \in M$ and a non-trivial elementary embedding $j : M \longrightarrow \mathrm{H}(\aleph_{\omega+1})$ satisfying $j(\mathrm{crit}(j)) = \omega_n$, $j(y) = y$, $j(\omega_\omega) = \omega_\omega$ and $\alpha \in \mathrm{ran}(j)$.

Then, there is $\bar{\alpha} < \omega_n$ such that $j(\bar{\alpha}) = \alpha$ and $\varphi(\bar{\alpha}, \kappa, y)$ holds in M.

But then Σ_1-upwards absoluteness implies that $\varphi(\bar{\alpha}, \kappa, y)$ also holds in V, a contradiction. □

Lemma

If ω_ω is ω_n-Rowbottom for some $0 < n < \omega$, then ω_ω has the $\Sigma_1(\omega_n)$-undefinability property.

Theorem

If ω_ω is Jónsson, then every infinite subset of $\{\omega_n \mid n < \omega\}$ is $\boldsymbol{\Sigma}_1$-stationary in ω_ω.

We can use the above methods to reduce the class of models of set theory in which ω_ω possesses strong partition properties.

Me specifically, we can show that ω_ω is not ω_2-Rowbottom in the standard models of strong forcing axioms, where the given axiom was forced over a model of the GCH by turning some large cardinal into ω_2.

Lemma

Assume that there are no special ω_2-Aronszajn trees and for all $2 < n < \omega$, there is a special ω_n-Aronszajn tree.

Then the set $\{\omega_2\}$ is definable by a Σ_1-formula with parameter ω_ω and the cardinal ω_ω is not ω_2-Rowbottom.

Consistency strength

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *Every unbounded subset of* $\{\omega_n \mid n < \omega\}$ *is* $\mathbf{\Sigma}_1(Ord)$*-stationary in* ω_ω*.*
- *There is a singular cardinal* κ *of countable cofinality and a subset of* κ *that consists of cardinals and is* $\mathbf{\Sigma}_1$*-stationary in* κ*.*
- *There is a measurable cardinal.*

In contrast, more measurable cardinals are required to obtain an analogous statement for singular cardinals of uncountable cofinality:

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There exists a singular cardinal* κ *of uncountable cofinality such that some non-stationary subset of* κ *is* $\mathbf{\Sigma}_1$*-stationary in* κ*.*
- *There exists a singular cardinal* κ *of uncountable cofinality such that some non-stationary subset of* κ *is* $\mathbf{\Sigma}_1(Ord)$*-stationary in* κ*.*
- *There exist uncountably many measurable cardinals.*

Disjoint $\Sigma_1(\mathsf{A})$-stationary sets

At cardinals κ of uncountable cofinality, Solovay's theorem ensures that existence of bistationary (i.e. stationary and costationary) subsets of κ.

In contrast, all stationary subsets of singular cardinals of countable cofinality are cobounded and hence there are no bistationary subsets of these cardinals.

We now consider the question how bistationarity behaves in the definable context.

We can show that there exists a cardinal δ such that ...

- **ZFC** proves that for every set A of cardinality less than δ and every singular cardinal κ of countable cofinality, there are disjoint $\Sigma_1(A)$-stationary subsets of κ.

- The following statements are equiconsistent over **ZFC**:
  - There is a singular cardinal κ of countable cofinality such that for every subset A of H_κ of cardinality δ, there are disjoint $\Sigma_1(A)$-stationary subsets of κ.
  - There is a measurable cardinal.

This cardinal is ... the *reaping number* $\mathfrak{r}$!

Definition

The reaping number $\mathfrak{r}$ is the least cardinality of a subset A of $[\omega]^\omega$ with the property that for every $b \in [\omega]^\omega$, there is $a \in A$ such that either $a \setminus b$ or $a \cap b$ is finite.

Proposition

Let κ be a singular cardinal of countable cofinality and let A be a set of cardinality less than $\mathfrak{r}$. Then there exists a subset E of κ with the property that both E and $\kappa \setminus E$ are $\Sigma_1(A)$-stationary in κ.

Proposition

Let κ be a singular cardinal of countable cofinality and let A be a set of cardinality less than $\mathfrak{r}$. Then there exists a subset E of κ with the property that both E and $\kappa \setminus E$ are $\Sigma_1(A)$-stationary in κ.

Proof.

Fix $C \subseteq [\kappa]^\omega$ of cardinality less than $\mathfrak{r}$ such that C consists of cofinal sets and every $\Sigma_1(A)$-definable club in κ contains a subset in C.

Let $\langle \kappa_n \mid n < \omega \rangle$ be cofinal in κ. Given $c \in C$, set

$$a_c \ = \ \{n < \omega \mid c \cap [\kappa_n, \kappa_{n+1}) \neq \emptyset\} \ \in \ [\omega]^\omega.$$

Then there is $b \in [\omega]^\omega$ with $a_c \setminus b$ and $a_c \cap b$ infinite for all $c \in C$.

We define $E \ = \ \bigcup\{[\kappa_n, \kappa_{n+1}) \mid n \in b\}$. $\square$

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a measurable cardinal.*
- *There is a singular cardinal* κ *of countable cofinality such that for every subset* A *of* $\mathrm{H}(\kappa)$ *of cardinality* $\mathfrak{r}$*, there exists a subset* E *of* κ *such that both* E *and* $\kappa \setminus E$ *are* $\Sigma_1(A)$*-stationary.*
- *There is a singular cardinal* κ *of countable cofinality such that there exists a subset* E *of* κ *such that both* E *and* $\kappa \setminus E$ *are* $\boldsymbol{\Sigma}_1(Ord)$*-stationary.*

Proof.

Assume that there is no inner model with a measurable cardinal and let κ be a singular cardinal of countable cofinality. Then there is a cofinal function $c : \omega \longrightarrow \kappa$ such that the set $\{c\}$ is definable by a Σ_1-formula with parameters in κ and $z \in \mathrm{H}(\kappa)$.

Pick $A \subseteq \mathrm{H}(\kappa)$ of size $\mathfrak{r}$ such that $z \in A$, $A \cap \kappa$ is unbounded in κ and $A \cap [\omega]^\omega$ is an unsplittable family.

Let S be $\Sigma_1(A)$-stationary in κ, set

$$b \;=\; \{n < \omega \mid c(n) \in S\} \in [\omega]^\omega$$

and pick $a \in A \cap [\omega]^\omega$ such that either $a \setminus b$ or $a \cap b$ is finite.

Since $\{c(n) \mid n \in a\}$ is a $\Sigma_1(A)$-definable club in κ, we know that $a \cap b$ is infinite and there is $k < \omega$ with $a \setminus b \subseteq k$.

Hence $\{c(n) \mid k < n \in a\}$ is a $\Sigma_1(A)$-definable club contained in S. □

Thank you for listening!