On Six Problems Posed by Jarik Nešetřil

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In this article we present several open problems posed (or co-posed) by Jarik Nešetřil. The choice was guided by two criteria. First, we restricted ourselves to problems that are simple to state and (therefore) possible to explain to a non-specialist in the given field. Second, we selected problems that, while being still open, did stimulate research by other people and have an interesting development behind them.

Most of the problems seem to be of fundamental nature and central. For all of them a simple argument might possibly solve them, yet this argument has eluded many researchers for many years. The inspiration from Jarik Nešetřil over the years is much appreciated!

⋆ The author was supported by DFG grant SCHA 1263/1-1.

⋆⋆ Partially supported by the Edmund Landau Center for Research in Mathematical Analysis, supported by the Minerva Foundation (Germany).

1 Minimally Asymmetric Oriented Graphs

Let G be an oriented graph. Suppose G is asymmetric, but every vertex-deleted subgraph $G - v$ fails to be asymmetric. Is it true that G must be K_1?

Jaroslav Nešetřil, Oberwolfach seminar, 1988

A (di)graph is *asymmetric* if its automorphism group is trivial, that is, contains only the identity element. A (di)graph is *symmetric* if it has at least one non-trivial automorphism. An asymmetric (di)graph G is *minimally asymmetric* if G is asymmetric, but $G - v$ is symmetric for every vertex v of G.

The problem was posed by Nešetřil at several conferences and according to Wójcik [Wój96] at least as early as during the Oberwolfach seminar in 1988. It is probably even older than that. At the Oberwolfach seminar in 1988 Nešetřil also conjectured that there are only finitely many minimally asymmetric undirected graphs.

In [NS92] and [Sab91] undirected minimally asymmetric graphs are studied. It turns out [Sab91] that a useful property to use in this context is the length λ of a longest induced path. It is shown in that paper that there are no minimally asymmetric graphs with $\lambda \geq 6$ and precisely two minimally asymmetric graphs with $\lambda = 5$. In [NS92] it is shown that there are exactly seven finite minimal asymmetric graphs with $\lambda = 4$.

Clearly, for every minimally asymmetric graph G one obtains a minimally asymmetric digraph D by replacing each edge of G by a directed 2-cycle. An oriented graph is a digraph obtained from a graph by orienting each of its edges. In particular an oriented cycle is obtained form an undirected cycle in this way. In [Wój96] Wójcik proved the following result. Recall that a cycle is symmetric if it has a non-trivial automorphism.

Theorem 1.1. *Every minimally asymmetric digraph contains a symmetric cycle.*

This implies in particular that there are no minimally asymmetric trees (a fact also proved earlier by Nešetřil) and that the conjecture holds for many classes of asymmetric acyclic digraphs. One example is a transitive tournament. Note that acyclic digraphs may contain symmetric cycles (e.g. $1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 1$ which has a non-trivial automorphism without fixpoints) so Theorem 1.1 does not immediately seem to imply that there are no minimally asymmetric acyclic digraphs.

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2 Partition into Induced Matchings alias The Strong Chromatic Index

The edges of a graph G of maximum degree Δ can be partitioned into at most $\frac{5}{4}\Delta^2$ colour classes, each of which induces a matching.

Paul Erdős and Jaroslav Nešetřil, a combinatorial seminar at Charles University, Prague, 1985

This conjecture was made by Erdős and Nešetřil at a combinatorial seminar at Charles University in Prague in 1985. One year later it was presented at Colloquium on Irregularities of Partitions in Fertőd, Hungary (see [EN89]). For every graph H, the chromatic number of H is at most $\Delta(H)+1$ and the chromatic number of its square is at most $\Delta(H)^2+1$. Vizing's theorem tells us that for line graphs, we can improve the first result, essentially by a factor of 2. The conjecture above suggests that a similar improvement is possible for the second result.

Erdős and Nešetřil had noticed that if we take a cycle of length five and replace each vertex by a stable set of size k, joining two new vertices precisely if the corresponding two vertices of the five cycle are adjacent, then the square of the line graph of the resultant graph is a clique. This shows that the above conjecture is tight when Δ is even. For odd Δ, Erdős and Nešetřil actually made the stronger conjecture that the chromatic number of the square of a line graph of maximum degree Δ is at most $\frac{5}{4}\Delta^2 - \frac{\Delta}{2} + \frac{1}{4}$, which again is tight because of a similar example.

The conjecture was proven for $\Delta = 3$ by Andersen [And92] and independently Horak et al [HQT93]. Cranston [Cra06$^+$] proved that the chromatic number of the square of a line graph of a graph of maximum degree $\Delta = 4$ is at most 22, improving on the bound of 23, obtained by Horak [Hor90]. Note that this does not quite match the conjectured bound of 20. For larger Δ, Molloy and Reed [MR97] showed that there is an $\varepsilon > 0$ such that the chromatic number of the square of the line graph of G is at most $(2-\varepsilon)\Delta(G)^2$.

It is not even known if the clique number of the square of the line graph of G is at most $\frac{5}{4}\Delta(G)^2$, although Chung et al. [CGTT90] did prove that a graph G whose line graph is a clique has at most $\frac{5}{4}\Delta(G)^2$ edges.

Inspired by the above conjecture, Faudree et al. [FGST90] proved that for bipartite G, the clique number of the line graph of G is at most Δ^2. $K_{\Delta,\Delta}$ shows that this bound is tight. They conjectured the same bound holds for the chromatic number ([FGST89] see also [BQ93]).

If every edge of G is in $\frac{3}{4}\Delta(G)^2$ cycles of length four, then the square of the line graph of G has maximum degree less than $\frac{5}{4}\Delta(G)^2 - 1$, so the result follows from Brooks' Theorem. Mahdian [Mah00] proved that if G has no C_4 then the square of its line graph has chromatic index $o(\Delta(G)^2)$. These two complementary results provide strong evidence that the conjecture holds, at least asymptotically. We refer the reader to Mahdian's M.Sc. thesis for a fuller discussion of this conjecture, including the origin of the use of the term *strong edge colouring* for a partition into induced matchings, and *strong chromatic index* for the chromatic number of the square of the line graph.

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3 A Ramsey-type Problem on the Integers

Let $k \geq 3$ be fixed. We ask if there exist a $\delta > 0$ and a set $A \subseteq \mathbb{N}$ with the following properties:

(i) for every integer $\ell \geq 2$ every finite partition $A_1 \dot{\cup} A_2 \dot{\cup} \cdots \dot{\cup} A_\ell = A$ yields one partition class containing a k-AP, i.e., an arithmetic progression of length k and

(ii) every finite subset $A' \subseteq A$ contains a dense subset $A'' \subseteq A'$, $|A''| \geq \delta |A'|$ containing no k-AP.

Paul Erdős, Jaroslav Nešetřil, and Vojtěch Rödl in [ENR90].

> *"It was in the summer of 1983, when Paul Erdős came to Prague and, as usual, described many new conjectures that he first introduced Jarik and me to the problem of Pisier. We were initially optimistic as we thought our experience with ramsey type constructions might yield some results. However, after several failed attempts we began to suspect that Pisier's problem was beyond the scope of our abilities. Still, despite our disapointment, we found the problem very compelling and so, together with Paul Erdős, we considered several alternate versions of the original conjecture. The question presented here is one of those variations that, like the original problem, resisted all our efforts."*
>
> *Vojtěch Rödl*

This problem is motivated by the well known theorems of van der Waerden [Wae27] and Szemerédi [Sze75]. The theorem of van der Waerden [Wae27] is one of the earliest results in Ramsey theory. It asserts that every finite partition of the integers yields one partition class containing an arithmetic progression of any fixed length. More precisely, for positive integers k and ℓ, we say a set of integers A has the *van-der-Waerden-property* $\mathbf{vdW}(k, \ell)$ if for every partition $A_1 \dot{\cup} A_2 \dot{\cup} \cdots \dot{\cup} A_\ell = A$ there is some i $(1 \leq i \leq \ell)$ such that A_i contains a k-AP, i.e., an arithmetic progression of length k. The theorem of van der Waerden can then be stated as follows.

Theorem 3.1 (van der Waerden (1927)). *For all integers $k \geq 3$ and $\ell \geq 2$ there exist $n_{\mathrm{vdW}} = n_{\mathrm{vdW}}(k, \ell)$ such that for every $n \geq n_{\mathrm{vdW}}$ the set $[n] = \{1, 2, \ldots, n\}$ has $\mathbf{vdW}(k, \ell)$.*

Solving a longstanding standing conjecture of Erdős and Turán [ET36] Szemerédi proved the following famous generalization of Theorem 3.1, which stimulated a lot of research and today several proofs using tools from quite diverse areas of mathematics are known [Fur77, Gow06$^+$, Gow01, NRS06, RS04, Tao06$^+$b] (see also [Tao06$^+$a] for a survey of those proofs).

Theorem 3.2 (Szemerédi (1975)). *For every integer $k \geq 3$ and $\delta > 0$ there exist $n_{\mathrm{Sz}} = n_{\mathrm{Sz}}(k, \delta)$ such that for every $n \geq n_{\mathrm{Sz}}$ every subset $A \subseteq [n]$ with $|A| \geq \delta n$ contains a k-AP.*

Similarly as above we say a finite set of integers A has the *Szemerédi-property* $\mathbf{Sz}(k,\delta)$ if every subset $A' \subseteq A$ with $|A'| \geq \delta|A|$ contains a k-AP. Then Szemerédi's theorem asserts that every sufficiently large subset of the first n integers has $\mathbf{Sz}(k,\delta)$. Moreover, since Theorem 3.2 implies Theorem 3.1, it implies, e.g., that every sufficiently large arithmetic progression A displays both properties $\mathbf{vdW}(k,\ell)$ and $\mathbf{Sz}(k,\delta)$ and one may wonder if all sets of integers admitting the van-der-Waerden-property may have the Szemerédi-property as well. That would be somewhat surprising and a proof of such a statement would give a new proof of Szemerédi's theorem. Erdős, Nešetřil, and Rödl [ENR06$^+$, ENR90] conjectured that this is not true. In other words, they conjectured that for fixed $k \geq 3$ there exist $\delta > 0$ and a set $A \subset \mathbb{N}$ which, on one hand, has the van-der-Waerden-property $\mathbf{vdW}(k,\ell)$ for every ℓ, but, on the other hand, no finite subset $A' \subseteq A$ has the Szemerédi-property $\mathbf{Sz}(k,\delta)$. For the case $k = 3$ a related question (motivated by this problem) was considered by Davenport, Hindman, and Strauss [DHS02].

The problem was also motivated by "negative" results concerning problems related to the well known problem of Pisier (see Problem 3.3 below). Suppose some family $\mathscr{I}$ of subsets of the integers is given. We call the elements $I \in \mathscr{I}$ *independent* sets. For an integer $k \geq 3$ let $\mathscr{I}_k = \{I \subseteq \mathbb{N}\colon I \text{ contains no } k\text{-AP}\}$. Then showing that no such set A with properties (i) and (ii) in the statement of the problem exists means to prove the following. *For every $\delta > 0$ and $A \subseteq \mathbb{N}$ there exist ℓ such that if every finite subset $A' \subseteq A$ contains an independent set $A'' \in \mathscr{I}_k$ of size $|A''| \geq \delta|A'|$, then $A = A_1 \dot\cup A_2 \dot\cup \ldots \dot\cup A_\ell$ can be partitioned into ℓ independent sets, i.e., $A_i \in \mathscr{I}_k$ for every $i = 1,2,\ldots,\ell$.* This formulation is formally related to Pisier's problem. To state this problem we say a set $I \subset \mathbb{N}$ is independent if all finite sums of I are distinct, i.e., for all finite, distinct subsets I_1, $I_2 \subseteq I$

$$\sum_{x\in I_1} x \neq \sum_{x\in I_2} x$$

and let $\mathscr{I}_P$ be the collection of all those sets. In [Pis83] Pisier asked whether the following is true.

Problem 3.3 (Pisier (1983)). For every $\delta > 0$ and $A \subseteq \mathbb{N}$ there exist ℓ such that if every finite subset $A' \subseteq A$ contains an independent set $A'' \in \mathscr{I}_P$ of size $|A''| \geq \delta|A'|$, then $A = A_1 \dot\cup A_2 \dot\cup \ldots \dot\cup A_\ell$ can be partitioned into ℓ independent sets, i.e., $A_i \in \mathscr{I}_P$ for every $i = 1,2,\ldots,\ell$.

The affirmative answer of Problem 3.3 would give an arithmetic characterization of *Sidon sets* in terms of this condition.

As pointed out in [ENR06$^+$] there are only very few non-trivial notions of independent families known, for which the Pisier-type problem was solved in the affirmative way. In [ENR06$^+$] a few "negative" examples were shown, i.e., results which are formally similar to the problem.

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4 The Pentagon Problem

Let G be a 3-regular graph that contains no cycle of length shorter than g. Is it true that for large enough g there is a homomorphism from G to C_5? Explicitly, is there a vertex coloring of G by $\{1, 2, 3, 4, 5\}$, such that colors of adjacent vertices differ by 1 modulo 5?

Jaroslav Nešetřil in [Neš99].

Apart from being published in [Neš99], this question was asked by Nešetřil at numerous problem sessions. By Brook's theorem any triangle-free cubic (i.e.

3-regular) graph is 3-colorable. Does a stronger assumption on girth of the graph imply existence of a more restricted coloring? (The girth of a graph G is the minimum length of a cycle in G.)

This problem is motivated by complexity considerations [GHN00] and also by exploration of density of the homomorphism order: We write $G \prec_h H$ if there is a homomorphism from G to H but not from H to G. It is known that whenever $G \prec_h H$ holds and H is not bipartite then there is a graph K satisfying $G \prec_h K \prec_h H$. In other words, the order $\prec_h$ is dense (if we do not consider edgeless graphs). A negative solution to the Pentagon problem would have the following density consequence: for each cubic graph H for which $C_5 \prec_h H$ holds, there exists a cubic graph K satisfying $C_5 \prec_h K \prec_h H$ (see [Neš99]).

If we replaced C_5 in the statement of the problem by a longer odd cycle, we would get a stronger statement. It is known that no such strenghthening is true. This was proved by Kostochka, Nešetřil, and Smolíková [KNS98] for C_{11} (hence for all C_l with $l \geq 11$), by Wanles and Wormald [WW01] for C_9, and recently by Hatami [Hat05] for C_7. Each of these results uses probabilistic arguments (random regular graphs), no constructive proof is known.

Häggkvist and Hell [HH93] proved that for every integer g there is a graph U_g with odd girth at least g (that is, U_g does not contain odd cycle of length less than g) such that every cubic graph of odd girth at least g maps homomorphically to U_g. Here, the graph U_g may have large degrees. This leads to a weaker version of the Pentagon problem: Is it true that for every k there exists a *cubic* graph H_k of girth k and an integer g such that every cubic graph of girth at least g maps homomorphically to H_k? A particular question in this direction: does a high-girth cubic graph map to the Petersen graph?

As an approach to this, we mention a result of DeVos and Šámal [DŠ06+]: a cubic graph of girth at least 17 admits a homomorphism to the Clebsch graph. In context of the Pentagon problem, the following reformulation is particularly appealing: If G is a cubic graph of girth at least 17, then there is a cut-continuous mapping from G to C_5; that is, there is a mapping $f : E(G) \to E(C_5)$ such that for any cut $X \subseteq E(C_5)$ the preimage $f^{-1}(X)$ is a cut. (Here by *cut* we mean the edge-set of a spanning bipartite subgraph. A more thorough exposition of cut-continuous mappings can be found in [DNR02].)

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5 Critical Graphs

Does every large k-critical graph contain a large $(k-1)$-critical subgraph?

Jaroslav Nešetřil and Vojtěch Rödl, International Colloquium on Finite and Infinite Sets, Keszthely, 1973.

> *"In 1973 Paul Erdős' 60th birthday was celebrated by the International Colloquium on Finite and Infinite Sets in Keszthely, Hungary. During the conference the participants had a memorable excursion by boat on Lake Balaton, with Erdős conducting a problem session onboard and the whole crowd visiting a vineyard on the northern coast. At the boat I met two young Czechoslovaks, Jaroslav Nešetřil and Vojtěch Rödl. They had asked Erdős whether every large k-critical graph always contains a large $(k-1)$-critical subgraph. Erdős obviously liked the problem, and knowing my interest in critical graphs [Toft70] he then got us in contact."*
>
> *Bjarne Toft*

A k-chromatic graph is k-critical if all proper subgraphs are $(k-1)$-colourable. For $k = 1$, 2 and 3 the k-critical graphs are the complete 1-graph K_1, the complete 2-graph K_2 and the odd cycles, respectively. For $k = 4$ the class of k-critical graphs is already very complicated. They are the forbidden subgraphs for 3-colourability, and it is an NP-complete problem type to decide about 3-colourability, as is well known. Thus for $k = 3$ the answer to the question in the title is obviously NO since 2-critical graphs have only two vertices and odd cycles may be large. However for $k = 4$ the situation is less clear. It turned out to be not so difficult to see that the answer for $k = 4$ is YES. However, for values of $k \geq 5$ the question is still unsettled.

The case $k = 4$

Does every large 4-critical graph contain a large odd cycle? Or more general: does every large 4-critical graph contain a large cycle? The answer to this second question was first given by Kelly and Kelly [KK54]. Let $L(n)$ denote the minimum taken over all 4-critical graphs G on n vertices of the maximum length of a cycle in G (this is called the circumference of G). Kelly and Kelly proved that indeed $L(n) \to \infty$ for $n \to \infty$. How fast does $L(n)$ tend to infinity? After subsequent improvements by Dirac [Dir55] and Reid [Reid57], Gallai [Gal63] obtained the so far best upper bound, namely that there is a constant c such that $L(n) < c \log n$. This means that the growth of the length of longest cycles in 4-critical graphs may be slow. It is seemingly still not known if this is best possible. The best lower bound is of order of magnitude $\sqrt{\log n}$, due to Alon, Krivelevich and Seymour [AKS00]. A large 4-critical graph therefore contains a long cycle. Since it is 2-connected and contains odd cycles, it is an easy exercise to show that it also must contain a long odd cycle. Thus the question of Nešetřil and Rödl has answer YES for $k = 4$. The case $k = 4$ was solved in a different manner by Voss [Voss77, Voss91]. He based his affirmative solution on the theory of bridges with respect to cycles in graphs.

The cases $k \geq 5$

Toft [Toft74] characterized the class of k-critical graphs in terms of the behaviour of the $(k-1)$-critical subgraphs they contain. One easy observation is that the $(k-1)$-critical subgraphs together cover the whole k-critical graph. In other words: any edge of a k-critical graph is contained in a $(k-1)$-critical subgraph. More generally, given two edges e_1 and e_2 of a k-critical graph, there is a $(k-1)$-critical subgraph containing e_1, but not e_2. The proof is simple, yet this seems to be useful. For example it follows easily from this that a k-critical graph is $(k-1)$-edge-connected, a result first obtained in a more complicated way by Dirac [Dir53]. Another consequence of the 'distinguishing property' of $(k-1)$-critical graphs was obtained by Stiebitz [Sti87]. He proved that if all $(k-1)$-critical subgraphs of a k-critical graph G are smallest possible, i.e. they are all complete $(k-1)$-graphs, then G is also smallest possible, i.e. G is the complete k-graph. This is related to the problem of Nešetřil and Rödl, giving an upper bound for the size of a k-critical graph in terms of its $(k-1)$-critical subgraphs, in a very special case. This problem was first thought of by Nešetřil and Toft during one of their later encounters, when they together visited G. A. Dirac at Aarhus in the mid 1970'ies. This special case, when all the $(k-1)$-critical subgraphs are complete, has the flavor of perfect graph theory, but is much, much easier to deal with (the main difference is that here we deal with all subgraphs, not just the induced ones). In connection with these results, the following is an interesting question: Given two arbitrary edges e_1 and e_2 of a k-critical graph with $k \geq 5$ is there a $(k-1)$-critical subgraph containing both e_1 and e_2? The answer is not known, even when the two edges e_1 and e_2 form a path of length 2. There seems to be no easy proof—this indicates that there may well be counterexamples. An example of a k-critical graph G, $k \geq 5$, without any $(k-1)$-critical subgraph containing two given edges e_1 and e_2 would be extremely interesting. Based on such a G one would be able to get a negative answer to the question of Nešetřil and Rödl, using copies of G and Hajós' construction [Haj61]:

Hajós' construction

Let G_1 and G_2 be disjoint graphs with edges x_1y_1 and x_2y_2 respectively. Remove x_1y_1 from G_1 and x_2y_2 from G_2, identify x_1 and x_2 to one new vertex x and join y_1 and y_2 by a new edge. Use this construction recursively on q disjoint copies G_1, G_2, …, G_q of the above G, with edges e_1 and e_2, removing edge e_2 from the copy G_i and edge e_1 from the copy G_{i+1}, $i = 1$, 2, …, $q-1$, identifying two endvertices from the removed edges and joining the two other ends by a new edge. The obtained k-critical graph H is large if q is large, yet any $(k-1)$-critical subgraph of H must be contained within two consecutive copies of G and hence be small (for $k \geq 5$).

Remarks

We saw in the previous sections that an example of a k-critical graph G, $k \geq 5$, containing edges e_1 and e_2 without any $(k-1)$-critical subgraph containing e_1 and e_2 would give the answer NO to the question of Nešetřil and Rödl. We know however that the answer is YES for $k = 4$. The case $k = 4$ behaves differently, and in fact for any 4-critical graph G and any path P of length 2 in G, there is an odd cycle in G containing P. This statement follows from an argument of Dirac [Dir64] and was also obtained by Wessel [Wes81]. The above potential counterexamples to the question of Nešetřil and Rödl have separating sets of size 2. It seems likely that such counterexamples exist. However most probably no counterexample is of connectivity at least 3 (or high enough). Is it possible (easy?) to prove that any large k-critical graph of connectivity at least 3 (or at least $c(k)$) contains a large $(k-1)$-critical subgraph? If we instead of just subgraphs ask for induced subgraphs, then it is not clear what to expect and what is known and what is not. The best way to look at this is perhaps to consider vertex-k-critical graphs, i.e. graphs G that are k-chromatic and $G - x$ is $(k-1)$-colourable for all vertices x in G. If all induced vertex-i-critical subgraphs of a vertex-k-critical graph G, $k \geq 4$, are smallest possible, i.e. complete i-graphs for all $i < k$, then G is small, more precisely G is either the complete k-graph or G is an odd cycle complement (with $2k-1$ vertices). As observed first by Wessel ([Wes77], see also [Toft85]) this is an equivalent statement to the very deep strong perfect graph conjecture, recently proved by Chudnovsky, Robertson, Seymour and Thomas [CRST06^{+}].

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6 The CLIQUE Problem in Geometric Intersection Graphs

Determine the computational complexity of the CLIQUE problem restricted to intersection graphs of straight line segments in the plane.

Jan Kratochvíl and Jaroslav Nešetřil in [KN90].

"This recollection illustrates Jarik Nešetřil's gentle understanding of students' feelings, as well as his excellent instinct in finding rewarding problems. Back in the beginning of the 1980's a group of undergraduate students of Charles University in Prague was discussing an urgent matter of selecting a research seminar. None of the officially offered ones was a winning favorite, and we had just three days to file our decision at the registrar. The discussion was held during a regular Graph Theory lecture of Jarik. Instead of expelling us from the class for disturbing, he quickly got himself involved in the discusion and on the spot offered to create a seminar especially for us. Who could resist such a generous offer? He also suggested a problem to work on. In the following year or two we learned a lot about research while working on the problem of characterizing string graphs. Though we did not manage to characterize these graphs, many side results led to a conference presentation and an undergraduate publication. I revived the problem for myself about 5 years later when I finally proved NP-hardness of the recognition problem. And when presenting this negative solution to Jarik, we started discussing the complexity of optimization problems in restricted classes of graphs and the CLIQUE question was born."

Jan Kratochvíl

An abstract graph G is a $\mathcal{K}$-intersection graph, for some class $\mathcal{K}$ of sets, if the vertices of G can be represented by sets in $\mathcal{K}$ such that two vertices in G are adjacent iff the corresponding sets have a nonempty intersection.

Intersection graphs for various classes of geometric objects (e.g., straight line segments, rectangles, or disks in the plane) have been studied extensively. On the one hand, they have numerous practical applications (for instance, in frequency assignment in cellular networks [Hale80, Mal97, AHKMS01], or in map labelling [AKS97]). On the other hand, geometric intersection graphs provide a rich source of classes of graphs with interesting properties, and of challenging problems that lie at the interface between graph theory and geometry.

Deciding for graph G if it is an intersection graph of a certain kind, or computing a representation, is often computationally hard. For instance, this recognition problem is $\mathcal{NP}$-hard for intersection graphs of disks [HK01] and of segments [KM94], respectively. Furthermore, representations may require coordinates that are exponential in the size of the graph [KM94], so it is not clear if these problems even belong to the class $\mathcal{NP}$; only $\mathcal{PSPACE}$ membership is known [BK98, KM94].

A circle of questions naturally arising in the applications concern the complexity of classical hard problems, such as CLIQUE or INDEPENDENT SET, for intersection graphs. In some cases, many of such problems become tractable; for instance, CLIQUE is polynomially solvable for intersection graphs of equal-radius disks [CCJ90], or of segments with a bounded number of directions [KN90]. For both results, it is assumed that a suitable form of geometric representation is provided as part of the input, because recognition remains $\mathcal{NP}$-hard also under the additional assumptions.

Even when the problem remains hard, the geometric structure might lead to better approximation algorithms. For instance, for general graphs, it is hard to approximate the size of a maximum independent set within a factor of $n^{1-\varepsilon}$ [Hås99], for any fixed $\varepsilon > 0$. Exactly solving INDEPENDENT SET remains $\mathcal{NP}$-hard in intersection graphs of segments [KN90] (even if the segments are restricted to 2 directions) and of disks [CCJ90] (even if the disks all have the same radius). However, the problem can be approximated in polynomial time within a factor of roughly $\sqrt{n}$ for intersection graphs of segments [AM04], and within $(1 + \varepsilon)$, for any fixed $\varepsilon > 0$, in the case of disks [HMRRRS98, EJS05, Chan03]. For unit disks, even the assumption that a representation is provided can be avoided [NHK04].

Among the most tantalizing unsolved problems in this area are the complexity of the CLIQUE problem for intersection graphs of segments and of disks, respectively. For segments, the question was first posed by Kratochvíl and Nešetřil in 1990, while for disk graphs, it seems to be folklore. We remark that the above-mentioned algorithm for equal-radius disks breaks down as soon as two radii are allowed, while for the case of segments with a bounded number d of directions, the runtime of the algorithm depends exponentially on d. As for results in the opposite direction, every complement of a planar graph can be represented as an intersection graph of convex polygons in the plane [KK98]. It follows that CLIQUE is $\mathcal{NP}$-hard for such graphs, because INDEPENDENT SET is hard for planar graphs. The polygons used in the representation are of nonconstant complexity. There are results for two types of geometric objects of constant complexity. The first type are intersection graphs of angles [MP92], where an angle consist of one horizontal and one vertical segment sharing a common endpoint. If all the angles are "upper" ones, say, then CLIQUE is polynomially solvable, but if opposite angles are allowed, then the problem is $\mathcal{NP}$-hard. The second type are intersection graphs of ellipses [AW05]. For these, CLIQUE is $\mathcal{NP}$-hard. In fact, for sufficiently small $\delta > 0$, even approximation within $(1+\delta)$ is hard. Moreover, this continues to hold even if for all ellipses, the ratio between the two principal radii is required to be any given constant ρ, $1 < \rho < \infty$. However, the "limit cases" of circles ("$\rho = 1$") and of segments ("$\rho = \infty$"), respectively, remain open.

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