ESSENTIALLY INFINITE COLOURINGS OF HYPERGRAPHS

B. BOLLOBÁS, Y. KOHAYAKAWA, V. RÖDL, M. SCHACHT AND A. TARAZ

Abstract

We consider edge colourings of the complete r-uniform hypergraph $K_n^{(r)}$ on n vertices. How many colours may such a colouring have if we restrict the number of colours locally? The local restriction is formulated as follows: for a fixed hypergraph H and an integer k we call a colouring (H,k)-local if every copy of H in the complete hypergraph $K_n^{(r)}$ receives at most k different colours.

We investigate the threshold for k that guarantees that every (H,k)-local colouring of $K_n^{(r)}$ must have a globally bounded number of colours as $n \to \infty$, and we establish this threshold exactly. The following phenomenon is also observed: for many H (at least in the case of graphs), if k is a little over this threshold, the unbounded (H,k)-local colourings exhibit their colourfulness in a "sparse way"; more precisely, a bounded number of colours are dominant while all other colours are rare. Hence we study the threshold k_0 for k that guarantees that every (H,k)-local colouring γ_n of $K_n^{(r)}$ with $k \le k_0$ must have a globally bounded number of colours after the deletion of up to εn^r edges for any fixed $\varepsilon > 0$ (the bound on the number of colours is allowed to depend on H and ε only); we think of such colourings γ_n as "essentially finite". As it turns out, every *essentially infinite* colouring is closely related to a non-monochromatic canonical Ramsey colouring of Erdős and Rado. This second threshold is determined up to an additive error of 1 for every hypergraph H. Our results extend earlier work for graphs by Clapsadle and Schelp [*Local edge colorings that are global*, J. Graph Theory **18** (1994), no. 4, 389–399] and by the first two authors and Schelp [*Essentially infinite colourings of graphs*, J. London Math. Soc. (2) **61** (2000), no. 3, 658–670]. We also consider a related question for colourings of the integers and arithmetic progressions.

1. *Introduction*

For an integer $r \ge 2$, let $K_n^{(r)}$ be the complete r-uniform hypergraph with vertex set $[n] = \{1,\dots,n\}$. We identify hypergraphs with their edge sets, e.g., $K_n^{(r)} = \binom{[n]}{r}$, the family of all subsets of $[n]$ with cardinality r. In the following, we consider colourings $\gamma_n\colon K_n^{(r)} \to \mathbb{Z}$ and the set of all such colourings will be denoted by $\mathcal{C}_n^{(r)}$. For a given colouring, we say that a vertex x *sees* colour i in this colouring if x is contained in an edge of colour i.

Fix an r-uniform hypergraph H and a positive integer k. A colouring $\gamma_n \in \mathcal{C}_n^{(r)}$ of $K_n^{(r)}$ is called (H,k)-*local* if every copy of H in $K_n^{(r)}$ has its edges coloured with *at most* k different colours. Local colourings were introduced by Truszczyński [14]. We shall denote the set of all such colourings by $\mathcal{L}_n^{(r)}(H,k)$.

We are interested in the structure of the colourings in $\mathcal{L}_n^{(r)}(H,k)$. In particular, we investigate what one can say about the *total* number of colours used in a colouring in $\mathcal{L}_n^{(r)}(H,k)$. It turns out that this total number is uniformly bounded (as $n \to \infty$) as long as k is less than or equal to a certain threshold $\mathrm{Fin}(H)$. Our first main result gives a simple, explicit expression for $\mathrm{Fin}(H)$ (see Theorem 2 below). This result generalizes a result of Clapsadle and Schelp [2], who investigated this problem for graphs, that is, the case $r = 2$.

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By definition, the (H,k)-local colourings γ_n of $K_n^{(r)}$ with k just above the threshold $\mathrm{Fin}(H)$ may use an unbounded number of colours (as $n \to \infty$). However, for many H, for k just a little above $\mathrm{Fin}(H)$, it turns out that only a uniformly bounded number of colours occur a large number of times in γ_n: if we restrict γ_n to some $(1-o(1))\binom{n}{r}$ edges of $K_n^{(r)}$, we again have only a uniformly bounded number of colours. We call such colourings γ_n "essentially finite". To be precise, we call a family of colourings $\{\gamma_n\}$ *essentially finite* if for any $\varepsilon > 0$ there is an integer T such that all but at most $\varepsilon\binom{n}{r}$ edges of $K_n^{(r)}$ are coloured by at most T colours by all colourings γ_n in the family.

We investigate a second threshold, which we denote by $\mathrm{EssFin}(H)$, related to essential finiteness of colourings. We have $\mathrm{EssFin}(H) = k_0$ if and only if k_0 is the maximal integer such that *every* (H,k)-local colouring γ_n of $K_n^{(r)}$ with $k \leq k_0$ is essentially finite. In what follows, we determine $\mathrm{EssFin}(H)$ up to an additive error of 1 (see Theorem 5). This result generalizes a result of the first and second authors together with Schelp [1], who investigated the parameter $\mathrm{EssFin}(H)$ for graphs H. As in that previous paper, most of the work will lie in identifying certain unavoidable substructures in *essentially infinite* colourings, that is, colourings that are not essentially finite. The main result that we obtain in this direction has, unfortunately, a somewhat technical look; see Theorem 8 in Section 2.3. Our estimate for $\mathrm{EssFin}(H)$ follows directly from Theorem 8 (see Section 4.4).

By definition, we have

$$\mathrm{Fin}(H) \leq \mathrm{EssFin}(H). \tag{1.1}$$

We shall show that, at least in the case of graphs, we have strict inequality in (1.1) in most cases (we also exhibit examples of graphs H for which equality holds). See Corollary 6.

We also consider essentially infinite colourings of the integers, and we prove that they necessarily contain arbitrarily long 'rainbow' (totally multicoloured) arithmetic progressions; see Theorem 10. It turns out that this result is of a much simpler form than the results for essentially infinite colourings of hypergraphs, and the proof is correspondingly more pleasant. In fact, we close this paper with a short section, Section 5, in which Theorem 10 is proved.

In the next section, we shall give a detailed account of our results, together with the necessary definitions, some of which will be introduced rather gently, as they do require some getting used to. Most of the work will be in the two sections that follow. In Section 3, we shall prove our explicit formula for $\mathrm{Fin}(H)$, and in Section 4 we shall investigate essentially infinite colourings of hypergraphs and prove our estimate on $\mathrm{EssFin}(H)$.

2. *Statement of the main results*

2.1. *Warm-up*

Suppose one tries to colour the edges of $K_n^{(r)}$ using as many colours as possible, and the only restriction is that it has to be an (H,k)-local colouring. Let us denote the maximum number of colours that one can achieve by

$$t(H,k,n) := \max\left\{|\operatorname{im}(\gamma)|\colon\ \gamma \in \mathcal{L}_n^{(r)}(H,k)\right\}.$$

For given H and k, we are interested in how $t(H,k,n)$ behaves as a function in n.

To warm up, consider the following example. Let $r=2$ and $H = K_5$. We have that

$$t(K_5,1,n) = 1 \qquad \text{and} \qquad t(K_5,2,n) = 2\,. \tag{2.1}$$

Indeed, the former is trivial and the latter is immediately verified as follows. Suppose for a contradiction that a colouring $\gamma \in \mathcal{L}_n^{(2)}(K_5,2)$ uses three different colours c_1, c_2, and c_3 on the edges $\{x_1,y_1\}$, $\{x_2,y_2\}$, and $\{x_3,y_3\}$. If these six vertices are not pairwise distinct, then they are contained in a copy of K_5 picking up 3 colours, which is forbidden. Also, the edge $\{x_1, x_2\}$ cannot have colour c_3, so w.l.o.g. it has colour c_1. But then the vertices x_1, x_2, y_2, x_3, y_3 span a K_5 with 3 colours. This shows that indeed $t(K_5, 2, n) = 2$.

Next we claim that

$$t(K_5, 3, n) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1\,.$$

This can be verified by considering the colouring $\gamma_{\mathrm{match},n} \in \mathcal{C}_n^{(2)}$, which assigns to each edge of a fixed matching of size $\lfloor n/2 \rfloor$ a new colour and colours all the other edges with an extra colour 0. It is clear that $\gamma_{\mathrm{match},n} \in \mathcal{L}_n^{(2)}(K_5, 3)$, because any copy of a K_5 can contain at most two matching edges, whereas $|\operatorname{im}(\gamma_{\mathrm{match},n})| = \lfloor n/2 \rfloor + 1$. Thus, when we move from $t(K_5, 2, n)$ to $t(K_5, 3, n)$, the function suddenly changes from bounded to unbounded.

2.2. *Finite local colourings*

One of the aims of this paper is to determine, for a given r-uniform hypergraph H, the maximal integer k for which $t(H, k, n)$ is bounded. Formally, we are interested in

$$\begin{aligned}\operatorname{Fin}(H) := \max \big\{ k \in \mathbb{N} \colon\ \exists T \in \mathbb{N} \text{ such that for every } n \in \mathbb{N}& \\ \text{every } \gamma \in \mathcal{L}_n^{(r)}(H, k) \text{ is such that } |\operatorname{im}(\gamma)| \leq T \big\}\,.&\end{aligned}$$

The earlier discussion shows that $\operatorname{Fin}(K_5) = 2$. A theorem by Clapsadle and Schelp gives a simple and elegant description of $\operatorname{Fin}(H)$ for any *graph* H.

Theorem 1 (Clapsadle & Schelp [2]). *Let H be a graph with at least two edges. Let $\nu(H)$ denote the cardinality of a maximum matching in H and $\Delta(H)$ the maximum degree of a vertex in H. Then*

$$\operatorname{Fin}(H) = \min\{\nu(H), \Delta(H)\}\,. \qquad \square$$

Clapsadle and Schelp were especially interested in the situation when $t(H, k, n) = k$. They observed that in that case H must contain every graph on k edges as a subgraph and conjectured that the converse is also true.

One of the aims of this paper is to generalize Theorem 1 to hypergraphs. For this we introduce the following definitions. An r-uniform *sunflower* (or Δ-*system*) with *core* L is an r-uniform hypergraph with set of edges $\{e_1, \ldots, e_s\}$ such that $e_i \cap e_j = L$ for all $i \neq j$. We allow $L = \emptyset$; in that case, a sunflower is simply a matching. The sets e_i are the edges and the sets $p_i := e_i \setminus L$ are the *petals*. The cardinality of the core $|L|$ is the *type* and s, the number of edges (or petals), is the *size* of the sunflower. If $\ell = |L|$ is the type and s is the size of the sunflower, we shall speak of an (ℓ, s)-sunflower and we shall denote it by $S = (L, p_1, \ldots, p_s)$.

Furthermore, for $\ell = 0, \ldots, r$ we denote by $\Delta_\ell(H)$ the maximum size of a sunflower of type ℓ in a hypergraph H. Obviously if H is a graph, i.e., $r = 2$, then we have $\Delta_1(H) = \Delta(H)$ and $\Delta_0(H) = \nu(H)$. Consequently, the following theorem is an extension of Theorem 1 from graphs to r-uniform hypergraphs.

Theorem 2. *For any r-uniform hypergraph H with at least two edges we have*

$$\operatorname{Fin}(H) = \min_{0 \leq \ell < r} \Delta_\ell(H)\,. \tag{2.2}$$

The upper bound, $\operatorname{Fin}(H) \leq \min_{0 \leq \ell < r} \Delta_\ell(H)$, is easy to verify and we give the proof below. The lower bound is harder to obtain; its proof will be given in Section 3.

Proof of the upper bound in Theorem 2. Suppose H is an r-uniform hypergraph with at least two edges. We shall show that

$$\mathrm{Fin}(H) < \min_{0\leq\ell<r} \Delta_\ell(H) + 1 =: k\,. \tag{2.3}$$

In order to verify (2.3) we give an example of a sequence of (H,k)-local colourings $\gamma_n \in \mathcal{C}_n^{(r)}$ such that $|\operatorname{im}(\gamma_n)|$ is unbounded.

By definition of k in (2.3) there is some $\ell_0 \in \{0,\dots,r-1\}$ so that $k > \Delta_{\ell_0}(H)$. Fix in $K_n^{(r)}$ an $(\ell_0,\bar{n})$-sunflower $S = (L, p_1,\dots,p_{\bar{n}})$, with $\bar{n} := \lfloor (n-\ell_0)/(r-\ell_0)\rfloor$. Now consider the following colourings $\gamma_n \in \mathcal{C}_n^{(r)}$: colour the edges of S with colours $1,\dots,\bar{n}$ and colour all other edges with colour 0. As H contains no (ℓ_0,k)-sunflower, every copy of H in $K_n^{(r)}$ cannot see more than $k-1$ colours from those appearing in S, and thus at most k different colours in total. Hence γ_n is (H,k)-local, but obviously $|\operatorname{im}(\gamma_n)| \to \infty$ as $n\to\infty$. □

2.3. *Essentially finite colourings*

Let us return to our warm-up example. Notice that in the $(K_5,3)$-local colouring $\gamma_{\mathrm{match},n}$ all but one colour was in fact only used once. In other words, $\gamma_{\mathrm{match},n}$ did use an unbounded number of colours, but only in a very sparse way. We would like to know how large we can make k before there exists a colouring in $\mathcal{L}_n^{(r)}(H,k)$ that uses a lot of colours in an "essential way", by which we mean that there are still unboundedly many colours after removing, say, some $f(n)$ edges.

For a moment suppose $f(n)$ is of order $o(n^2)$. We modify the colouring $\gamma_{\mathrm{match},n}$ and consider $\gamma'_{\mathrm{match},n} \in \mathcal{C}_n^{(2)}$, where we have $n^2/(8f(n))$ vertex disjoint copies of the complete bipartite graph $K_{4f(n)/n,4f(n)/n}$, each of its own colour, and the other edges receive colour 0. It is easy to check that $\gamma'_{\mathrm{match},n}$ uses an unbounded number of colours, even after the deletion of any $f(n)$ edges. On the other hand, $\gamma'_{\mathrm{match},n}$ is still $(K_5,3)$-local. Summarizing the above, we note that while the original colouring $\gamma_{\mathrm{match},n}$ was an example of a $(K_5,3)$-local colouring which remains unbounded after deleting up to cn edges for any $c<\frac{1}{2}$, the modified colouring $\gamma'_{\mathrm{match},n}$ witnesses that the same remains true if we remove up to $o(n^2)$ edges. Hence, if we want to guarantee that our colouring γ uses boundedly many colours after deleting up to $o(n^2)$ edges, we cannot allow more colours locally. Hence for $r=2$ let us consider functions $f(n)=\varepsilon\binom{n}{2}$ for some $\varepsilon>0$ and, more generally, we allow the deletion of up to $\varepsilon\binom{n}{r}$ edges in $K_n^{(r)}$.

Definition 3. Let $r\geq 2$ be an integer, $t\in\mathbb{N}$ and $\varepsilon>0$. We say a colouring $\gamma\in\mathcal{C}_n^{(r)}$ is *(ε,t)-bounded* if there exists a subgraph $G\subseteq K_n^{(r)}$ such that $|G|\geq(1-\varepsilon)\binom{n}{r}$ and $|\gamma(G)|\leq t$. Moreover, we say that a family of colourings $\{\gamma_n\in\mathcal{C}_n^{(r)}\colon n\in\mathbb{N}\}$ is *essentially finite* if for every $\varepsilon>0$ there is an integer T such that any γ_n in the family is (ε,T)-bounded. Otherwise, we say that the family is *essentially infinite*. When there is no danger of confusion, we refer to the colourings themselves as essentially finite and essentially infinite.

For a given r-uniform hypergraph H, we are interested in the maximum integer k that guarantees that every (H,k)-local colouring is (ε,T)-bounded for every $\varepsilon>0$ and $T=T(\varepsilon)$. More precisely, we define

$$\mathrm{EssFin}(H) := \max\Big\{k\in\mathbb{N}\colon\ \forall\varepsilon>0\,\exists T\in\mathbb{N} \text{ such that for every } n\in\mathbb{N}$$
$$\text{every } \gamma\in\mathcal{L}_n^{(r)}(H,k) \text{ is } (\varepsilon,T)\text{-bounded}\Big\}\,.$$

Although the definition of $\mathrm{EssFin}(H)$ looks a little overwhelming at first, observe that it is similar to that of $\mathrm{Fin}(H)$, except that we are now allowed to remove $\varepsilon\binom{n}{r}$ edges *before* we count the colours. This way we may be able to allow for a larger number of colours locally while remaining *essentially finite* globally.

In order to get used to $\mathrm{EssFin}(H)$, we return to our example $H = K_5$ and show that

$$\mathrm{EssFin}(K_5) = 3\,. \tag{2.4}$$

For that we consider the following two colourings $\gamma_{\min,n}$ and $\gamma_{\mathrm{bip},n} \in \mathcal{C}_n^{(2)}$. For every edge $e = \{x, y\} \in \binom{[n]}{2}$ with $x < y$, let

$$\gamma_{\min,n}(e) := x,$$
$$\gamma_{\mathrm{bip},n}(e) := \begin{cases} x & \text{if } x \le \frac{n}{2} < y\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Observe that both $\{\gamma_{\min,n} \colon n \in \mathbb{N}\}$ and $\{\gamma_{\mathrm{bip},n} \colon n \in \mathbb{N}\}$ are essentially infinite. Moreover, $\gamma_{\min,n}$ is $(K_5, 4)$-local, but not $(K_5, 3)$-local; $\gamma_{\mathrm{bip},n}$ is not even $(K_5, 4)$-local. Therefore, $\gamma_{\min,n}$ shows that $\mathrm{EssFin}(K_5) < 4$.

On the other hand, let us sketch the proof of $\mathrm{EssFin}(K_5) \ge 3$. We need to show that for every $\varepsilon > 0$ there exists an integer T so that every $(K_5, 3)$-local colouring γ is (ε, T)-bounded. So suppose $\{\gamma_n \in \mathcal{C}_n^{(r)} \colon n \in \mathbb{N}\}$ is essentially infinite. Then it follows from the results in [1] that for sufficiently large n the colouring γ_n must exhibit a "local spot" that is (in some sense) at least as rich in colours as either $\gamma_{\min,n}$ or $\gamma_{\mathrm{bip},n}$. But then γ_n cannot be $(K_5, 3)$-local, as neither $\gamma_{\min,n}$ nor $\gamma_{\mathrm{bip},n}$ are, which yields $\mathrm{EssFin}(K_5) \ge 3$.

In order to formalize this for arbitrary hypergraphs, we generalize the colourings $\gamma_{\min,n}$ and $\gamma_{\mathrm{bip},n}$ and describe a family $\mathcal{CEIC}_n^{(r)} \subseteq \mathcal{C}_n^{(r)}$ of *canonical essentially infinite colourings* of $K_n^{(r)}$, which turn out to be unavoidable for every essentially infinite colouring.

DEFINITION 4. Let $r \ge 2$ and $\ell \in [r]$. A vector $\tau = (\tau_1, \dots, \tau_\ell) \in \mathbb{N}_0^\ell$ of non-negative integers is an *ℓ-type* if $\sum_{i \in [\ell]} \tau_i = r$. We call τ *degenerate* if $\tau_i = 0$ for some $i \in [l]$ and *non-degenerate* otherwise. We denote the set of all non-degenerate types by

$$\mathcal{T}^{(r)} = \bigcup_{\ell \in [r]} \left\{ \tau = (\tau_1, \dots, \tau_\ell) \colon \sum_{i \in [\ell]} \tau_i = r \text{ and } \tau_i > 0 \text{ for all } i \in [\ell] \right\}$$

For a family of mutually disjoint sets $W_1, \dots, W_\ell \subseteq [n]$ and an ℓ-type τ we say an edge $e \in K_n^{(r)}$ *has type* τ if $|e \cap W_i| = \tau_i$ for every $i \in [\ell]$. We denote the family of all edges of type τ by $(W_1, \dots, W_\ell)\langle\tau\rangle$.

For fixed integers r and n we consider for every $\ell \in [r]$ a partition Π_ℓ of $[n]$ with ℓ partition classes $I_i(\ell, n)$ for $i \in [\ell]$ defined by

$$I_i(\ell, n) := \left\{ \left\lfloor \frac{(i-1)n}{\ell} \right\rfloor + 1, \dots, \left\lfloor \frac{in}{\ell} \right\rfloor \right\} \quad 1 \le i \le \ell \le r\,.$$

Now we define the canonical essentially infinite colourings $\chi_{\tau,j_1,n}^{(r)}$ for every non-degenerate ℓ-type $\tau = (\tau_1, \dots, \tau_\ell)$ and $j_1 \in [\tau_1]$ by setting, for every $e = \{v_1 < \dots < v_r\} \in K_n^{(r)}$,

$$\chi_{\tau,j_1,n}^{(r)}(e) := \begin{cases} v_{j_1} & \text{if } e \in (I_1(\ell, n), \dots, I_\ell(\ell, n))\langle\tau\rangle\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.5}$$

We let

$$\mathcal{CEIC}_n^{(r)} := \left\{ \chi_{\tau,j_1,n}^{(r)} \colon \tau \in \mathcal{T}^{(r)} \text{ and } j_1 \in [\tau_1] \right\}.$$

Note that for example $\gamma_{\min,n} = \chi_{\tau,j_1,n}^{(2)}$ for the 1-type $\tau = (2)$ with $j_1 = 1 \in [2]$, and $\gamma_{\mathrm{bip},n}$ corresponds to $\chi_{\tau,j_1,n}^{(2)}$ for the 2-type $\tau = (1, 1)$ with $j_1 = 1 \in [1]$.

It is easy to see that for any $\tau \in \mathcal{T}^{(r)}$ and $j_1 \in [\tau_1]$ the family $\{\chi^{(r)}_{\tau,j_1,n}\colon n \in \mathbb{N}\}$ is essentially infinite. (Note that $\tau \in \mathcal{T}^{(r)}$ yields $\tau_1 > 0$ here.) Consequently,

$$\mathrm{EssFin}(H) < \max\left\{\left|\chi^{(r)}_{\tau,j_1,n}(H_0)\right|\colon H_0 \text{ is a copy of } H \text{ in } K_n^{(r)}\right\} \tag{2.6}$$

for any $\tau \in \mathcal{T}^{(r)}$, $j_1 \in [\tau_1]$, and $n \geq r \cdot v_H$. Let us set

$$\Xi(H) := \min_{\tau,\, j_1} \max_{H_0} \left|\chi^{(r)}_{\tau,j_1,r\cdot v_H}(H_0)\right|,$$

where the minimum is taken over all $\tau \in \mathcal{T}^{(r)}$ and $j_1 \in [\tau_1]$ and the maximum is taken over all copies H_0 of H in $K^{(r)}_{r\cdot v_H}$. The following theorem states that the upper bound in (2.6) is almost tight.

Theorem 5. *For every r-uniform hypergraph H on v_H vertices with at least two edges*

$$\Xi(H) - 2 \leq \mathrm{EssFin}(H) \leq \Xi(H) - 1. \tag{2.7}$$

Moreover, if $r = 2$, then

$$\mathrm{EssFin}(H) = \min\left\{\max_{H_0}\left\{|\gamma_{\min,2v_H}(H_0)|\right\},\ \max_{H_0}\left\{|\gamma_{\mathrm{bip},2v_H}(H_0)|\right\}\right\} - 1\,, \tag{2.8}$$

where the maxima are taken over all copies H_0 of H in $K^{(2)}_{r\cdot v_H}$.

By definition $\mathrm{EssFin}(H) \geq \mathrm{Fin}(H)$ for every hypergraph H. The next corollary says that, in fact, the inequality is strict for "most" graphs ($r = 2$). For an integer $\ell \geq 2$ we denote by MC_ℓ the "matched clique" of order ℓ, i.e., the graph with 2ℓ vertices $\{v_1, \dots, v_\ell, u_1, \dots, u_\ell\}$ with $v_1, \dots, v_\ell$ spanning a complete graph K_ℓ and additional matching edges $\{v_i, u_i\}$ for every $i \in [\ell]$.

Corollary 6. *Suppose H is a connected graph with at least two edges and $v_H \geq 6$ vertices. If, moreover, one of the following holds:*
(i) $\max\{\nu(H), \Delta(H)\} \geq \min\{\nu(H)\,, \Delta(H)\} + 2$, *or*
(ii) v_H *is odd, or*
(iii) v_H *is even, but H is not a subgraph of $MC_{v_H/2}$,*
then $\mathrm{EssFin}(H) > \mathrm{Fin}(H)$.
On the other hand, $\mathrm{EssFin}(MC_\ell) = \mathrm{Fin}(MC_\ell)$ *for every $\ell \geq 2$.* □

Corollary 6 follows from Theorems 1 and 5. While (*i*) and the last statement are immediate, (*ii*) and (*iii*) require some additional arguments, which will be omitted.

Recall from the short discussion about $\mathrm{EssFin}(K_5) = 3$ (see (2.4)) that the main work in determining $\mathrm{EssFin}(H)$ and thus in establishing Theorem 5 is needed for the lower bound, and that our approach is to show that any essentially infinite colouring must exhibit a local spot that is at least as colourful as a colouring in $\mathcal{CEIC}^{(r)}_m$ for some sufficiently large m. To make this precise, we need a few more definitions. For any edge $e = \{v_1, \dots, v_r\} \subseteq [n]$ with $v_1 < \dots < v_r$ and any set of indices $J = \{j_1, \dots, j_\ell\} \subseteq [r]$ we let $e[J] := \{v_{j_1}, \dots, v_{j_\ell}\}$. Moreover, if $J = \emptyset$, then $e[J] = \emptyset$. With that notation a classical theorem of Erdős and Rado may be stated as follows.

Theorem 7 (Erdős & Rado [5]). *For all integers $q \geq r \geq 2$, there exists an integer $n = n(q, r)$ so that for every colouring $\gamma \in \mathcal{C}^{(r)}_n$ there is a set $W \subseteq [n]$ with $|W| = q$ and there is a set $J \subseteq [r]$ such that*

$$\gamma(e) = \gamma(e') \;\Leftrightarrow\; e[J] = e'[J]$$

for all edges e, $e' \in \binom{W}{r}$. □

In this context, Ramsey's theorem [12] says that if the total number of colours used by γ is bounded, then one can ask for $J = \emptyset$ or, equivalently, for a monochromatic complete subgraph of order q. With the aim of proving Theorem 5, among others, we shall prove a complementary result: if γ is sufficiently rich in colours, then we can ask for $J \neq \emptyset$ or, equivalently, for a multicoloured subgraph. As we shall see in Section 4.4, Theorem 5 is a simple consequence of the following theorem, which is one of the main results of this paper.

THEOREM 8. *For all integers $q \geq r \geq 2$ and for every $\varepsilon > 0$, there are integers T and n_0 so that for every $n \geq n_0$ and every colouring $\gamma \in \mathcal{C}_n^{(r)}$ that is not (ε, T)-bounded, there exist an integer $\ell \in [r]$, a non-degenerate ℓ-type $\tau = (\tau_1, \ldots, \tau_\ell)$, a set $\emptyset \neq J_1 \subseteq [\tau_1]$, and a family $\mathcal{W} = \{W_1, \ldots, W_\ell\}$ of mutually disjoint sets, each of cardinality q, such that for all edges e, $e' \in (W_1, \ldots, W_\ell)\langle\tau\rangle$*

$$\gamma(e) = \gamma(e') \;\Rightarrow\; (e \cap W_1)[J_1] = (e' \cap W_1)[J_1]\,.$$

Moreover, if $e \in (W_1, \ldots, W_\ell)\langle\tau'\rangle$ for a degenerate ℓ-type τ', then

$$\gamma(e) \notin \big\{\gamma(f)\colon\, f \in (W_1, \ldots, W_\ell)\langle\tau\rangle\big\}. \tag{2.9}$$

Theorem 8 extends earlier results of Bollobás, Kohayakawa, and Schelp [1] from graphs to hypergraphs. For the proof of Theorem 8, presented in Section 4, we shall develop a partite version of the result of Erdős and Rado, which might be of independent interest (see Theorem 24). Theorem 5 may be deduced from Theorem 8; see Section 4.4.

2.4. *Rainbow colourings of arithmetic progressions*

We also obtain a very much related result for arithmetic progressions. The following result of Erdős and Graham (see also [11] for an elementary proof) may be viewed as an analogue of Theorem 7 for arithmetic progressions.

THEOREM 9 (Erdős & Graham [4]). *For every integer $k \geq 3$ there exists an integer n_0 such that for every $n \geq n_0$ and every colouring $\gamma\colon [n] \to \mathbb{Z}$ there exists a k-term arithmetic progression $A \subseteq [n]$ which is either monochromatic or injective, i.e., $|\gamma(A)|$ is either 1 or k.* □

This may be viewed as a canonical version of van der Waerden's theorem [15], which says that if $|\operatorname{im}(\gamma)|$ is bounded (independent of n), then one is guaranteed to have a monochromatic arithmetic progression. Following the approach in the preceding section, we wish to obtain a condition on the colouring that guarantees an injective arithmetic progression.

Let us first observe that it is not enough to simply require that the colouring should use an unbounded number of colours. Consider the colouring $\gamma_n^{\mathrm{AP}}\colon [n] \to \mathbb{Z}$, which assigns colour i to every integer $m = 3^i x$, where x is not divisible by 3. Clearly, $|\operatorname{im}\gamma_n^{\mathrm{AP}}| \to \infty$ as $n \to \infty$. Moreover, let us observe that γ_n^{AP} yields no 3-term arithmetic using three colours. Indeed suppose for a contradiction that the integers $3^a x < 3^b y < 3^c z$ receive the three distinct colours a, b, and c and form a 3-term arithmetic progression. Suppose first that $a < c$. Then $2 \cdot 3^b y = 3^a x + 3^c z = 3^a(3^{c-a} z + x)$. As y and x are not divisible by 3, this implies that $b = a$. The same argument works for the case $a > c$.

Hence, similarly to the graph and hypergraph cases, we need a condition that guarantees that the colouring uses a lot of colours in an "essential way". Following our previous approach, we introduce the following definition. A colouring $\gamma\colon [n]\to\mathbb{Z}$ is (ε,t)-*bounded* if there exists a set $X\subseteq[n]$ with $|X|\geq n-\varepsilon n$, such that $|\gamma(X)|\leq t$. This may be viewed as a natural extension of Definition 3 to "1-uniform hypergraphs".

Theorem 10. *For every integer $k\geq 3$ and for every real $\varepsilon>0$, there exist integers n_0 and T such that for every $n\geq n_0$ and every colouring $\gamma\colon [n]\to\mathbb{Z}$ the following holds. If γ is not (ε,T)-bounded, then there exists an injective k-term arithmetic progression in $[n]$.*

Notice that for every function $f(n)$ of order $o(n)$ and $X\subseteq[n]$ with $|X|\geq n-f(n)$ we have that the colouring γ_n^{AP} defined above satisfies $|\gamma_n^{\mathrm{AP}}(X)|\geq T$ for any fixed T as long as n is sufficiently large. Consequently, the hypothesis on γ in Theorem 10 is best possible. The proof of Theorem 10, to be presented in Section 5, is based on a quantitative version of Szemerédi's theorem [13].

3. *Globally bounded local colourings*

In this section we prove Theorem 2. We split this section in a few subsections to make the reading a little easier. In Section 3.1, we give some further definitions and state the auxiliary lemmas that we shall need. In particular, we state Lemmas 14 and 16, which are central to the proof. In this section, we also sketch the approach we take in the proof of Theorem 2. The actual proof of this theorem is given in Section 3.2. Finally, we give the proofs of Lemmas 14 and 16 in Section 3.3.

3.1. *Auxiliary lemmas*

We first recall and extend some of the definitions given earlier. A *sunflower* with *core* L is an r-uniform hypergraph whose edges $e_1,\dots,e_s$ satisfy the property $e_i\cap e_j=L$ for all $i\neq j$. The sets $p_i:=e_i\setminus L$ are the *petals*, $|L|$ is the *type*, and the number of edges (or petals) is the *size* of the sunflower. If $\ell=|L|$ is the type and s is the size of the sunflower, we shall speak of an (ℓ,s)-sunflower and we shall denote it by $S=(L,p_1,\dots,p_s)$. Observe that we shall be talking about sunflowers both in $K_n^{(r)}$ and in H. To differentiate between those two kinds of sunflowers, we shall follow the convention that sub-hypergraphs of H will have dashes, e.g., $S'=(L',p'_1\dots,p'_s)$. Moreover, the letter k (as well as $\widehat{k}$, $\widetilde{k}$, $\bar{k}$) will denote bounds on the *local* number of colours in sunflowers contained in $K_n^{(r)}$, whereas T will give bounds on the global number of colours used in $K_n^{(r)}$, i.e., $|\operatorname{im}(\gamma)|$.

Definition 11. For a given colouring $\gamma\in\mathcal{C}_n^{(r)}$, an (ℓ,k)-sunflower $S(L,p_1,\dots,p_k)\subseteq K_n^{(r)}$ will be called *injective* if all its k edges receive different colours. We say γ is (ℓ,k)-*local* if it yields no injective $(\ell,k+1)$-sunflower in $K_n^{(r)}$. In other words, γ is (ℓ,k)-local if it is (S_ℓ,k)-local for every sunflower S_ℓ of type ℓ. Moreover, if γ is (ℓ,k)-*local* for every $\ell=0,\dots,r-1$, then it will be called k-*local*.

To prove Theorem 2, it suffices to verify the lower bound in (2.2). (For the proof of the upper bound, see the paragraph following Theorem 2 in Section 2.2.) In other words, we have to show that for every r-uniform hypergraph H with at least two edges

$$\operatorname{Fin}(H)\geq\min_{0\leq\ell<r}\Delta_\ell(H)=:s_H\,,\tag{3.1}$$

where $\Delta_\ell(H)$ is the maximum size of a sunflower of type ℓ in H. This means we have to show that for every n, every (H,s_H)-local colouring $\gamma\in\mathcal{C}_n^{(r)}$ is T-bounded, i.e., $|\operatorname{im}(\gamma)|\leq T$ for some constant $T=T(H)$ independent of n. The next proposition shows that it is sufficient to show that every (H, s_H)-local colouring γ is k-local for some constant $k = k(H)$, i.e, it does not yield an injective $(\ell, k+1)$-sunflower for all $0 \le \ell < r$.

Proposition 12. *For all integers k, $r \ge 2$ there exists an integer $T = T(k,r)$ such that for every n and every k-local colouring $\gamma \in \mathcal{C}_n^{(r)}$ we have $|\operatorname{im}(\gamma)| \le T$.*

We easily deduce Proposition 12 from the following theorem of Erdős and Rado.

Theorem 13 (Erdős & Rado [6]). *If an r-uniform hypergraph contains more than $r!k^r$ edges, then it contains an $(\ell, k+1)$-sunflower for some $0 \le \ell < r$.* □

In fact for $k = 3$ Erdős offered \$1000 for the proof that $r!$ can be replaced by c^r for some constant c independent of r. This conjecture is still open and currently the best bound for $k = 3$ is due to Kostochka [9].

Proof of Proposition 12. Let integers k, $r \ge 2$ be given. Set $T = r!k^r$ and suppose that $\gamma \in \mathcal{C}_n^{(r)}$ is k-local, but fails to satisfy $|\operatorname{im}(\gamma)| \le T$. Then Theorem 13 immediately implies that any collection of $|\operatorname{im}(\gamma)|$ mutually different coloured hyperedges of $K_n^{(r)}$ contains an injective $(\ell, k+1)$-sunflower for some $0 \le \ell < r$, which is a contradiction to the assumption that γ is k-local. □

We deduce (3.1) from Lemmas 14 and 16. Before we formally state these somewhat "dry" lemmas let us briefly describe them and discuss their relevance for the proof of (3.1) under the assumption $s_H \ge 2$. Recall that $\mathcal{L}_n^{(r)}(H, s_H)$ denotes the set of all (H, s_H)-local colourings of $K_n^{(r)}$. In view of Proposition 12 it suffices to show that every colouring $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ is k-local for some constant $k = k(H)$. Lemma 14 roughly says that if $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ is such that it yields an injective (i, k_i)-sunflower in $K_n^{(r)}$ for some "large" k_i, then it either admits an injective $(j, k_i - r)$-sunflower with $j > i$ (see part (a) of Lemma 14) or we infer that H contains a subhypergraph H' with a special structure (see part (b)). The structure of H' and the existence of a "large" injective (i, k_i)-sunflower in $K_n^{(r)}$ under γ, then (see Lemma 16) also imply that there is an injective $(j, \widetilde{k})$-sunflower with $j > i$, where $\widetilde{k}$ is of similar order as k.

In other words, Lemmas 14 and 16 show that if an (H, s_H)-local colouring γ is not k-local for some "large" k, i.e., γ admits a "large" injective sunflower of type i for some $i = 0, \dots, r-1$, then it necessarily admits a similarly "large" sunflower of type $j > i$ and, consequently, by repeated application of both lemmas, a "large" sunflower of type $r-1$. On the other hand, Lemma 14 also bounds the maximum size of an injective sunflower of type $r-1$ for any $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ by some constants $\widetilde{k}_{r-1} = \widetilde{k}_{r-1}(H)$. Hence, it follows that every $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ must be k-local for some $k = k(H)$.

Lemma 14. *Let H be an r-uniform hypergraph and suppose $2 \le \min_{0 \le \ell < r} \Delta_\ell(H) = s_H =: s$. For every $i = 0, \dots, r-1$ there exists an integer $\widetilde{k}_i = \widetilde{k}_i(H) > r$ such that for every $k_i \ge \widetilde{k}_i$, for every positive integer n, and for every colouring $\gamma \in \mathcal{L}_n^{(r)}(H, s)$ that yields an injective (i, k_i)-sunflower S_i in $K_n^{(r)}$, one of the following is true:*

(a) there exists $j > i$ and an injective $(j, k_i - r)$-sunflower S_j in $K_n^{(r)}$, or

(b) there exists a subgraph $H_i' = S' + e' \subseteq H$ with the following properties:

- *$(b1)$ S' is an (i, s)-sunflower in H, and we write $S' = (L', p_1', \dots, p_s')$,*
- *$(b2)$ e' contains at least i vertices outside the petals of S', i.e., $|e' \setminus \bigcup_{\sigma=1}^{s} p_\sigma'| \ge i$, and*
- *$(b3)$ e' intersects at least two petals, i.e., there are σ_1 and σ_2, $1 \le \sigma_1 < \sigma_2 \le s$, so that $e' \cap p_{\sigma_1}' \ne \emptyset$ and $e' \cap p_{\sigma_2}' \ne \emptyset$.*

In particular, for $i = r-1$ the above $\widetilde{k}_{r-1} = \widetilde{k}_{r-1}(H) > r$ is such that for every positive integer n every $\gamma \in \mathcal{L}_n^{(r)}(H,s)$ is also $(r-1, \widetilde{k}_{r-1}-1)$-local.

Remark 15. To see that the last part of Lemma 14 also holds, note that if $i = r-1$, then e' (in part (b)) cannot have $r-1$ vertices outside the petals and intersect two petals at the same time. Furthermore, conclusion (a) of Lemma 14 cannot hold either since $k_{r-1} \geq \widetilde{k}_{r-1} > r$. Consequently, the assumptions of Lemma 14 can never hold for $i = r-1$.

Lemma 16. *Let H be an r-uniform hypergraph and suppose $2 \leq \min_{0 \leq \ell < r} \Delta_\ell(H) = s_H =: s$. For every $0 \leq i \leq r-2$ and every integer $\bar{k}$ there exists a positive integer $\widehat{k}_i = \widehat{k}_i(s, \bar{k})$ such that the following is true for every positive integer n. If*
(i) H contains a subgraph $H_i' = S' + e'$ satisfying $(b1)$–$(b3)$ of Lemma 14 and
(ii) $\gamma \in \mathcal{L}_n^{(r)}(H,s)$ yields an injective $(i, \widehat{k}_i)$-sunflower,
then γ gives rise to an injective $(j, \bar{k})$-sunflower in $K_n^{(r)}$ for some $j > i$.

We defer the proofs of Lemmas 14 and 16 to Section 3.3. We close this section with the following simple but useful observation, to be used in the proof of (3.1) in the next section.

Proposition 17. *Suppose $n \geq 3r-1$ and $\gamma \in \mathcal{C}_n^{(r)}$ is a colouring such that $|\operatorname{im}(\gamma)| \geq 2$. Then for every $i = 0, \ldots, r-1$ there are two edges e_1, $e_2 \in K_n^{(r)}$ satisfying*

$$|e_1 \cap e_2| = i \quad \text{and} \quad \gamma(e_1) \neq \gamma(e_2).$$

Proof. Let $n \geq 3r-1$ and $\gamma \in \mathcal{C}_n^{(r)}$ be a colouring such that $|\operatorname{im}(\gamma)| \geq 2$. First we consider the case $i = 0$. Since $|\operatorname{im}(\gamma)| \geq 2$, there are two edges e_1 and e_2 in $K_n^{(r)}$ such that $\gamma(e_1) \neq \gamma(e_2)$. If $e_1 \cap e_2 = \emptyset$ then we are done. On the other hand, if $e_1 \cap e_2 \neq \emptyset$ then $|e_1 \cup e_2| \leq 2r-1$. Since $n \geq 3r-1$ there is some edge $e_3 \in K_n^{(r)}$ disjoint from both e_1 and e_2 and either $\gamma(e_1) \neq \gamma(e_3)$ or $\gamma(e_2) \neq \gamma(e_3)$, which concludes the case $i = 0$.

We now proceed by induction. Let $0 < i \leq r-1$ be fixed. By induction assumption there are two edges e_1 and e_2 in $K_n^{(r)}$ such that $|e_1 \cap e_2| = i-1$ and $\gamma(e_1) \neq \gamma(e_2)$. Let $v_1 \in e_1 \setminus e_2$ and $v_2 \in e_2 \setminus e_1$. Clearly, $|(e_1 \cap e_2) \cup \{v_1, v_2\}| = i+1 \leq r$. Now simply consider some edge $e_3 \in K_n^{(r)}$ which contains $(e_1 \cap e_2) \cup \{v_1, v_2\}$ and $r-(i+1)$ points from $[n] \setminus (e_1 \cup e_2)$. Then, $|e_3 \cap e_1| = |(e_1 \cap e_2) \cup \{v_1\}| = i$ and, similarly, $|e_3 \cap e_2| = |(e_1 \cap e_2) \cup \{v_2\}| = i$. (Such an edge e_3 exists indeed since $(2r-(i-1)) + (r-(i+1)) = 3r-2i < 3r-1 \leq n$.) Clearly, $\gamma(e_3)$ must differ from either $\gamma(e_1)$ or $\gamma(e_2)$, which finishes the proof. □

We mention that a slightly more refined argument shows that the hypothesis $n \geq 2r+1$ suffices in Proposition 17, which is best possible.

3.2. *Proof of Theorem 2*

Recall that all we have left to do to complete the proof of Theorem 2 is to prove the lower bound (3.1).

Proof of (3.1). Let H be an r-uniform hypergraph with at least two edges. In order to verify (3.1), we have to show that there exists some constant $T = T(H)$ such that for every integer n and every colouring $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ (see (3.1) for the definition of s_H)

$$|\operatorname{im}(\gamma)| \leq T\,. \tag{3.2}$$

We distinguish two cases depending on the size of s_H.

Case 1 We have $s_H = 1$. Here we set $T = \binom{3r-1}{r}$. Now let n be some positive integer and let $\gamma \in \mathcal{L}_n^{(r)}(H,1)$ be given. Clearly, $|\operatorname{im}(\gamma)| \le T$ as long as $n \le 3r-1$. So let $n > 3r-1$ and suppose for the moment that $|\operatorname{im}(\gamma)| \ge 2$. Then Proposition 17 implies that γ yields an injective $(\ell, 2)$-sunflower for every $\ell = 0, \dots, r-1$. From the fact that H has at least two edges, it then follows that γ is not $(H,1)$-local, i.e., $\gamma \notin \mathcal{L}_n^{(r)}(H,1)$. Consequently, if $n > 3r-1$, then $|\operatorname{im}(\gamma)| \le 1 < T$. $\Diamond$

Case 2 We have $s_H > 1$. In this case the definition of $T = T(H)$ is a little more complicated. We first recursively define integers $k_{r-1}, \dots, k_0$ as follows:

$$k_i = \begin{cases} \widetilde{k}_{r-1}\big(\text{Lem.}14(H)\big) & \text{if} \quad i = r-1\,, \\ \max\Big\{k_{i+1} + r,\, \widehat{k}_i\big(\text{Lem.}16(s = s_H, \bar{k} = k_{i+1})\big), & \\ \qquad\qquad\qquad\qquad \widetilde{k}_i\big(\text{Lem.}14(H)\big)\Big\} & \text{if} \quad i = r-2, \dots, 0\,, \end{cases}$$

where $\widetilde{k}_{r-1}$, $\widetilde{k}_i$, and $\widehat{k}_i$ for $i = r-2, \dots, 0$ are given by Lemmas 14 and 16, respectively. Note that by definition the sequence $k_0, \dots, k_{r-1}$ is not only monotone decreasing, but also satisfies

$$k_{i+1} \le k_i - r \quad \text{for} \quad i = r-2, \dots, 0\,. \tag{3.3}$$

We then define the promised constant T by setting

$$T = T\big(\text{Prop.}12(k = k_0 - 1, r)\big)\,. \tag{3.4}$$

Now let n be some positive integer and let $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ be given. We first show the following.

Claim 18. *The colouring γ is $(i, k_i - 1)$-local for every $i = 0, \dots, r-1$.*

Proof. Assume for a contradiction that i_0 is the largest index i so that γ is not $(i, k_i - 1)$-local. Due to the definition of k_{r-1} and the last part of Lemma 14 we have that $i_0 < r-1$. Furthermore, by definition of i_0 there exists an injective (i_0, k_{i_0})-sunflower, and as $k_{i_0} \ge \widetilde{k}_{i_0}\big(\text{Lem.}14(H)\big)$, we can apply Lemma 14. Now part (a) of Lemma 14 is impossible, since for any $j > i_0$ we have $k_j \le k_{i_0} - r$ (cf. (3.3)) and thus an injective $(j, k_{i_0} - r)$-sunflower would contain an injective (j, k_j)-sunflower, contradicting the maximality of i_0.

Hence case (b) of Lemma 14 must occur. By definition of k_{i_0} we have $k_{i_0} \ge \widehat{k}_{i_0}\big(\text{Lem.}16(s = s_H, \bar{k} = k_{i_0+1})\big)$. Hence both assumptions (i) and (ii) of Lemma 16 are satisfied for $\bar{k} = k_{i_0+1}$. Thus Lemma 16 yields an injective (j, k_{i_0+1})-sunflower. Again, as $j > i_0$, we have $k_j \le k_{i_0+1}$, and thus we have an injective (j, k_j)-sunflower, contradicting the maximality of i_0 again. This proves Claim 18. $\square$

Now Claim 18 and (3.3) assert that γ is a $(k_0 - 1)$-local colouring and, therefore, the choice of T in (3.4) and Proposition 12 now imply $|\operatorname{im}(\gamma)| \le T$ in this case, Case 2. $\Diamond$

Having verified (3.1) in both cases, we have concluded the proof of the lower bound in Theorem 2, based on Lemmas 14 and 16. $\square$

3.3. *Proofs of Lemmas 14 and 16*

In this section we prove Lemmas 14 and 16 stated in Section 3.1 and used in Section 3.2.

3.3.1. *Proof of Lemma 14* Let H be an r-uniform hypergraph and

$$s := s_H = \min_{0\le\ell<r} \Delta_\ell(H) \ge 2\,. \tag{3.5}$$

Let i be a fixed integer in the interval $[0, r-1]$ and set

$$\widetilde{k}_i = \max\{s+1+r+i^2, 3r-1\}\,. \tag{3.6}$$

Moreover, let integers $k_i \ge \widetilde{k}_i$ and n and a colouring $\gamma \in \mathcal{L}_n^{(r)}(H,s)$ be given. Suppose $S_i = (L, p_1, \dots, p_{k_i}) \subseteq K_n^{(r)}$ is an injective (i,k_i)-sunflower under γ.

For the rest of the proof we assume that γ does not contain an injective $(j, k_i - r)$-sunflower for any $j > i$, i.e., we assume that conclusion (a) of Lemma 14 fails and we are going to deduce (b). By the definition of s there exists an (i,s)-sunflower $S' = (L', p'_1, \dots, p'_s)$ in H, as claimed in $(b1)$. We first show that there is an edge $e' \in H \setminus S'$ which satisfies property $(b2)$.

Claim 19. *There is an edge $e' \in H \setminus S'$ with $\left|e' \setminus \bigcup_{\sigma=1}^s p'_\sigma\right| \ge i$.*

Proof. If $i = 0$, then it follows from $s \ge 2$ that $H \setminus S' \ne \emptyset$ (otherwise H contains no (j,s)-sunflower for $j \ge 1$, which contradicts the assumption $s = s_H \ge 2$) and, hence, there is an edge e' which trivially satisfies the conclusion of the claim.

So let $i > 0$. By the definition of s there exists a matching $M' \subseteq H$ of size s. On average the edges of M' have at least

$$\frac{1}{|M'|}\Big|\bigcup_{f'\in M'} f' \setminus \bigcup_{\sigma\in[s]} p'_\sigma\Big| \ge \frac{1}{s}\big(sr - s(r-i)\big) = i$$

vertices outside the petals of S'. Consequently, there is an edge $e' \in M'$ which has at least i vertices outside the petals of S'. If $e' \notin S'$ then we found our edge. If, however, $e' \in S' \cap M'$, then we can repeat the argument with $M' \setminus \{e'\}$ and $S' \setminus \{e\}'$. Indeed, on average the edges of $M' \setminus \{e'\}$ have at least

$$\frac{1}{s-1}\big((s-1)r - (s-1)(r-i)\big) = i$$

vertices outside the petals of $S' \setminus \{e'\}$. Hence, there must be an edge $e'' \in M' \setminus \{e'\}$ which has at least i vertices outside the petals of $S' \setminus \{e'\}$. Moreover, since $e' \cap e'' = \emptyset$ (both are edges in the matching M') and since we assumed that $e' \in S'$, we have that $e'' \notin S'$. □

Fix e' as in Claim 19. It remains to show that e' has non-empty intersection with at least two petals of S'. Our proof is by contradiction. So let us first assume that

$$e' \cap p'_\sigma = \emptyset \quad \text{for every} \quad \sigma \in [s]\,. \tag{3.7}$$

In this case, let e be an edge of $K_n^{(r)}$ which satisfies $|e \cap L| = |e' \cap L'|$. Since $k_i \ge \widetilde{k}_i \ge s+1+r$ (cf. (3.6)), after removing those edges f from S_i for which $\gamma(f) = \gamma(e)$ or $(f \setminus L) \cap e \ne \emptyset$ there must be an injective (i,s)-sunflower $S_i^* \subseteq S_i \subseteq K_n^{(r)}$ for which $\gamma(e) \notin \gamma(S_i^*)$ and $e \cap V(S_i^*) = e \cap L$. Consequently, $e \cup S_i^*$ (which is a copy of $e' \cup S' \subseteq H$) picks up $s+1$ colours, which contradicts the assumption $\gamma \in \mathcal{L}_n^{(r)}(H,s)$. Hence, assumption (3.7) must fail.

Next we assume that e' intersects precisely one petal of S'. With an appropriate relabelling we assume

$$e' \cap p'_1 \ne \emptyset \quad \text{and} \quad e' \cap p'_\sigma = \emptyset \quad \text{for every} \quad \sigma = 2, \dots, s\,. \tag{3.8}$$

Set

$$i_L = |e' \cap L'|\,, \qquad i_O = |e' \setminus V(S')|\,, \qquad \text{and} \qquad i_1 = |e' \cap p_1|\,. \tag{3.9}$$

Note that $r = i_L + i_O + i_1$ and since $e' \notin S'$ (see Claim 19), we have

$$i_O > 0 \qquad \text{and, consequently,} \qquad i_L + i_1 < r\,. \tag{3.10}$$

We shall need the following claims to derive a contradiction from assumption (3.8).

Claim 20. *For every edge e of $K_n^{(r)}$ satisfying $|e \cap L| = i_L$ and $|e \cap p_\lambda| = i_1$ for some $\lambda \in [k_i]$, we have $\gamma(e) = \gamma(p_\lambda \cup L)$.*

Proof. Let e and p_λ be as in the hypothesis of the claim. Since $k_i \geq \widetilde{k}_i \geq s+1+r \geq s+1+i_O$, there is an injective $(i, s-1)$-sunflower $S_i^* \subseteq S_i$ satisfying the following:
- S_i^* does not contain the petal p_λ,
- none of the petals of S_i^* intersects e, and
- $\gamma(e) \notin \gamma(S_i^*)$.

What can we say about $e \cup S_i^* \cup \{L \cup p_\lambda\}$? (We observe that in this last expression we are mixing the standard notation with the convention of omitting $\{\ \}$ for singletons when the meaning is clear.) Note first that $e \cup S_i^* \cup \{L \cup p_\lambda\}$ forms a copy of $e' \cup S_i' \subset H$. Hence, if $\gamma(e) \neq \gamma(p_\lambda \cup L)$, then $e \cup S_i^* \cup \{L \cup p_\lambda\}$ uses $s+1$ colours, which contradicts the fact that $\gamma \in \mathcal{L}_n^{(r)}(H, s)$. Therefore $\gamma(e) = \gamma(p_\lambda \cup L)$, as claimed. □

The simple observation in Claim 20 implies our next claim, Claim 21. This latter claim asserts that $i_L + i_O = i$ and, more importantly, that any set L^* of i vertices in $K_n^{(r)}$ is, roughly speaking, the core of a 'large' injective sunflower.

Claim 21. *We have $i_L + i_O = i$ and for all sets $L^* \subseteq [n]$ with $|L^*| = i$ there is an injective $(i, s+1+r)$-sunflower S_i^* with core L^*.*

Proof. First we show that $i_L + i_O = i$. Note that

$$i_L + i_O = i \qquad \overset{(3.9)}{\Longleftrightarrow} \qquad i_1 = r - i\,. \tag{3.11}$$

Clearly, $i_L + i_O = |e' \cap L'| + |e' \setminus V(S')| \geq i$ since by Claim 19 the edge e' contains at least i vertices outside the petals of S'. If $i_L + i_O > i$, then fix some set O of cardinality i_O in $[n] \setminus L$ and some set $\bar{L}$ of cardinality i_L in L. Moreover, for every $\lambda \in [k_i]$ fix i_1 vertices I_λ in every petal p_λ. Then, apply Claim 20 for every $e_\lambda = O \cup \bar{L} \cup I_\lambda$ for which $p_\lambda \cap O = \emptyset$. Since there are at least $k_i - i_O \geq k_i - r$ such petals, the above yields an injective $(j, k_i - r)$-sunflower S_j for $j = i_L + i_O > i$, which is a contradiction, as we assumed that (*a*) does not hold. Thus we do indeed have $i_L + i_O = i$, as claimed in the first part of Claim 21.

We now focus on the second part of the claim. For that let $L^* \subseteq [n]$ be a set of size i. We fix a sequence of sets $L_1, \ldots, L_b$ in $[n]$ with $b \leq i+1$ so that

$$L_1 = L\,, \quad |L_a| = i\,, \quad |L_a \cap L_{a+1}| = i_L \text{ for } a = 1, \ldots, b-1, \quad \text{and} \quad L_b = L^*\,.$$

Note that such a sequence exists since $i_L = i - i_O < i$ (cf. (3.10)). For convenience we define for $a = 1, \ldots, b$

$$k(a) = k_i - (a-1)i_O\,.$$

We now show inductively that for every $a = 1, \ldots, b$ there exists an injective $(i, k(a))$-sunflower $S(a)$ with core L_a. As $k(b) = k_i - (b-1)i_O \geq k_i - i^2 \geq \widetilde{k}_i - i^2 \geq s+1+r$ this yields Claim 21.

Setting $S(1) = S_i$ gives the induction start. So suppose there is an injective $(i, k(a))$-sunflower $S(a)$ with core L_a and petals $p_1^a, \ldots p_{k(a)}^a$. Note that $|L_{a+1} \setminus L_a| = i - i_L = i_O$. We set $\Lambda = \{\lambda \in [k(a)] \colon p_\lambda^a \cap L_{a+1} = \emptyset\}$. Obviously, $|\Lambda| \geq k(a) - i_O$. For every $\lambda \in \Lambda$ set $p_\lambda^{a+1} := p_\lambda^a$. It is easy to see that the p_λ^{a+1} together with the core L_{a+1} form an injective $(i, |\Lambda|)$-sunflower $S(a+1)$. Indeed, simply apply Claim 20 with $L := L_a$ for every edge $e := p_\lambda^{a+1} \cup L_{a+1}$ and

$p_\lambda := p_\lambda^a$. This will yield that $\gamma(p_\lambda^{a+1} \cup L_{a+1}) = \gamma(p_\lambda^a \cup L_a)$, and hence the injectivity of $S(a+1)$ is inherited from that of $S(a)$, and the induction step follows from the definition of $k(a+1)$. $\square$

Based on Claim 21 we now show that our assumption (3.8) contradicts $\gamma \in \mathcal{L}_n^{(r)}(H, s)$, thus finishing the proof of Lemma 14. Since by (3.6) we have $k_i \geq 3r-1$, we have $n \geq 3r-1$ and $|\operatorname{im}(\gamma)| > 1$. Therefore, Proposition 17 ensures the existence of two edges $e, f \in K_n^{(r)}$ satisfying $|f \cap e| = i_L + i_1 < r$ and $\gamma(f) \neq \gamma(e)$. Let $\bar{p} \cup \bar{L}$ be a partition of $e \cap f$ with

$$|\bar{p}| = i_1 \qquad \text{and} \qquad |\bar{L}| = i_L\,.$$

Set $L^* = \bar{L} \cup (f \setminus e)$ and note that

$$(e \cup f) \setminus L^* \subseteq e \qquad \text{and} \qquad |L^*| = i_L + (r - i_L - i_1) = r - i_1 = i_O + i_L = i\,,$$

where we used the first part of Claim 21 for the last identity. We then apply the second part of Claim 21 with L^*, which yields an injective $(i, s+1+r)$-sunflower S_i^* with core L^*. Therefore, after removing those edges of S_i^* which have the colour of e or f and those which intersect $(e \cup f) \setminus L^*$ there still exists an injective $(i, s-1)$-sunflower $S_i^{**} \subseteq S_i^*$ with core L^* satisfying

$$\gamma(S_i^{**}) \cap \{\gamma(f), \gamma(e)\} = \emptyset \qquad \text{and} \qquad V(S_i^{**}) \cap ((e \cup f) \setminus L^*) = \emptyset\,.$$

Consequently, $S_i^{**} \cup f$ is an injective (i, s)-sunflower with core L^* and additional petal $f \setminus L^* = \bar{p}$. Moreover, the definitions of $\bar{p}, \bar{L} \subseteq e \cap f$, $L^* = \bar{L} \cup (f \setminus e)$, and S_i^{**} imply that $|e \cap \bar{p}| = |\bar{p}| = i_1$, $|e \cap L^*| = |\bar{L}| = i_L$, and $|e \setminus (V(S_i^{**}) \cup f)| = |e \setminus f| = r - i_1 - i_L = i_O$. In other words, $e \cup S_i^{**} \cup f$ is isomorphic to $e' \cup S'$. Since $|\gamma(e \cup S_i^{**} \cup f)| = s+1$ this contradicts the fact that $\gamma \in \mathcal{L}_n^{(r)}(H, s)$. Therefore, assumption (3.8) cannot hold and e' must intersect at least two petals of S'.

As observed in Remark 15, the last assertion in Lemma 14 follows easily from the first part. Therefore, the proof of Lemma 14 is complete.

3.3.2. *Proof of Lemma 16* Let an r-uniform hypergraph H satisfying

$$s := s_H = \min_{0 \leq \ell < s} \Delta_\ell(H) \geq 2 \tag{3.12}$$

and integers i, $0 \leq i \leq r-2$, and $\bar{k}$ be given. We set

$$\widetilde{k} = \max_{2 \leq u \leq r} R^{(u)}(\bar{k} + r - 1; u) \qquad \text{and} \qquad \widehat{k}_i = \widetilde{k} + r\,, \tag{3.13}$$

where $R^{(u)}(\bar{k} + r - 1; u)$ is the Ramsey number which ensures that every u-colouring of the complete u-uniform hypergraph on $R^{(u)}(\bar{k} + r - 1; u)$ vertices yields a monochromatic copy of $K_{\bar{k}+r-1}^{(u)}$.

Let $H_i' = S' + e'$ be a subhypergraph of H which satisfies $(b1)$–$(b3)$ of Lemma 14. Moreover, let $\gamma \in \mathcal{L}_n^{(r)}(H, s)$ be an (H, s)-local colouring of $K_n^{(r)}$ which yields an injective $(i, \widehat{k}_i)$-sunflower. We have to ensure the existence of an injective $(j, \bar{k})$-sunflower in $K_n^{(r)}$ for some $j > i$.

Consider first the sub-hypergraph $H_i' = S' + e'$ of H. By property $(b1)$ the hypergraph $S' = (L', p_1', \ldots, p_s')$ is an (i, s)-sunflower, with core L' and petals $p_1', \ldots, p_s'$. We set

$$i_L = |e' \cap L'|\,, \quad i_O = |e' \setminus V(S')|\,, \quad \text{and} \quad i_\sigma = |e' \cap p_\sigma'| \text{ for every } \sigma \in [s]\,. \tag{3.14}$$

We may assume w.l.o.g. that $i_1 \geq \cdots \geq i_u > 0$ and $i_{u+1} = \cdots = i_s = 0$, We know from $(b3)$ that $u \geq 2$. Observe that

$$i_O + i_L + i_1 + \cdots + i_u = r \tag{3.15}$$

and clearly $u \leq r$.

Now we turn back to $K_n^{(r)}$ and γ. Let L be the core of an injective $(i,\widehat{k}_i)$-sunflower in $K_n^{(r)}$. First fix a set O of i_O vertices in $V(K_n^{(r)})\setminus L$ and a set $\bar{L}$ of i_L vertices inside the core L. Since $i_O < r$ (cf. (3.15)) and $\widehat{k}_i = \widetilde{k} + r$, there still exists an injective $(i,\widetilde{k})$-sunflower $S \subseteq K_n^{(r)}$ with core L satisfying $V(S)\cap O = \emptyset$. Let $p_1,\dots,p_{\widetilde{k}}$ be the petals of that sunflower, i.e., $S = (L, p_1, \dots, p_{\widetilde{k}})$.

Appealing to the fact that $\gamma \in \mathcal{L}_n^{(r)}(H,s)$ and following the line of proof of Claim 20 one can show the following claim.

Claim 22. *Suppose $\Lambda = \{\lambda_1,\dots,\lambda_u\} \subseteq [\widetilde{k}]$, and suppose e is an edge of $K_n^{(r)}$ satisfying $|e\cap L| = i_L$ and $|e \cap p_{\lambda_\sigma}| = i_\sigma$ for every $\sigma \in [u]$. Then there exists $\sigma(\Lambda) \in [u]$ such that $\gamma(e) = \gamma(p_{\lambda_{\sigma(\Lambda)}} \cup L)$.* □

For every $\lambda \in [\widetilde{k}]$ we fix u not necessarily disjoint subsets $B_{\lambda,1},\dots,B_{\lambda,u} \subseteq p_\lambda$ in such a way that

$$|B_{\lambda,\sigma}| = i_\sigma \quad \text{for every } \sigma\in[u] \text{ and } \lambda \in [\widetilde{k}]\,. \tag{3.16}$$

From Claim 22 we infer that for every $\Lambda = \{\lambda_1 < \dots < \lambda_u\} \subseteq [\widetilde{k}]$ we have

$$\gamma\Big(\bar{L}\cup O \cup \bigcup_{\sigma\in[u]} B_{\lambda_\sigma,\sigma}\Big) = \gamma(L \cup p_{\lambda_{\sigma(\Lambda)}}) \quad \text{for some } \sigma(\Lambda)\in[u]\,. \tag{3.17}$$

Note that the assertion above states that for every set $\Lambda = \{\lambda_1 < \dots < \lambda_u\} \subseteq [\widetilde{k}]$ there exists a $\sigma(\Lambda)$ determining the colour of $\bar{L}\cup O\cup\bigcup_{\sigma\in[u]} B_{\lambda_\sigma,\sigma}$. While the above $\sigma(\Lambda)$ depends on Λ, a Ramsey type argument ensures a strengthening in which $\sigma(\Lambda)$ is independent of $\Lambda \subseteq X$ for a suitable subset $X \subseteq [\widetilde{k}]$. More precisely, we shall prove the following.

Claim 23. *There exist a subset $X \subseteq [\widetilde{k}]$ with $|X| = \bar{k}+u-1$ and a $\sigma_0 \in [u]$ such that for every $\{\lambda_1 < \dots < \lambda_u\} \subseteq X$ we have*

$$\gamma\Big(\bar{L}\cup O \cup \bigcup_{\sigma\in[u]} B_{\lambda_\sigma,\sigma}\Big) = \gamma(L\cup p_{\lambda_{\sigma_0}})\,.$$

We prove Claim 23 momentarily, but first we deduce Lemma 16 from it. Let $X = \{x_1 < \dots < x_{\bar{k}+u-1}\}$ and $\sigma_0 \in [u]$ be as in Claim 23. Set

$$L^* = \bar{L}\cup O\cup \bigcup_{\sigma=1}^{\sigma_0-1} B_{x_\sigma,\sigma} \cup \bigcup_{\sigma=\sigma_0+1}^{u} B_{x_{\bar{k}+\sigma-1},\sigma}$$

and

$$p^*_\tau = B_{x_{\sigma_0+\tau-1},\sigma_0} \quad \text{for } \tau = 1,\dots,\bar{k}\,.$$

Recall that $B_{x,\sigma} \subseteq p_x$ and, therefore, $B_{x,\sigma}\cap B_{x',\sigma'} = \emptyset$ whenever $x \neq x'$. Moreover, by (3.16)

$$\big|L^*\big| = i_L + i_O + \sum_{\sigma\in[u]\setminus\{\sigma_0\}} i_\sigma = r - i_{\sigma_0} =: j$$

and $j > i$ since $i_L + i_O = i$, $u \geq 2$, and $i_\sigma > 0$ for every $\sigma \in [u]$. Moreover, the choice of p^*_τ and the definition of L^* imply that $|L^*\cup p^*_\tau| = j + i_{\sigma_0} = r$ and, hence,

$$S^* = (L^*, p^*_1, \dots, p^*_{\bar{k}})$$

is a $(j,\bar{k})$-sunflower in $K_n^{(r)}$. Furthermore, it follows from Claim 23 that

$$\gamma(L^*\cup p^*_\tau) = \gamma(L\cup p_{x_{\sigma_0+\tau-1}})$$

for every $\tau \in [\bar{k}]$. Since S is injective by assumption this implies that S^* is an injective $(j,\bar{k})$-sunflower in $K_n^{(r)}$ and the proof of Lemma 16 is complete, except for the proof of Claim 23.

Proof of Claim 23. Recall that Claim 22 guarantees for every $\Lambda = \{\lambda_1 < \dots < \lambda_u\} \subseteq [\widetilde{k}]$ a $\sigma(\Lambda) \in [u]$ such that

$$\gamma\Big(\bar{L} \cup O \cup \bigcup_{\sigma\in[u]} B_{\lambda_\sigma,\sigma}\Big) = \gamma(L \cup p_{\lambda_{\sigma(\Lambda)}})\,.$$

In other words we may view σ as a u-edge colouring of the complete u-uniform hypergraph with vertex set $[\widetilde{k}]$. By the choice of $\widetilde{k}$ in (3.13) we infer from Ramsey's theorem [12] that there exist a subset $X \subseteq [\widetilde{k}]$ of size $|X| = \bar{k} + r - 1$ and a $\sigma_0 \in [u]$ such that $\sigma(\Lambda) = \sigma_0$ for every $\Lambda = \{\lambda_1 < \dots < \lambda_u\} \subseteq X$. □

4. *Essentially unbounded colourings*

In this section we prove Theorem 8 (Section 4.3) and Theorem 5 (Section 4.4). Behind the scene we shall need a partite version of the canonical theorem of Erdős and Rado, Theorem 7; see Theorem 24 below.

4.1. *A partite version of the Erdős–Rado canonical theorem*

For a given ℓ-type τ (see Definition 4) we call a vector $\mathcal{J} = (J_1,\dots,J_\ell)$ of sets an *τ-trace* if $J_i \subseteq [\tau_i]$ for every $i \in [\ell]$. Finally, we recall that for a set $(e \cap W_i) = \{v_1 < \dots < v_{\tau_i}\}$ and $J_i = \{j_1,\dots,j_x\} \subseteq [\tau_i]$ we write $(e\cap W_i)[J_i]$ to denote the set $\{v_{j_1},\dots,v_{j_x}\}$ and $(e\cap W_i)[J_i] = \emptyset$ if and only if $J_i = \emptyset$.

THEOREM 24. *For all integers $q \geq r \geq 2$ and $\ell \in [r]$ and every ℓ-type τ there exists an integer $n = n(q,r,\ell,\tau)$ so that for every colouring $\gamma \in \mathcal{C}^{(r)}_{\ell\cdot n}$ and every partition of the vertex set into classes $V_1,\dots,V_\ell$ of cardinality $|V_i| = n$ each, there exists a family $W_1,\dots,W_\ell$ of disjoint sets $W_i \subset V_i$ with $|W_i| = q$ and a τ-trace $\mathcal{J} = \mathcal{J}(\tau) = (J_1,\dots,J_\ell)$, such that for all edges e, $e' \in (W_1,\dots,W_\ell)\langle\tau\rangle$*

$$\gamma(e) = \gamma(e') \;\Leftrightarrow\; (e \cap W_i)[J_i] = (e' \cap W_i)[J_i] \quad \forall i \in [\ell]\,.$$

Observe that the case $\ell = 1$ of Theorem 24 is exactly Theorem 7, since then $\tau = (r)$ is the only 1-type and then Theorem 24 guarantees for every colouring γ a set W and a set $J \subseteq [r]$ so that two edges $e, e' \subseteq W$ receive the same colour iff $e[J] = e'[J]$.

Proof of Theorem 24. Let integers q, r, and ℓ and an ℓ-type $\tau = (\tau_1,\dots,\tau_\ell)$ be given. We set n to be the integer $n(q\ell, r)$ guaranteed by Theorem 7 applied with $q\cdot\ell$ and r. Let γ be colouring $K^{(r)}_{\ell\cdot n} \to \mathbb{Z}$ and let $V_1,\dots,V_\ell$ be an arbitrary partition of the vertex set of $K^{(r)}_{\ell\cdot n}$.

We treat the sets $V_1,\dots,V_\ell$ as (pairwise disjoint) copies of $[n]$ and denote by $\widehat{V}$ another copy of $[n]$. Consider the natural projection $\bigcup_{i\in[k]} V_i \to \widehat{V}$, where all the ℓ copies of $x \in [n]$ in $\bigcup_{i\in[k]} V_i$ are mapped onto the same $x \in \widehat{V}$. Restricting that projection to $(V_1,\dots,V_\ell)\langle\tau\rangle$ gives rise to

$$\pi\colon\; (V_1,\dots,V_\ell)\langle\tau\rangle \to \binom{\widehat{V}}{\leq r}\,, \tag{4.1}$$

where $\binom{\widehat{V}}{\leq r}$ is the family of all subsets of $\widehat{V}$ with cardinality at most r.

Let us define an "inverse" π^{-1} of π on $\binom{\widehat{V}}{r}$ as follows. Lift $\widehat{e} \in \binom{\widehat{V}}{r}$ to the element $\pi^{-1}(\widehat{e}) = e \in (V_1, \dots, V_\ell)\langle\tau\rangle$ such that $\pi(e) = \widehat{e}$ and

$$\pi(e \cap V_1) < \cdots < \pi(e \cap V_\ell)\,,$$

where as usual we write $X < Y$ for two sets $X, Y \subseteq [n]$ to denote $\max X < \min Y$.

Based on π^{-1} and the given colouring γ, we define an auxiliary colouring $\widehat{\gamma}\colon \binom{\widehat{V}}{r} \to \mathbb{Z}$ by setting for every $\widehat{e} \in \binom{\widehat{V}}{r}$

$$\widehat{\gamma}(\widehat{e}) := \gamma(\pi^{-1}(\widehat{e}))\,. \tag{4.2}$$

Apply Theorem 7 to $\widehat{\gamma}$. We obtain a subset $\widehat{W} \subset \widehat{V}$ with $|\widehat{W}| = q\ell$ and a set $\widehat{J} \subset [r]$ such that for all $\widehat{e}, \widehat{e}' \in \binom{\widehat{W}}{r}$

$$\widehat{\gamma}(\widehat{e}) = \widehat{\gamma}(\widehat{e}') \Leftrightarrow \widehat{e}[\widehat{J}] = \widehat{e}'[\widehat{J}]\,. \tag{4.3}$$

View $\widehat{J}$ as the corresponding characteristic vector in $\{0,1\}^r$, and partition this vector by letting J_1 consist of the first τ_1 components, J_2 of the next τ_2 components, up to J_ℓ. Finally view the sets J_i as subsets of τ_i and fix the promised τ-trace $\mathcal{J} = \mathcal{J}(\tau) = (J_1, \dots, J_\ell)$. We obtain the sets $W_i \subseteq V_i$ from $\widehat{W}$ in a similar manner: simply partition $\widehat{W}$ into ℓ sets $\widehat{W}_1, \dots \widehat{W}_\ell$ of the same cardinality q so that for every $i = 1, \dots, \ell - 1$

$$\widehat{W}_1 < \cdots < \widehat{W}_\ell\,,$$

and lift $\widehat{W}_i$ to V_i in the natural way, i.e., W_i equals to the copy of $\widehat{W}_i$ in V_i. Thus we obtain $W_i \subset V_i$ for all $i \in [\ell]$.

Observe that

$$\pi \text{ is injective on } (W_1, \dots, W_\ell)\langle\tau\rangle \tag{4.4}$$

and that, since $\widehat{W}_i \cap \widehat{W}_j = \emptyset$, we have $\pi(e) \in \binom{\widehat{V}}{r}$ for every $e \in (W_1, \dots, W_\ell)\langle\tau\rangle$. Moreover, for every $e \in (W_1, \dots, W_\ell)\langle\tau\rangle$ we have

$$\pi^{-1}(\pi(e)) = e\,. \tag{4.5}$$

Also for every $\widehat{e} \in \binom{\widehat{V}}{r}$

$$\widehat{e}[\widehat{J}] = \Big((\widehat{e} \cap \widehat{W}_1)[J_1] < \cdots < (\widehat{e} \cap \widehat{W}_\ell)[J_\ell]\Big)\,. \tag{4.6}$$

Finally, we show that the $W_1, \dots, W_\ell$ together with $\mathcal{J} = (J_1, \dots, J_\ell)$ satisfy the conclusion of Theorem 24. For all edges $e, e' \in (W_1, \dots, W_\ell)\langle\tau\rangle$ we have

$$\begin{aligned}
\gamma(e) = \gamma(e') &\Leftrightarrow \widehat{\gamma}(\pi(e)) = \widehat{\gamma}(\pi(e')) && \text{by (4.5) and (4.2)}\\
&\Leftrightarrow \pi(e)[\widehat{J}] = \pi(e')[\widehat{J}] && \text{by (4.3) and (4.4)}\\
&\Leftrightarrow \forall i \in [\ell]\colon\ (\pi(e) \cap \widehat{W}_i)[J_i] = (\pi(e') \cap \widehat{W}_i)[J_i] && \text{by (4.6)}\\
&\Leftrightarrow \forall i \in [\ell]\colon\ (e \cap W_i)[J_i] = (e' \cap W_i)[J_i] && \text{by choice of } W_i.
\end{aligned}$$

□

4.2. *Further auxiliary lemmas*

Besides Theorem 24 from the last section, we need a few technical lemmas for the proof of Theorem 8. We start with an auxiliary result relating $(r-1,k)$-local colourings (see Definition 11) and (ε, T)-bounded colourings (see Definition 3). Roughly speaking, Lemma 25 asserts that unbounded colourings are not local.

LEMMA 25. *For all integers $r \geq 2$ and $k \geq 1$ and every $\varepsilon > 0$ there exists an integer $T = T(r, k, \varepsilon)$ such that for every $n \in \mathbb{N}$, every $(r-1, k)$-local colouring $\gamma \in \mathcal{C}_n^{(r)}$ is (ε, T)-bounded.*

Proof. Let $r \geq 2$, $k \geq 1$, and $\varepsilon > 0$ be given and set

$$T = \left\lfloor \left(\frac{kr^r}{\varepsilon}\right)^r \right\rfloor + 1\,.$$

Assume for a contradiction that for some $n \in \mathbb{N}$ there exists an $(r-1,k)$-local colouring $\gamma \in \mathcal{C}_n^{(r)}$ which is not (ε, T)-bounded. Denote by c_i the number of edges of colour i. After renumbering we may assume that $c_i = 0$ for every $i \leq 0$ and $c_i \geq c_{i+1}$ for every $i \geq 1$. Moreover,

$$\sum_{i>T} c_i > \varepsilon \binom{n}{r}, \tag{4.7}$$

since otherwise γ would be (ε, T)-bounded.

As there are c_i edges of colour i, by the Kruskal–Katona theorem [8, 10] there are at least $c_i^{(r-1)/r}$ sets $L \in \binom{[n]}{r-1}$ seeing colour i, i.e., each such L is contained in some edge of colour i. On the other hand, since γ is $(r-1,k)$-local, each such set L sees at most k different colours, and so combining these two arguments we have that

$$\sum_{i\geq 1} c_i^{1-1/r} \leq \sum_{L \in \binom{[n]}{r-1}} \#\{\text{ different colours seen by } L\ \} \leq k\binom{n}{r-1} \leq kn^{r-1}. \tag{4.8}$$

Furthermore, for every $i > T$ we have

$$c_i \leq c_T \leq \frac{1}{T}\sum_{j\in[T]} c_j \leq \frac{1}{T}\binom{n}{r} \leq \frac{n^r}{T}\,. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we obtain

$$\varepsilon\binom{n}{r} \overset{(4.7)}{\leq} \sum_{i>T} c_i = \sum_{i>T} c_i^{1/r} c_i^{1-1/r} \overset{(4.9)}{\leq} \frac{n}{\sqrt[r]{T}} \sum_{i>T} c_i^{1-1/r} \overset{(4.8)}{\leq} \frac{kn^r}{\sqrt[r]{T}} \leq \frac{kr^r}{\sqrt[r]{T}}\binom{n}{r},$$

which contradicts the choice of T. □

Suppose $\gamma \in \mathcal{C}_n^{(r)}$ and $L \in \binom{[n]}{r-1}$. Let $C_{L,i}$ be the set of those vertices $v \in [n] \setminus L$ for which $\gamma(L \cup \{v\}) = i$. Again we may assume (after renumbering if necessary) that $C_{L,i} = \emptyset$ for $i \leq 0$ and $i \geq n+1$ and $|C_{L,i}| \geq |C_{L,i+1}|$ for every $i \geq 1$. For a given integer $k \geq 1$ and $\alpha > 0$ we call L (k, α, γ)-*good*, if

$$\sum_{i>k} |C_{L,i}| \geq \alpha n, \tag{4.10}$$

and (k, α, γ)-*bad* otherwise. In other words, a set L is good if its "smaller colour classes" $C_{L,i}$ $(i > k)$ add up a positive fraction. We first show (see Proposition 26) that, in this case $[n] \setminus L$ can be partitioned into classes of sensible sizes with disjoint colour ranges. Then we prove (see Lemma 27) that every unbounded colouring must contain many good sets L.

PROPOSITION 26. *For all integers $r \geq 2$ and $k \geq 1$, every $\alpha > 0$ and every colouring $\gamma \in \mathcal{C}_n^{(r)}$ the following holds. If $L \in \binom{[n]}{r-1}$ is (k, α, γ)-good, then $[n] \setminus L$ can be partitioned into classes $U_1, \ldots, U_k$ such that*
(i) $|U_i| \geq \alpha n/(2k)$ and
(ii) for all $1 \leq i < j \leq k$ and all $x \in U_i$ and $y \in U_j$ we have $\gamma(L \cup \{x\}) \neq \gamma(L \cup \{y\})$.

Proof. Let constants $r \geq 2$, $k \geq 1$, $\alpha > 0$, a colouring $\gamma \in \mathcal{C}_n^{(r)}$ and a (k,α,γ)-good set $L \in \binom{[n]}{r-1}$ be given. Moreover, let $C_{L,i}$ be defined as before.

First note that if $|C_{L,k}| \geq \alpha n/(2k)$ then we are done by setting

$$U_i = \begin{cases} C_{L,i} & \text{if} \quad i = 1, \dots, k-1, \\ \bigcup_{j \geq k} C_{L,j} & \text{if} \quad i = k\,. \end{cases}$$

Therefore, assume that $|C_{L,k}| < \alpha n/(2k)$. Let $\{X_1, \dots, X_k\}$ be a partition of $\{k+1, \dots, n\}$ such that

$$M := \max_{1 \leq i < j \leq k} \left| \sum_{x \in X_i} |C_{L,x}| - \sum_{y \in X_j} |C_{L,y}| \right|$$

is minimized. Note that, $|C_{L,x}| \leq |C_{L,k}| < \alpha n/(2k)$ for any $x > k$, we have

$$M \leq \frac{\alpha n}{2k}\,. \tag{4.11}$$

Assume for a contradiction that $|\sum_{x \in X_{i_0}} |C_{L,x}| < \alpha n/(2k)$ for some $i_0 \in [k]$. Then (4.11) would imply that

$$\sum_{x \in X_i} |C_{L,x}| < |\sum_{x \in X_{i_0}} |C_{L,x}| + \frac{\alpha n}{2k} \leq \frac{\alpha n}{k}$$

for every $i \in [k]$, and, consequently,

$$\sum_{i > k} |C_{L,i}| < \frac{\alpha n}{2k} + (k-1)\frac{\alpha n}{k} < \alpha n\,,$$

which contradicts the fact that L is (k,α,γ)-good. Hence, $\sum_{x \in X_i} |C_{L,x}| \geq \alpha n/(2k)$ for every $i \in k$ and setting for every $i \in [k]$

$$U_i = \bigcup_{x \in X_i} C_{L,x} \cup C_{L,i}$$

satisfies (i) and (ii). $\square$

Lemma 27. *For all integers $r \geq 2$ and $k \geq 1$, and every $\varepsilon > 0$ there exists an integer $T = T(r,k,\varepsilon)$ and a real $\alpha = \alpha(r,k,\varepsilon) > 0$ such that for every $n \in \mathbb{N}$ and every colouring $\gamma \in \mathcal{C}_n^{(r)}$ which is not (ε, T)-bounded, there are more than $\frac{\varepsilon}{3r^r}\binom{n}{r-1}$ sets in $\binom{[n]}{r-1}$, which are (k,α,γ)-good.*

Proof. Let $r \geq 2$, $k \geq 1$, and $\varepsilon > 0$ be given. Set $T = T(r, k+1, \varepsilon/3)$ as given by Lemma 25 and set $\alpha = \varepsilon/(3r^r)$. Assume for a contradiction that for some not (ε, T)-bounded colouring $\gamma \in \mathcal{C}_n^{(r)}$ there are at most $(\varepsilon/(3r^r))\binom{n}{r-1}$ sets $L \in \binom{[n]}{r-1}$ which are (k,α,γ)-good.

For simplicity we assume that $\mathrm{im}(\gamma) \subseteq \mathbb{N}$ and for every $L \in \binom{[n]}{r-1}$ let $\pi = \pi_L \colon \mathbb{N} \to \mathbb{N}$ be a bijection for which $|C_{L,\pi(1)}| \geq |C_{L,\pi(2)}| \dots$, where as above, $C_{L,\pi(i)} = \{v \in [n] \setminus L \colon \gamma(L \cup \{v\}) = \pi(i)\}$. This way for every (k,α,γ)-bad set $L \in \binom{[n]}{r-1}$ we have

$$\sum_{i > k} |C_{L,\pi(i)}| < \alpha n = \frac{\varepsilon}{3r^r} n\,. \tag{4.12}$$

We define an auxiliary colouring $\bar{\gamma}$ by setting for every $e \in K_n^{(r)}$

$$\bar{\gamma}(e) = \begin{cases} 0 & \begin{cases} \text{if} \quad e \text{ contains a } (k,\alpha,\gamma)\text{-good set or} \\ \text{if} \quad \gamma(e) = \pi_L(i) \text{ for some } i > k \\ \qquad \text{and some } (k,\alpha,\gamma)\text{-bad set } L \in \binom{e}{r-1}, \end{cases} \\ \gamma(e) & \text{otherwise}\,. \end{cases}$$

Since by assumption there are at most $(\varepsilon/(3r^r))\binom{n}{r-1}$ different (k,α,γ)-good sets and since (4.12) holds, we have

$$|\bar{\gamma}^{-1}(0)| \leq \frac{\varepsilon}{3r^r}\binom{n}{r-1} \times n + \alpha n \times \binom{n}{r-1} \leq \frac{2\varepsilon n^r}{3r^r} \leq \frac{2\varepsilon}{3}\binom{n}{r}.$$

Thus in total we recoloured at most $(2/3)\varepsilon\binom{n}{r}$ edges in $\bar{\gamma}$. On the other hand, by definition the colouring $\bar{\gamma}$ is $(r-1,k+1)$-local and, hence, by Lemma 25 it is $(\varepsilon/3,T)$-local. But this implies that the original colouring γ must be (ε,T)-bounded (as it differs from $\bar{\gamma}$ on at most $(2/3)\varepsilon\binom{n}{r}$ edges), which contradicts our assumption. □

4.3. *Proof of Theorem 8*

In this section we prove Theorem 8. However, we shall first prove a slightly weaker result, namely, Lemma 29 below. For the proof of this lemma, we need the following well known result of Erdős, which says that every sufficiently large and dense r-uniform hypergraph contains every r-partite r-uniform hypergraph of fixed order. We denote by $K^{(r)}(k;r)$ the complete r-partite r-uniform hypergraph with vertex classes of size k.

Theorem 28 (Erdős [3]). *For all integers $r \geq 2$ and $k \geq 1$ and every $\delta > 0$ there is some $n_0 = n_0(r,k,\delta)$ such that every r-uniform hypergraph G on $|V(G)| = n \geq n_0$ vertices with at least $\delta\binom{n}{r}$ edges, contains a copy of $K^{(r)}(k;r)$.* □

We now state and prove Lemma 29, which deals with edges of the unique, non-degenerate r-type $1^r = (1,\dots,1)$ and proves the first part of Theorem 8.

Lemma 29. *For all integers $q \geq r \geq 2$ and every $\varepsilon > 0$, there exist integers $T = T(r,q,\varepsilon)$ and $n_0 = n_0(r,q,\varepsilon)$ so that for every $n \geq n_0$ and every colouring $\gamma \in \mathcal{C}_n^{(r)}$ which is not (ε,T)-bounded the following holds. There exists a family $\mathcal{V} = \{V_1,\dots,V_r\}$ of mutually disjoint sets, each of cardinality q, such that with $\tau = (1,\dots,1) \in \mathbb{N}^r$ for all edges e, $e' \in (V_1,\dots,V_r)\langle\tau\rangle$*

$$\gamma(e) = \gamma(e') \;\Rightarrow\; e \cap V_1 = e' \cap V_1\,.$$

Proof. Let $q \geq r \geq 2$ and $\varepsilon > 0$ be given. Fix an integer k sufficiently large so that

$$k^{qr-1} < \binom{k}{q}^r. \tag{4.13}$$

We set the promised constant T to $T(r,k,\varepsilon)$ given by Lemma 27. Moreover, let $\alpha = \alpha(r,k,\varepsilon)$ be given by Lemma 27. We fix auxiliary constants s and δ by letting

$$s = \left\lceil \frac{\varepsilon}{3r^r}\binom{n}{r-1} \right\rceil \qquad \text{and} \qquad \delta = \frac{\varepsilon}{3r^r}\left(\frac{\alpha}{2k}\right)^k. \tag{4.14}$$

Finally, we set n_0 to $n_0(r-1,k,\delta)$ given by Theorem 28.

After we fixed the promised constants T and n_0, let $\gamma \in \mathcal{C}_n^{(r)}$ for $n \geq n_0$ be a not (ε,T)-bounded colouring. Due to the choice of the constants above, Lemma 27 implies that there exist at least s sets $L^1,\dots,L^s \in \binom{n}{r-1}$, which are (k,α,γ)-good. For each such L^σ, $\sigma \in [s]$, we are guaranteed by Proposition 26 to have a partition $\{U_1^\sigma,\dots,U_k^\sigma\}$ of $[n] \setminus L^\sigma$ satisfying properties (i) and (ii) of Proposition 26. In particular, property (ii) implies that for any set $P = \{p_1^\sigma,\dots,p_k^\sigma\} \in U_1^\sigma \times \dots \times U_k^\sigma$ the $(r-1,k)$-sunflower $S_P^\sigma = (L^\sigma,p_1^\sigma,\dots,p_k^\sigma)$ is an injective sunflower. Since by property (i) the sets $|U_i^\sigma| \geq \alpha n/(2k)$ for every $\sigma \in [s]$ and $i \in [k]$, we thus obtain

$$s \times \left(\frac{\alpha n}{2k}\right)^k \overset{(4.14)}{\geq} \delta n^k \binom{n}{r-1}$$

distinct, injective $(r-1,k)$-sunflowers. As there are less than n^k ways to choose k petals, there must be a set $W_1 = \{p_1, \dots, p_k\} \in \binom{[n]}{k}$ with more than $\delta\binom{n}{r-1}$ such injective $(r-1,k)$-sunflowers using $p_1, \dots, p_k$, the elements of W_1, for the k petals. The kernels of those sunflowers give rise to an auxiliary $(r-1)$-uniform hypergraph G on the vertex set $[n]$ with $\delta\binom{n}{r-1}$ edges. By the choice of n_0 and $n \geq n_0$ appealing to Theorem 28, we infer that G contains a copy of the complete $(r-1)$-partite hypergraph $K^{(r-1)}(k; r-1)$. Let $W_2, \dots, W_r \subseteq [n]$ be the vertex classes of cardinality k of that copy of $K^{(r-1)}(k; r-1)$. Recalling that the edges of G are actually kernels of $(r-1,k)$-sunflowers with the k petals coming from $W_1 = \{p_1, \dots, p_k\}$ implies that $W_1 \cap W_i = \emptyset$ for every $i = 2, \dots, r$ and, hence, $W_1, \dots, W_r$ is a family of mutually disjoint sets of cardinality k. Moreover, for every $L \in W_2 \times \dots \times W_r$ the $(r-1,k)$-sunflower $S = (L, p_1, \dots, p_k)$ is injective, thus for all x, $x' \in W_1$ with $x \neq x'$ we have

$$\gamma(L \cup \{x\}) \neq \gamma(L \cup \{x'\}) . \tag{4.15}$$

Our aim is to find sets $V_i \in \binom{W_i}{q}$ for all $i \in [r]$ such that for all not necessarily disjoint L, $L' \in V_2 \times \dots \times V_r$ and all distinct $x \neq x' \in W_1$ we have

$$\gamma(L \cup \{x\}) \neq \gamma(L' \cup \{x'\}) . \tag{4.16}$$

For that we call a family $\mathcal{V} = \{V_1, \dots, V_r\}$ of sets $V_i \in \binom{W_i}{q}$ *faulty* if the above condition is not satisfied. We count all faulty families. By definition, every faulty family contains two sets L, $L' \in V_2 \times \dots V_r$ and two points x, $x' \in V_1$ so that $\gamma(L \cup \{x\}) = \gamma(L' \cup \{x'\})$. There are at most $k^{|L \cup L'|+1}$ ways to choose L, L' and x. Once these are given, there is only one choice for x', because if there were two distinct choices, say x' and x'', then $\gamma(L \cup \{x\}) = \gamma(L' \cup \{x'\})$ and $\gamma(L \cup \{x\}) = \gamma(L' \cup \{x''\})$ would imply $\gamma(L' \cup \{x'\}) = \gamma(L' \cup \{x''\})$, which contradicts (4.15). So our choice of x' is forced. Now the remaining points in the family can be chosen arbitrarily, and there are at most k^{q-2} ways to complete V_1 and $k^{(r-1)q - |L \cup L'|}$ ways to complete $V_2, \dots, V_r$. But since

$$k^{|L \cup L'|+1} \times k^{q-2} \times k^{(r-1)q-|L \cup L'|} = k^{qr-1} \overset{(4.13)}{<} \binom{k}{q}^r,$$

there is at least one family $\mathcal{V} = \{V_1, \dots, V_r\}$, with $V_i \in \binom{W_i}{q}$ for $i \in [r]$, which is not faulty, i.e., it satisfies (4.16). □

We are finally able to give the proof of Theorem 8, which is based on Lemma 29 and Theorem 24.

Proof of Theorem 8. Let $q \geq r \geq 2$ and $\varepsilon > 0$ be given. First we define the constants T and n_0. For that let $\tau(1), \dots, \tau(\xi)$ be any list of all non-degenerate types (for r) in which each ℓ-type ($\ell \in [r]$) appears $\binom{r}{\ell}$ times. It will be convenient to assume that $\tau(\xi) = (1, \dots, 1)$ is the single copy of the unique non-degenerate r-type. Furthermore, let $\ell(i) \in [r]$ be so that $\tau(i)$ is an $\ell(i)$-type, i.e., let $\ell(i)$ denote the dimension of the vector $\tau(i)$. Finally, let $\Lambda(i) = \{\lambda_1(i) < \dots < \lambda_{\ell(i)}(i)\} \subseteq [r]$ be an ordered subset of $\ell(i)$ indices in $[r]$ so that every two copies $\tau(i_1)$ and $\tau(i_2)$ of the same type get different sets, i.e., $\Lambda(i_1) \neq \Lambda(i_2)$.

We define the following sequence of integers $q(\xi) \leq \dots \leq q(1)$ recursively by setting

$$q(i) = \begin{cases} q + \xi & \text{if} \quad i = \xi, \\ n\Big(\text{Thm.24}\big(q(i+1), r, \ell(i), \tau(i)\big)\Big) & \text{if} \quad i = \xi - 1, \dots, 1, \end{cases} \tag{4.17}$$

where $n(q,r,\ell,\tau)$ is given by Theorem 24. Finally, we fix the promised constants T and n_0 by appealing to Lemma 29 with $q(1)$ and ε. In fact, we set

$$T = T\big(\text{Lem.29}(q(1),\varepsilon)\big) \qquad \text{and} \qquad n_0 = n_0\big(\text{Lem.29}(q(1),\varepsilon)\big)\,. \tag{4.18}$$

Having defined the constants T and n_0, we let $\gamma \in \mathcal{C}_n^{(r)}$, for some $n \geq n_0$, be a not (ε, T)-bounded colouring.

Clearly, by our choice of T and n_0 in (4.18) we can apply Lemma 29. Consequently, there exists a family $\mathcal{V}(1) = \{V_1(1), \dots, V_r(1)\}$ of mutually disjoint sets, with

$$|V_1(1)| = \dots = |V_r(1)| = q(1)\,, \tag{4.19}$$

so that for all edges e, $e' \in (V_1(1), \dots, V_r(1))\langle\tau(\xi)\rangle$

$$\gamma(e) = \gamma(e') \;\Rightarrow\; e \cap V_1(1) = e' \cap V_1(1)\,. \tag{4.20}$$

Notice that this would already prove the first assertion of the theorem by choosing $\ell = r$ and $\tau = \tau(\xi) = (1, \dots, 1) \in \mathbb{N}^r$. However, at this point we cannot guarantee that all edges of degenerate r-type receive a colour different from the ones used so far, which we need for the moreover-part of Theorem 8. The idea to find the right value for ℓ is, roughly spoken, to go down with $\ell = r, r-1, \dots$ and stop just before all $J_j(i) = \emptyset$.

Next we apply Theorem 24 consecutively for $i = 1, \dots, \xi - 1$ to obtain a family $\mathcal{V}(i+1) = \{V_1(i+1), \dots, V_r(i+1)\}$, each of cardinality at least $q(i+1)$ and $V(i+1) \subseteq V(i)$. More precisely, given a family $\mathcal{V}(i) = \{V_1(i), \dots, V_r(i)\}$ of mutually disjoint sets, each of size $q(i)$, which exist for $i = 1$ due to (4.19), we apply Theorem 24 with $q(i+1)$, r, $\ell(i)$, and $\tau(i)$ to the family of sets $\{V_j \colon j \in \Lambda(i)\}$ and γ restricted to the union of those sets. Theorem 24 then gives rise to subsets $W_j(i) \subseteq V_j(i)$ for $j \in \Lambda(i) = \{\lambda_1(i) < \dots < \lambda_{\ell(i)}(i)$ and a $\tau(i)$-trace $\mathcal{J}(\tau(i)) = (J_1(i), \dots, J_{\ell(i)}(i))$, so that for all edges e, $e' \in (W_{\lambda_1(i)}(i), \dots, W_{\lambda_{\ell(i)}}(i))\langle\tau(i)\rangle$

$$\gamma(e) = \gamma(e') \;\Leftrightarrow\; (e \cap W_j(i))[J_j] = (e' \cap W_j(i))[J_j] \quad \forall j \in [\ell(i)]\,. \tag{4.21}$$

We conclude the inductive definition of $\mathcal{V}(i)$ by setting

$$V_j(i+1) = \begin{cases} W_j(i) & \text{if} \quad j \in \Lambda(i), \\ V_j(i) & \text{if} \quad j \notin \Lambda(i). \end{cases}$$

We call a $\tau(i)$-trace $\mathcal{J}(\tau(i)) = (J_1(i), \dots, J_{\ell(i)}(i))$ *monochromatic*, if $J_j(i) = \emptyset$ for every $j \in [\ell(i)]$, as in this case all $e \in (W_{\lambda_1(i)}(i), \dots, W_{\lambda_{\ell(i)}}(i))\langle\tau(i)\rangle$ receive the same colour. Fixing the $(\tau(\xi) = (1, \dots, 1))$-trace $\mathcal{J}(\tau(\xi)) = (\{1\}, \dots, \{1\})$, we have, in view of (4.20), a non-monochromatic trace for the unique non-degenerate r-type. Therefore, there exists a minimum integer $\ell_0 \in [r]$ for which there exists an ℓ_0-type, say $\tau(i_0)$ with corresponding index set $\Lambda(i_0)$, with a non-monochromatic trace $\mathcal{J}(\tau(i_0))$.

From the choice of ℓ_0 it follows that if $\Lambda(i) \subsetneq \Lambda(i_0)$, then $\mathcal{J}(\tau(i))$ is monochromatic. In particular, there exists a relabelling $U_1, \dots, U_{\ell_0}$ of the sets $W_j(i_0) = V_j(i_0+1)$ for $j \in \Lambda(i_0)$ such that for every degenerate ℓ_0-type τ the colouring γ is monochromatic on $(U_1, \dots, U_{\ell_0})\langle\tau\rangle$ and if $U_1 = W_j(i_0)$ then $J_j(i_0) \neq \emptyset$, which is possible since $\mathcal{J}(\tau(i_0))$ is non-monochromatic. Let $\tau^* = (\tau_1^*, \dots, \tau_{\ell_0}^*)$ be the vector which we obtain from $\tau(i_0) = (\tau_1(i_0), \dots, \tau_{\ell_0}(i_0))$ after reshuffling the entries corresponding to the relabelling above, i.e., if $U_j^* = W_{\lambda_j(i_0)}(i_0)$, then $\tau_j^* = \tau_j(i_0)$. Similarly, let $\mathcal{J}(\tau^*) = (J_1^*, \dots, J_{\ell_0}^*)$ be the corresponding reshuffling of $\mathcal{J}(\tau(i_0))$, where $J_1^* \neq \emptyset$. Therefore, from (4.21) we infer the first part of Theorem 8, i.e, for all edges e, $e' \in (U_1, \dots, U_{\ell_0})\langle\tau^*\rangle$

$$\gamma(e) = \gamma(e') \;\Rightarrow\; (e \cap U_1)[J_1^*] = (e' \cap U_1)[J_1^*]\,.$$

Moreover, due to the choice of the integers $q(i)$ in (4.17), we have $|U_j| \geq q(i_0+1) \geq q + \xi$ for all $j \in [\ell_0]$. Since there are less than ξ colours used by degenerate ℓ_0-types, the deletion of at most ξ many vertices from each U_j will produce the final family $\mathcal{W}$. □

4.4. *Proof of Theorem 5*

In this section, we deduce Theorem 5 from Theorem 8.

Proof of Theorem 5. Let H be an r-uniform hypergraph with at least two edges and v_H vertices and set

$$k := \Xi(H) = \min_{\substack{\tau\in\mathcal{T}^{(r)}\\ j_1\in[\tau_1]}} \max\left\{\left|\chi^{(r)}_{\tau,j_1,r\cdot v_H}(H_0)\right|\colon\ H_0\subseteq K^{(r)}_{r\cdot v_H}\right\}. \tag{4.22}$$

In (4.22) above as well as later in this proof, H_0 denotes a copy of H in some "large enough" complete hypergraph. We have to show that $k-2\le \mathrm{EssFin}(H)<k$. We first prove the upper bound. For that it suffices to give an example of a family of (H,k)-local colourings, that are not (ε,T)-bounded for a given $\varepsilon>0$ and every T. For that we note that for fixed $\varepsilon<r!/r^r$ and given T the colouring $\chi^{(r)}_{\tau,j_1,n}$ is not (ε,T)-bounded for any $\tau\in\mathcal{T}^{(r)}$, $j_1\in[\tau_1]$, and $n=n(\varepsilon,T)$ sufficiently large. Moreover, by definition of k in (4.22) there is some $\tau_0\in\mathcal{T}^{(r)}$ and some $j_1\in[\tau_1]$ such that $\chi^{(r)}_{\tau_0,j_1,n}$ is (H,k)-local and, hence,

$$\mathrm{EssFin}(H)<k\,. \tag{4.23}$$

We prove the lower bound by contradiction. So assume $\mathrm{EssFin}(H)<k-2$, i.e., there is an $\varepsilon>0$ such that for every T there exist an n and a colouring $\gamma\in\mathcal{L}^{(r)}_n(H,k-2)$ that is not (ε,T)-bounded. Let such an $\varepsilon>0$ be given. For $q=v_H$, r, and ε Theorem 8 yields T and n_0. Now suppose for some $n\ge n_0$ there exist some $\gamma\in\mathcal{L}^{(r)}_n(H,k-2)\subseteq\mathcal{C}^{(r)}_n$ which is not (ε,T)-bounded. Then by Theorem 8 there exist an integer $\ell_0\in[r]$, a non-degenerate ℓ_0-type $\tau=(\tau_1,\dots,\tau_{\ell_0})$, a set $\emptyset\ne J_1\subseteq[\tau_1]$, and a family $\mathcal{W}=\{W_1,\dots,W_{\ell_0}\}$ of mutually disjoint sets of cardinality q such that for all edges e, $e'\in(W_1,\dots,W_{\ell_0})\langle\tau\rangle$

$$\gamma(e)=\gamma(e')\ \Rightarrow\ (e\cap W_1)[J_1]=(e'\cap W_1)[J_1]\,. \tag{4.24}$$

Consequently, for $j_1=\min J_1$ we have

$$\begin{aligned}\max_{H_0\subseteq K^{(r)}_n}\left|\gamma(H_0)\right| &\ge \max\left\{\left|\gamma(H_0)\right|\colon\ H_0 \text{ induced on } \bigcup_{i\in[\ell_0]} W_i\right\}\\ &\overset{(4.24)}{\ge} -1+\max_{H_0\subseteq K^{(r)}_{\ell_0\cdot q}}\left|\chi^{(r)}_{\tau,j_1,\ell_0\cdot q}(H_0)\right|.\end{aligned} \tag{4.25}$$

Note that the "-1" is needed, because H_0 may contain edges of a non-degenerate ℓ_0-type $\tau'\ne\tau$. Theorem 8 gives us no control over the colour of those edges, but $\chi^{(r)}_{\tau,j_1,\ell_0\cdot q}(H_0)$ insists on a colour different from those used for the edges of type τ. However, if $r=2$, then there exist only one non-degenerate 1-type ($\tau=(2)$) and only one non-degenerate 2-type ($\tau=(1,1)$). Hence, for $r=2$ we infer

$$\max_{H_0\subseteq K^{(2)}_n}\left|\gamma(H_0)\right|\ge\max_{H_0\subseteq K^{(2)}_{\ell_0\cdot q}}\left|\chi^{(2)}_{\tau,j_1,\ell_0\cdot q}(H_0)\right|. \tag{4.26}$$

Moreover, since $\tau\in\mathcal{T}^{(r)}$ and $q\ge v_H$, we infer from (4.25) that

$$\max_{H_0\subseteq K^{(r)}_n}\left|\gamma(H_0)\right|\ge -1+\min_{\substack{\tau\in\mathcal{T}^{(r)}\\ j_1\in[\tau_1]}}\ \max_{H_0\subseteq K^{(r)}_{r\cdot v_H}}\left|\chi^{(r)}_{\tau,j_1,r\cdot v_H}(H_0)\right|.$$

But by definition of k in (4.22) this contradicts $\gamma\in\mathcal{L}^{(r)}_n(H,k-2)$. Hence $\mathrm{EssFin}(H)\ge k-2$ and (2.7) follows from (4.23) and (4.22).

The moreover-part of Theorem 5 for $r = 2$ follows in the same way. Colourings $\chi^{(2)}_{(2),1,n}$ and $\chi^{(2)}_{(2),2,n}$ are equivalent in the sense that

$$\max_{H_0 \subseteq K_n^{(2)}} \left|\chi^{(2)}_{(2),1,n}(H_0)\right| = \max_{H_0 \subseteq K_n^{(2)}} \left|\chi^{(2)}_{(2),2,n}(H_0)\right|$$

for every integer n. Recalling, that $\gamma_{\min,n} = \chi^{(2)}_{(2),1,n}$ and $\gamma_{\mathrm{bip},n} = \chi^{(2)}_{(1,1),1,n}$ we infer from (4.23) and (4.22) that

$$\mathrm{EssFin}(H) \le -1 + \min\left\{ \max_{H_0} |\gamma_{\min,2v_H}(H_0)|\,,\ \max_{H_0} |\gamma_{\mathrm{bip},2v_H}(H_0)| \right\} = k - 1\,,$$

where the H_0 range over all copies of H in $K_n^{(2)}$. Similarly, repeating the analysis as in the proof of $\mathrm{EssFin}(H) \ge k - 2$ for general r above, but using (4.26) instead of (4.25), we infer $\mathrm{EssFin}(H) \ge k - 1$ for $r = 2$. □

5. *Essentially unbounded colourings of the integers*

In this short, final section, we present the proof of Theorem 10. We shall use the following quantitative version of Szemerédi's theorem, which was proved for 3-term arithmetic progressions by Varnavides [16] and for k-term progressions by Frankl, Graham, and Rödl [7].

THEOREM 30 (Quantitative version of Szemerédi's theorem). *For every integer $k \ge 3$ and $\varepsilon > 0$ there exists $d = d(k, \varepsilon)$ and $n_1 = n_1(k, \varepsilon)$ such that for every $n \ge n_1$, every subset $X \subseteq [n]$ with $|X| \ge \varepsilon n$ contains at least dn^2 arithmetic progressions with k elements.* □

Proof of Theorem 10. We start with an argument similar to the one in the proof of Lemma 25. Let $k \ge 3$ and $\varepsilon > 0$ be given. We set

$$n_0 = n_1(k, \varepsilon), \qquad T = \left\lfloor \frac{1}{d(k,\varepsilon)} \binom{k}{2} \right\rfloor + 1\,, \tag{5.1}$$

where $n_1(k, \varepsilon)$ and $d(k, \varepsilon)$ are given by Theorem 30.

Let $n \ge n_0$ and $\gamma\colon [n] \to \mathbb{Z}$ be a colouring that is not (ε, T)-bounded. We denote by $C_i \subseteq [n]$ the set of integers that receive colour i, i.e., $C_i = \gamma^{-1}(i)$ and let $c_i := |C_i|$. Without loss of generality we may assume that $c_i = 0$ for every $i \le 0$ and $c_i \ge c_{i+1}$ for every $i \ge 1$. Moreover, for every $i \ge T$ we have $T \cdot c_i \le \sum_{j=1}^{T} c_j \le n$ and hence

$$c_i \le \frac{n}{T} \quad \text{for all} \quad i \ge T\,. \tag{5.2}$$

Next let $Y = C_1 \cup \cdots \cup C_T$. Clearly, $|\gamma(Y)| \le T$ and since γ is not (ε, T)-bounded

$$|Y| = \sum_{i=1}^{T} c_i < n - \varepsilon n.$$

Therefore $\sum_{i>T} c_i > \varepsilon n$ and we may apply Theorem 30 to the set $X = \bigcup_{i>T} C_i$. By Theorem 30 we obtain dn^2 arithmetic progressions with k elements inside X, where $d = d(k, \varepsilon)$. If one of them is injective, i.e., uses k colours, then we are done. Suppose that none of them is injective, so that each of them contains a monochromatic pair. In general, every monochromatic pair can prevent at most $\binom{k}{2}$ different k-term arithmetic progressions from being injective, which implies the following bounds:

$$dn^2 \binom{k}{2}^{-1} \le \#\{\text{ monochromatic pairs in } X\ \} \le \sum_{i>T} \binom{c_i}{2} \le \sum_{i>T} c_i^2 \le T \left(\frac{n}{T}\right)^2,$$

where for the last step we used the fact that the above sum is maximized when as many summands as possible take the maximum possible value as given by (5.2). This yields that $T \le \binom{k}{2}/d$, contradicting our choice of T in (5.1). $\square$

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B. Bollobás
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
USA
and
Trinity College
Cambridge CB2 1TQ
UK

`bollobas@msci.memphis.edu`

Y. Kohayakawa
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010
05508–090 São Paulo, SP
Brazil

`yoshi@ime.usp.br`

V. Rödl
Department of Mathematics and Computer Science
Emory University
Atlanta, GA 30322
USA

`rodl@mathcs.emory.edu`

M. Schacht
Institut für Informatik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
Germany

`schacht@informatik.hu-berlin.de`

A. Taraz
Zentrum Mathematik
Technische Universität München
Boltzmannstraße 3
D-85747 Garching bei München
Germany

taraz@ma.tum.de