# ON PERFECT MATCHINGS IN UNIFORM HYPERGRAPHS WITH LARGE MINIMUM VERTEX DEGREE

HIỆP HÀN, YURY PERSON, AND MATHIAS SCHACHT

Abstract. We study sufficient $\ell$-degree ($1 \leq \ell < k$) conditions for the appearance of perfect and nearly perfect matchings in $k$-uniform hypergraphs. In particular, we obtain a minimum vertex degree condition ($\ell = 1$) for 3-uniform hypergraphs, which is approximately tight, by showing that every 3-uniform hypergraph on $n$ vertices with minimum vertex degree at least $(5/9 + o(1))\binom{n}{2}$ contains a perfect matching.

## 1. Notations and Results

Our notation follows [2]. We refer to the set $\{1, 2, \dots, n\}$ with $n \in \mathbb{N}$ by $[n]$. For a set $M$ and an integer $k$, we denote by $\binom{M}{k} = \{A \subseteq M\colon |A| = k\}$ the set of all $k$-element subsets of $M$ and we denote by $(M)_k = \{(v_1, v_2, \dots, v_k)\colon \{v_1, \dots, v_k\} \in \binom{M}{k}\}$ the set of all ordered $k$-tuples of $M$. We often write $v_1 v_2 \dots v_k \in \binom{M}{k}$ instead of $\{v_1, v_2, \dots, v_k\} \in \binom{M}{k}$. Throughout this paper $\mathcal{H}$ denotes a $k$-uniform hypergraph, that is a pair $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ with vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H}) \subseteq \binom{V(\mathcal{H})}{k}$. Often we write $V$ instead of $V(\mathcal{H})$ and identify $\mathcal{H}$ with its edge set, i.e., $\mathcal{H} \subseteq \binom{V}{k}$. A $k$-uniform hypergraph is called $k$-partite if there is a partition of the vertex set $V$ into $k$ sets $V = V_1 \dot\cup \dots \dot\cup V_k$ such that every edge intersects every $V_i$ in exactly one vertex.

For a $k$-uniform hypergraph $\mathcal{H}$ and a set $T = \{v_1, \dots, v_\ell\} \in \binom{V(\mathcal{H})}{\ell}$ let $\deg(T) = \deg(v_1 \dots v_\ell)$ denote the number of edges containing $v_1 \dots v_\ell$ and let $\delta_\ell(\mathcal{H})$ be the minimum $\ell$-degree of $\mathcal{H}$, i.e., the minimum of $\deg(v_1 \dots v_\ell)$ over all $\ell$-element sets of vertices in $\mathcal{H}$. Moreover, by a matching of $\mathcal{H}$ we mean a subset $M \subseteq \mathcal{H}$ of pairwise disjoint edges of $\mathcal{H}$ and a perfect matching is a matching covering all vertices of $\mathcal{H}$. Of course, such a matching can only exist, if $n = |V|$ is a multiple of $k$, which we indicate by $n \in k\mathbb{Z}$.

**Definition 1.** *For all integers $k > \ell \geq 1$ and $n \in k\mathbb{Z}$ let $t(k, \ell, n)$ denote the minimum $t$ such that every $k$-uniform hypergraph $\mathcal{H}$ on $n$ vertices satisfying $\delta_\ell(\mathcal{H}) \geq t$ contains a perfect matching.*

For $k = 2$, in case of graphs, it is easily seen that $t(2, 1, n) = n/2$. Indeed, the complete bipartite graph $K_{n/2+1, n/2-1}$ serves as lower bound and the upper bound is an obvious consequence of Dirac's theorem on the existence of Hamilton cycles.

For $k \geq 3$, $\ell = k - 1$ and $n \in k\mathbb{Z}$ the number $t(k, k-1, n)$ was investigated by Kühn and Osthus [5] and Rödl et al. [12, 10, 9]. In particular, Rödl, Ruciński, and

The first author was supported by DFG within the research training group "Methods for Discrete Structures".

The second author was supported by GIF grant no. I-889-182.6/2005.

Szemerédi [10] determined $t(k, k-1, n)$ for arbitrary $k \geq 3$ and sufficiently large $n$ and showed

$$t(k, k-1, n) = n/2 - k + c_{k,n}\,, \tag{1}$$

where $c_{k,n} \in \{3/2, 2, 5/2, 3\}$ depending on the parities of $n$ and $k$. Another notable phenomenon is that nearly perfect matchings, i.e., matchings covering all but a constant number, say $rk$ (for $r \geq k-2$), of the vertices, already appear at minimum $(k-1)$-degree $n/k - r$ (see [12]). Furthermore, for $k \geq 4$ and $\lceil k/2 \rceil \leq \ell \leq k-1$, Pikhurko [8] showed

$$\frac{1}{2}\binom{n}{k-\ell} - O(n^{k-\ell-1}) \leq t(k, \ell, n) \leq \frac{1}{2}\binom{n}{k-\ell} + O(n^{k-\ell-1/2}\sqrt{\log n}). \tag{2}$$

Observe from (1) and (2) that $t(k, \ell, n)$ is roughly $\binom{n}{k-\ell}/2$ for $\lceil k/2 \rceil \leq \ell \leq k-1$. However, the approach in [8] breaks down for $1 \leq \ell < k/2$ and for this regime no sharp bounds are known so far. For example, for $\ell = 1$ it was asked by Kühn and Osthus [5] to determine $t(k, 1, n)$. The best known upper bound we are aware of is due to Daykin and Häggkvist [3], who showed $t(k, 1, n) \leq \frac{k-1}{k}\binom{n-1}{k-1} + 1/k$.

In the first part of this paper we will provide general upper bounds on the minimum $\ell$-degree which ensure the existence of perfect and nearly perfect matchings in $k$-uniform hypergraphs. First, we show an upper bound for the existence of nearly perfect matchings in $k$-uniform, $k$-partite hypergraphs. Here the minimum $\ell$-degree $\delta_\ell(\mathcal{H})$ of a $k$-uniform, $k$-partite hypergraph with vertex partition $V_1 \dot\cup \ldots \dot\cup V_k$ is $\min \deg(v_{i_1}, \ldots, v_{i_\ell})$, where the minimum runs over all index sets $\{i_1, \ldots, i_\ell\} \in \binom{[k]}{\ell}$ and all $\ell$-sets of vertices $v_{i_j} \in V_{i_j}$ for $j = 1, \ldots, \ell$.

**Theorem 2.** *Let $\mathcal{H}$ be a $k$-uniform, $k$-partite hypergraph with partition classes $V_1, \ldots, V_k$ each of size $|V_i| = n$ and suppose the minimum $\ell$-degree of $\mathcal{H}$ is*

$$\delta_\ell(\mathcal{H}) > \frac{k-\ell}{k} n^{k-\ell} + kn^{k-\ell-1}.$$

*Then $\mathcal{H}$ contains a matching covering all but $(\ell-1)k$ vertices. In particular, for $\ell = 1$ the matching is perfect.*

Using this we obtain the following bound for the existence of (nearly) perfect matchings for general $k$-uniform hypergraphs.

**Theorem 3.** *For all integers $k > \ell > 0$ there is an $n_0$ such that for all $n > n_0$ the following holds: Suppose $\mathcal{H}$ is a $k$-uniform hypergraph on $n > n_0$ vertices, $n \in k\mathbb{Z}$ with minimum $\ell$-degree*

$$\delta_\ell(\mathcal{H}) \geq \frac{k-\ell}{k}\binom{n}{k-\ell} + k^{k+1}(\ln n)^{1/2} n^{k-\ell-1/2},$$

*then $\mathcal{H}$ contains a matching covering all but $(\ell-1)k$ vertices. In particular, for $\ell = 1$ the matching is perfect.*

For $\ell = 1$ slightly better bounds, compared to Theorems 2 and 3, were obtained by Daykin and Häggkvist [3]. Those authors showed that the minimum degree condition $\delta_1(\mathcal{H}) > \frac{k-1}{k}(n^{k-1} - 1)$ yields perfect matchings in the partite case and $\delta_1(\mathcal{H}) > \frac{k-1}{k}(\binom{n-1}{k-1} - 1)$ yields perfect matchings in the general case.

Theorem 3 together with the absorbing technique, developed by Rödl, Ruciński, and Szemerédi, yields the following theorem about the existence of perfect matchings in $k$-uniform hypergraphs.

**Theorem 4.** *For all $\gamma > 0$ and all integers $k > \ell > 0$ there is a $n_0$ such that for all $n > n_0$, $n \in k\mathbb{Z}$ the following holds: Suppose $\mathcal{H}$ is a $k$-uniform hypergraph on $n > n_0$ vertices with minimum degree*

$$\delta_\ell(\mathcal{H}) \geq \left(\max\left\{\frac{1}{2}, \frac{k-\ell}{k}\right\} + \gamma\right)\binom{n}{k-\ell}$$

*then $\mathcal{H}$ contains a perfect matching.*

In other words the theorem says

$$t(k,\ell,n) \leq \left(\max\left\{\frac{1}{2}, \frac{(k-\ell)}{k}\right\} + o(1)\right)\binom{n}{k-\ell}$$

for any $k > \ell > 0$. For $\ell \geq k/2$ the maximum is $1/2$ and this bound, which is best possible up to the error term $o(1)$, was already shown by Pikhurko [8]. For $\ell < k/2$, however, there is a gap between currently known upper and lower bound, since the best lower bounds follow from well known constructions (see, e.g., [3, 5, 8, 10]).

**Fact 5.** *For all $k > 0$ and all $n \in k\mathbb{Z}$ there are $k$-uniform hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ on $n$ vertices with minimum $\ell$-degrees $(0 < \ell < k)$*

$$\delta_\ell(\mathcal{H}_1) = \binom{n-\ell}{k-\ell} - \binom{\frac{(k-1)n}{k} - \ell + 1}{k-\ell} = \left(1 - \left(\frac{k-1}{k}\right)^{k-\ell} - o(1)\right)\binom{n}{k-\ell}$$

$$\delta_\ell(\mathcal{H}_2) = \frac{1}{2}\binom{n}{k-\ell} + O(n^{k-\ell-1})$$

*which do not contain a perfect matching.*

*Proof.* In $\mathcal{H}_1$ we split the vertex set into sets $A$ and $B$ of size $|A| = \frac{n}{k} - 1$ and $|B| = \frac{(k-1)}{k}n + 1$ and take as edges of $\mathcal{H}_1$ all those $k$-tuples intersecting $A$ in at least one vertex. It is easily seen that $\delta_\ell(\mathcal{H}_1) = \binom{n-\ell}{k-\ell} - \binom{(k-1)n/k-\ell+1}{k-\ell}$. However, since every edge of a matching covers at least one vertex in $A$ and $|A| = \frac{n}{k} - 1$ there cannot exist a perfect matching.

For the second hypergraph $\mathcal{H}_2$ we split the vertex set into sets $A$ and $B$ such that $|A|$ is the maximal odd integer which does not exceed $n/2$. Further we take all edges intersecting $A$ in a even number of vertices. Then, due to parity, $\mathcal{H}_2$ does not contain a perfect matching and the minimum $\ell$-degree is $\frac{1}{2}\binom{n}{k-\ell} + O(n^{k-\ell-1})$. □

We believe that for small $\ell$ (compared to $k$) the lower bound given by $\mathcal{H}_1$ in Fact 5 is the right one. Indeed, the main result of this paper, justifies this for the case $k = 3$ and $\ell = 1$. Note that in this case $\delta_\ell(\mathcal{H}_1) = (5/9 - o(1))\binom{n}{2}$.

**Theorem 6** (Main result)**.** *For all $\gamma > 0$ there is an $n_0$ such that for all $n > n_0$, $n \in 3\mathbb{Z}$ the following holds: Suppose $\mathcal{H}$ is a 3-uniform hypergraph on $n$ vertices with*

$$\delta_1(\mathcal{H}) \geq \left(\frac{5}{9} + \gamma\right)\binom{n}{2}.$$

*Then $\mathcal{H}$ contains a perfect matching.*

In view of Fact 5, Theorem 6 is, up to the error term $\gamma\binom{n}{2}$, best possible and this answers the question of Kühn and Osthus [5] asymptotically in the case $k = 3$. Combining Theorem 6 with some previous results we give a classification of the existence of perfect and nearly perfect matchings in 3-uniform hypergraphs in terms of both minimum degrees $\delta_1$ and $\delta_2$ in Section 5.

**Organisation.** In Section 2 we introduce a few auxiliary results. In particular, we prove the Absorbing Lemma (Lemma 10). Section 3 contains the proofs of the upper bounds for $k$-uniform hypergraphs, i.e., Theorem 2, Theorem 3, and Theorem 4. Section 4 contains the proof of our main result, Theorem 6, and in Section 5 we study the interplay of $\delta_1$ and $\delta_2$ in view of perfect and nearly perfect matchings in 3-uniform hypergraphs. We close with a few open problems in Section 6.

## 2. Preliminary Results

2.1. **Partitioning uniform hypergraphs.** In this section we show, by a simple probabilistic argument, that there exists a partition of the vertex set of a hypergraph which distributes the vertex degrees fairly (similar results appeared in [5, 8]). We start with a folklore observation.

**Proposition 7.** *Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices. Then there is a decomposition of the edge set of $\mathcal{H}$ into $kn^{k-1}$ pairwise edge disjoint matchings.*

*Proof.* Consider the auxiliary graph $G$ on the vertex set $E(\mathcal{H})$ in which $A, B \in E(\mathcal{H})$ are connected if and only if $A$ and $B$ have nonempty intersection. Then the maximum degree of $G$ is at most $k\binom{n-1}{k-1}$. Thus $G$ has a proper colouring using $k\binom{n}{k-1}$ colours. And since the colour classes correspond to pairwise edge disjoint matchings we obtain the proposition. □

Next, let $V = V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_k$ be an equipartition of the vertex set of a $k$-uniform hypergraph $\mathcal{H}$, i.e., $|V_i| = |V_j|$ for all $i, j \in [k]$. For a set $T \subset V$ we say $T$ is crossing (with respect to $V_1, \dots, V_k$) if for all $i \in [k]$ we have $|T \cap V_i| \leq 1$. For a crossing $\ell$-set $T = \{v_1, \dots, v_\ell\}$ let $\deg'(T) = |\{E \in \mathcal{H}\colon T \subset E \text{ and } E \text{ is crossing}\}|$ denote its $k$-partite degree.

**Lemma 8.** *For all $k > \ell \geq 1$ there is a $n_0$ such that for all $n > n_0$, $n \in k\mathbb{Z}$ and every $k$-uniform hypergraph $\mathcal{H}$ on $n$ vertices there is an equipartition of $V(\mathcal{H}) = V_1 \dot\cup \dots \dot\cup V_k$ satisfying*

$$\deg'(T) \geq \frac{(k-\ell)!}{k^{k-\ell}} \deg(T) - 2(k \ln n)^{1/2} n^{k-\ell-1/2}$$

*for each crossing $\ell$-set $T \in \binom{V}{\ell}$.*

A similar lemma appeared in [8, Corollary 2], for completeness we include a short elementary proof.

*Proof.* First set $m = k - \ell$ and let $V = U_1 \dot\cup \dots \dot\cup U_k$ be a random partition of $V$, where each vertex appears in vertex class $U_j$ $(j = 1, \dots, k)$ independently with probability $1/k$. For a fixed $\ell$-set $T = \{v_1, \dots, v_\ell\}$ let $\mathcal{L} = \mathcal{L}(T)$ denote the link hypergraph of $T$ which consists of the vertex set $V(\mathcal{H})$ and the edge set $\mathcal{L} = \{E \in \binom{V}{m}\colon E \cup T \in \mathcal{H}\}$. Then $\mathcal{L}$ is an $m$-uniform hypergraph with $\deg(v_1, \dots, v_\ell)$ edges. Using Proposition 7 we decompose the edge set of $\mathcal{L}$ into at most $i_0 \leq mn^{m-1}$ nonempty pairwise edge disjoint matchings which we denote by $M_1, \dots, M_{i_0}$.

For every $i \in [i_0]$, every edge $E \in M_i$, and every index set $J \in \binom{[k]}{m}$, we say $E$ survived (in the partition $\bigcup_{j \in J} U_j$), if $|E \cap U_j| = 1$ for all $j \in J$. Since the partition $U_1, \dots, U_k$ was chosen randomly we have for fixed $J \in \binom{[k]}{m}$

$$\mathbb{P}[E \text{ survived}] = \frac{m!}{k^m}.$$

Thus, for $X_{i,J} = X_{i,J}(T) = |\{E \in M_i \colon E \text{ survived}\}|$ we have

$$\mu_{i,J} = \mu_{i,J}(T) = \mathbb{E}[X_{i,J}] = \frac{m!}{k^m}|M_i|.$$

Now call a matching $M_i$ bad (with respect to the chosen partition $U_1, \dots, U_k$) if there exists a set $J \in \binom{[k]}{m}$ such that

$$X_{i,J} \le \left(1 - \left(\frac{(4k-2)\ln n)}{\mu_{i,J}}\right)^{1/2}\right)\mu_{i,J}$$

and call $T$ a bad set (with respect to $U_1, \dots, U_k$) if there is at least one bad $M_i = M_i(T)$. Otherwise call $T$ a good set. For a fixed $M_i$ the events "$E$ survived" with $E \in M_i$ are jointly independent, hence we can apply Chernoff's inequality (see, e.g., [1]) and we obtain

$$\mathbb{P}[M_i \text{ is bad}] \le \tbinom{k}{m}\exp(-(2k-1)\ln n) = \tbinom{k}{m}n^{-2k+1}.$$

Summing over all matchings $M_i$ and recalling $i_0 \le mn^{m-1}$ and $m \le k-1$ yields

$$\mathbb{P}[\text{there is at least one bad } M_i] \le i_0\tbinom{k}{m}n^{-2k+1} \le n^{-k}$$

and summing over all $\ell$-sets $T$ we obtain

$$\mathbb{P}[\text{there is at least one bad } T] \le n^\ell n^{-k} \le n^{-1}.$$

Moreover, Chernoff's inequality yields

$$\mathbb{P}\left[\exists\, k_0 \in [k] \colon |U_{k_0}| > n/k + n^{1/2}(\ln n)^{1/4}/k\right] \le k\exp(-(\ln n)^{1/2}/(3k)) = o(1)\,.$$

Thus, with positive probability there is a partition $U_1, \dots, U_k$ such that all $\ell$-sets $T$ are good and such that

$$|U_j| \le n/k + n^{1/2}(\ln n)^{1/4}/k \quad \text{for every } j \in [k]\,.$$

Consequently, by redistributing at most $n^{1/2}(\ln n)^{1/4}$ vertices of the partition $U_1, \dots, U_k$ we obtain an equipartition partition $V = V_1 \dot\cup \dots \dot\cup V_k$ with

$$|V_j| = n/k \quad \text{and} \quad |U_j \setminus V_j| \le n^{1/2}(\ln n)^{1/4}/k \quad \text{for every } j \in [k]\,.$$

To verify that the partition $V_1, \dots, V_k$ satisfies the claim of the lemma note that for a crossing $\ell$ set $T$ and the $m$-set $J = \{j \in [k] \colon T \cap V_j = \emptyset\}$ we have

$$\begin{aligned}
\deg'(T) &\ge \sum_{i\in[i_0]}\left(1 - \left(\frac{(4k-2)\ln n)}{\mu_{i,J}(T)}\right)^{1/2}\right)\mu_{i,J}(T) - m\frac{n^{1/2}(\ln n)^{1/4}}{k}n^{m-1}\\
&\ge \sum_{i\in[i_0]}\mu_{i,J}(T) - ((4k-2)\ln n)^{1/2}\sum_{i\in[i_0]}(\mu_{i,J}(T))^{1/2} - (\ln n)^{1/4}n^{m-1/2}\\
&= \frac{m!}{k^m}\deg(T) - ((4k-2)\ln n)^{1/2}\sum_{i\in[i_0]}(\mu_{i,J}(T))^{1/2} - (\ln n)^{1/4}n^{m-1/2}\,.
\end{aligned}$$

The Cauchy-Schwarz inequality then gives

$$\sum_{i\in[i_0]}(\mu_{i,J}(T))^{1/2} \le \left(i_0\sum_{i\in[i_0]}\mu_{i,J}(T)\right)^{1/2} \le \left(mn^{m-1}\binom{n}{m}\right)^{1/2} \le n^{m-1/2}.$$

This implies that for the partition $V_1, \dots, V_k$ every crossing $\ell$-set $T$ satisfies

$$\begin{aligned}\deg'(T) &\geq \frac{m!}{k^m}\deg(T) - ((4k-2)^{1/2} + (\ln n)^{-1/4})(\ln n)^{1/2} n^{m-1/2} \\ &\geq \frac{m!}{k^m}\deg(T) - 2(k\ln n)^{1/2} n^{m-1/2}\,,\end{aligned}$$

which proves the lemma. □

2.2. **Absorbing Lemma.** In this section we prove an *absorbing lemma*, Lemma 10. The idea was introduced by Rödl, Ruciński, and Szemerédi, e.g., in [11] (see also [10]). The Lemma asserts the existence of a small and powerful matching in a hypergraph with high minimum degree which, by "absorbing" vertices, creates a perfect matching provided a nearly perfect matching was founded.

First consider the following simple proposition

**Proposition 9.** *Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices. For all $x \in [0,1]$ and all integers $m \leq \ell$ we have, if*

$$\delta_\ell(\mathcal{H}) \geq x\binom{n}{k-\ell}\,, \quad \text{then} \quad \delta_m(\mathcal{H}) \geq x\binom{n}{k-m} - O(n^{k-m-1})\,,$$

*where the constant in the error term only depends on $k$, $\ell$, and $m$.*

*Proof.* Consider a arbitrary $m$-set $T = \{v_1, \dots, v_m\} \in \binom{V(\mathcal{H})}{m}$. Then the condition on $\delta_\ell(\mathcal{H})$ implies that $T$ is contained in at least

$$\begin{aligned}\binom{k-m}{\ell-m}^{-1} \sum_{v_{m+1},\dots,v_\ell \in \binom{V\setminus T}{\ell-m}} \deg(v_1,\dots,v_\ell) \quad &\geq \quad \binom{k-m}{\ell-m}^{-1}\binom{n-m}{\ell-m} x \binom{n}{k-\ell} \\ &\geq \quad x\binom{n}{k-m} - O(n^{k-m-1})\end{aligned}$$

edges, and the proposition follows. □

**Lemma 10** (Absorbing lemma)**.** *For all $\gamma > 0$ and integers $k > \ell > 0$ there is an $n_0$ such that for all $n > n_0$ the following holds: Suppose $\mathcal{H}$ is a $k$-uniform hypergraph on $n$ vertices with minimum $\ell$-degree $\delta_\ell(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-\ell}$, then there exists a matching $M$ in $\mathcal{H}$ of size $|M| \leq \gamma^k n/k$ such that for every set $W \subset V \setminus V(M)$ of size at most $\gamma^{2k} n \geq |W| \in k\mathbb{Z}$ there exists a matching covering exactly the vertices in $V(M) \cup W$.*

*Proof.* Let $\mathcal{H}$ be a $k$-uniform hypergraph with $\delta_\ell(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-\ell}$. From Proposition 9 we know $\delta_1(\mathcal{H}) \geq \left(\frac{1}{2} + \gamma\right)\binom{n}{k-1}$ (for all large $n$) and it suffices to prove the lemma for $\ell = 1$.

Throughout the proof we assume (without loss of generality) that $\gamma \leq 1/10$ and let $n_0$ be chosen sufficiently large. Further set $m = k(k-1)$ and call a set $A \in \binom{V}{m}$ of size $m$ an **absorbing** $m$-set for $T = \{v_1, \dots, v_k\} \in \binom{V}{k}$ if $A$ spans a matching of size $k-1$ and $A \cup T$ spans a matching of size $k$, i.e., $\mathcal{H}[A]$ and $\mathcal{H}[A \cup T]$ both contain a perfect matching.

**Claim 11.** *For every $T = \{v_1, \dots, v_k\} \in \binom{V}{k}$ there are at least $\gamma^{k-1}\binom{n}{k-1}^k/2$ absorbing $m$-sets for $T$.*

*Proof.* Let $T = \{v_1, \dots, v_k\}$ be fixed. Since $n_0$ was chosen large enough there are at most $(k-1)\binom{n}{k-2} \le \gamma\binom{n}{k-1}$ edges which contain $v_1$ and $v_j$ for some $j \in \{2, \dots, k\}$. Due to the minimum degree of $\mathcal{H}$ there are at least $\binom{n}{k-1}/2$ edges containing $v_1$ but none of the vertices $v_2, \dots, v_k$. We fix one such edge $\{v_1, u_2, \dots, u_k\}$ and set $U_1 = \{u_2, \dots, u_k\}$. For each $i = 2, 3, \dots, k$ and each pair $u_i, v_i$ suppose we succeed to choose a set $U_i$ such that $U_i$ is disjoint to $W_{i-1} = \bigcup_{j\in[i-1]} U_j \cup T$ and both $U_i \cup \{u_i\}$ and $U_i \cup \{v_i\}$ are edges in $\mathcal{H}$. Then, for a fixed $i = 2, \dots, k$ we call such a choice $U_i$ good, motivated by $W_k = \bigcup_{i\in[k]} U_i$ being an absorbing $m$-set for $T$.

Note that in each step $2 \le i \le k$ there are $k + (i-1)(k-1) \le k^2$ vertices in $W_{i-1}$, thus the number of edges intersecting $u_i$ (or $w_i$ respectively) and at least one other vertex in $W_{i-1}$ is at most $k^2\binom{n}{k-2}$. So the restriction on the minimum degree implies that for each $i \in \{2, \dots, k\}$ there are at least $2\gamma\binom{n}{k-1} - 2k^2\binom{n}{k-2} \ge \gamma\binom{n}{k-1}$ choices for $U_i$ and in total we obtain $\gamma^{k-1}\binom{n}{k-1}^k/2$ absorbing $m$-sets for $T$. $\square$

Continuing the proof of the Lemma 10, let $\mathcal{L}(T)$ denote the family of all those $m$-sets absorbing $T$. From Claim 11 we know $|\mathcal{L}(T)| \ge \gamma^{k-1}\binom{n}{k-1}^k/2$.

Now, choose a family $\mathcal{F}$ of $m$-sets by selecting each of the $\binom{n}{m}$ possible $m$-sets independently with probability

$$p = \gamma^k n/\Delta \quad \text{with} \quad \Delta = 2\binom{n}{k-1}^k \ge 2n\binom{n}{m-1} \ge 2m\binom{n}{m}. \tag{3}$$

Then, by Chernoff's bound (see, e.g., [1]), with probability $1 - o(1)$, as $n \to \infty$ the family $\mathcal{F}$ fulfills the following properties:

$$|\mathcal{F}| \le \gamma^k n/m \tag{4}$$

and

$$|\mathcal{L}(T) \cap \mathcal{F}| \ge \gamma^{2k-1} n/5 \quad \forall T \in \binom{V}{k}. \tag{5}$$

Furthermore, using (3) we can bound the expected number of intersecting $m$-sets by

$$\binom{n}{m} \times m \times \binom{n}{m-1} \times p^2 \quad \le \quad \gamma^{2k} n/4.$$

Thus, using Markov's bound, we derive that with probability at least 3/4

$$\mathcal{F} \text{ contains at most } \gamma^{2k} n \text{ intersecting pairs.} \tag{6}$$

Hence, with positive probability the family $\mathcal{F}$ has all the properties stated in (4), (5) and (6). By deleting all the intersecting and non-absorbing $m$-sets in such a family $\mathcal{F}$ we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint absorbing $m$-sets which, due to $\gamma \le 1/10$, satisfies

$$|\mathcal{L}(T) \cap \mathcal{F}'| \ge \gamma^{2k-1} n/5 - \gamma^{2k} n \ge \gamma^{2k} n \quad \forall T \in \binom{V}{m}.$$

So, since $\mathcal{F}'$ consists of pairwise disjoint absorbing $m$-sets, $\mathcal{H}[V(\mathcal{F}')]$ contains a perfect matching $M$ of size at most $\gamma^k n/k$. Further, for any subset $W \subset V \setminus V(M)$ of size $\gamma^{2k} n \ge |W| \in k\mathbb{Z}$ we can partition $W$ into at most $\gamma^{2k} n/k$ sets of size $k$ and successively absorb them using a different absorbing $m$-set each time. Thus there exists a matching covering exactly the vertices in $V(\mathcal{F}') \cup W$. $\square$

As a consequence we obtain the following.

**Corollary 12.** *For all $\gamma > 0$ and $k > \ell \geq 1$ there is an $n_0$ such that for all $n_0 \leq n \in k\mathbb{Z}$ the following holds: If $\mathcal{H}$ is a $k$-uniform hypergraph on $n$ vertices with minimum $\ell$-degree $\delta_\ell(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-\ell}$ and for any set $U \subset V$ of size $|U| \leq \gamma^k n$ the remaining hypergraph $\mathcal{H}[V \setminus U]$ has a matching covering all but at most $\gamma^{2k} n$ vertices. Then $\mathcal{H}$ has a perfect matching.*

*Proof.* Let $\gamma$, $k$, and $\ell$ be given. Then, applying Lemma 10 yields $n_0$. Now let $\mathcal{H}$ be a $k$-uniform hypergraph on $n \geq n_0$ vertices with minimum $\ell$-degree $\delta_\ell(\mathcal{H}) \geq (1/2+2\gamma)\binom{n}{k-\ell}$. Then using Lemma 10 we can remove a matching $M$ of size $\gamma^k n/k$ from $\mathcal{H}$. Then, according to the assumption, the remaining hypergraph $\mathcal{H}[V\setminus V(M)]$ contains a matching $M'$ such that, $W$, the set of the uncovered vertices has size at most $\gamma^{2k} n \geq |W| \in k\mathbb{Z}$. But due to Lemma 10 there is a matching covering exactly those vertices in $V(M) \cup W$, which together with $M'$ forms a perfect matching of $\mathcal{H}$. □

## 3. General upper bounds for $k$-uniform hypergraphs

In this section we prove Theorems 2, 3, and 4. For this we verify general upper bounds on the minimum $\ell$-degree, which guarantee the existence of a perfect matching and nearly perfect matching in a $k$-uniform hypergraphs $\mathcal{H}$.

Let $\mathcal{H}$ be a $k$-uniform, $k$-partite hypergraph on the partition classes $V_0, \dots V_{k-1}$ and $M$ a matching in $\mathcal{H}$. For an edge $E \in \mathcal{H}$ we denote the unique vertex in $E \cap V_i$ by $v_i(E)$ and for notational convenience below we consider all additions in $\mathbb{Z}/k\mathbb{Z}$. Further let $T_i = (v_i, v_{i+1}, \dots, v_{i+\ell-1})$ with $i \in \mathbb{Z}/k\mathbb{Z}$ and $v_j \in V_j$ for all $j \in \{i, \dots, i+\ell-1\}$ and let $\mathcal{E} = (E_0, E_1, \dots, E_{k-\ell-1}) \in [M]_{k-\ell}$ be a $(k-\ell)$-tuple of matching edges. We say $T_i$ and $\mathcal{E}$ are **adjacent** if $\{v_i, \dots, v_{i+\ell-1}, v_{i+\ell}(E_0), \dots, v_{i+k-1}(E_{k-\ell-1})\} \in \mathcal{H}$. The set $N(T_i, (E_0, \dots, E_{k-\ell-1})) = \{v_{i+\ell}(E_0), \dots, v_{i+k-1}(E_{k-\ell-1})\}$ is called the **neighbour** of $T$ **with respect to** $\mathcal{E}$ and by $\deg(T_i, [M]_{k-\ell})$ we denote the number of $(k-\ell)$-tuples $\mathcal{E} \in [M]_{k-\ell}$ the tuple $T_i$ is adjacent to.

*Proof of Theorem 2.* For the proof keep in mind that all additions are considered in $\mathbb{Z}/k\mathbb{Z}$. Take $M$ to be a largest matching in $\mathcal{H}$. By adding arbitrary $k$-tuples if necessary, without loss of generality we may assume $|M| = n-\ell$. Then there are $\ell k$ unmatched vertices which we divide into $k$ pairwise disjoint sets $T_0, \dots, T_{k-1}$ with $T_i = \{v_i, v_{i+1} \dots, v_{i+\ell-1}\}$ where $v_j \in V_j$.

For an arbitrary edge $E \in \mathcal{H}$ say $E$ is $M$-non-crossing if there is an $F \in M$ such that $|E \cap F| \geq 2$. Then, for a fixed $i = 1, 2, \dots, k-1$, the number of $M$-non-crossing edges $E$ with $T_i \subset E$ and $T_j \cap E = \emptyset$ for all $j \neq i$ is at most $kn^{k-\ell-1}$. Hence, the restriction on the minimum $\ell$-degree implies

$$\deg(T_i, [M]_{k-\ell}) \geq \delta_\ell(\mathcal{H}) - kn^{k-\ell-1} > \frac{k-\ell}{k} n^{k-\ell}.$$

And since this is true for each $T_i, i \in \{0, \dots, k-1\}$ the total degree is

$$\deg(T_0 \dots T_{k-1}, [M]_{k-\ell}) := \sum_{i \in \{0,\dots,k-1\}} \deg(T_i, [M]_{k-\ell}) > (k-\ell)n^{k-\ell}.$$

Then, by averaging, we conclude that there must be a $(k-\ell)$-tuple of matching edges $(E_0, \dots, E_{k-\ell-1})$ which is adjacent to at least $(k-\ell+1)$ tuples $T_i$. And without loss of generality let those $T_i$ be $T_0, \dots, T_{k-\ell}$. From the definition note that $N(T_i, (E_0, \dots, E_{k-\ell-1})) = \{v_{i+\ell}(E_0), \dots, v_{i+k-1}(E_{k-\ell-1})\}$, the neighbours of those $T_i$ with respect to $(E_0, \dots, E_{k-\ell-1})$, are pairwise disjoint. And since each pair $T_i$ and $N(T_i, (E_0, \dots, E_{k-\ell-1}))$ form an edge in $\mathcal{H}$ the $(k-\ell+1)$ tuples $T_i$ and their neighbours $N(T_i, (E_0, \dots, E_{k-\ell-1}))$ form a matching of size $(k-\ell+1)$ in $\mathcal{H}$. Replacing $E_0, \dots, E_{k-\ell-1}$ by this matching we obtain a larger matching. □

*Proof of Theorem 3.* Let $n_0$ be as asserted by Lemma 8 for given $k$ and $\ell$. Next let $\mathcal{H}$ be a $k$-uniform hypergraph on $n > n_0$ vertices, $n \in k\mathbb{Z}$, with minimum $\ell$-degree

$$\delta_\ell(\mathcal{H}) \geq \frac{k-\ell}{k}\binom{n}{k-\ell} + k^{k+1}(\ln n)^{1/2} n^{k-\ell-1/2}.$$

According to Lemma 8 there is a partition of $V = V(\mathcal{H})$ into $k$ partition classes $V = V_0 \dot\cup \dots \dot\cup V_{k-1}$ such that $|V_i| = |V_j| = n/k =: m$ for all $i, j$ and every crossing $\ell$-set $T$ satisfies

$$\deg'(T) \geq \frac{(k-\ell)!}{k^{k-\ell}}\delta_\ell(\mathcal{H}) - 2(k \ln n)^{1/2} n^{k-\ell-1/2}.$$

Using $(m)_{k-\ell} \geq m^{k-\ell} - m^{k-\ell-1}\sum_{i\in[k-\ell]} i$ a simple calculation yields

$$\deg'(T) \geq \frac{k-\ell}{k} m^{k-\ell} + k m^{k-\ell-1}$$

for all crossing $\ell$-sets $T$. By Theorem 2 this ensures a matching covering all but $(\ell-1)k$ vertices. □

*Proof of Theorem 4.* Let $\gamma > 0$ and integers $k > \ell > 0$ be given. Applying Corollary 12 with $\gamma_1 = \gamma/(4k)$ and $k, \ell$ we obtain $n_0'$. Applying Theorem 3 with the same $k$ and $\ell$ we obtain $n_0''$. Set $n_0 = \max\{n_0', 2n_0'', 4k^{4k}/\gamma^2\}$ and let $\mathcal{H}$ be a $k$-uniform hypergraph on $k\mathbb{Z} \ni n > n_0$ vertices with minimum $\ell$-degree

$$\delta_\ell(\mathcal{H}) \geq \left(\max\left\{\frac{1}{2}, \frac{k-\ell}{k}\right\} + \gamma\right)\binom{n}{k-\ell}.$$

We want to apply Corollary 12 and pick a set $U$ of size $|U| \leq \gamma_1^k n$. Then the remaining graph $\mathcal{H}_U = \mathcal{H}[V \setminus U]$ has minimum degree

$$\delta_\ell(\mathcal{H}_U) \geq \delta_\ell(\mathcal{H}) - \gamma_1^k n\binom{n}{k-\ell-1} \geq \left(\max\left\{\frac{1}{2}, \frac{k-\ell}{k}\right\} + \frac{\gamma}{2}\right)\binom{n}{k-\ell}$$

According to Theorem 3 there is a matching in $\mathcal{H}_U$ covering all but $(\ell-1)k \leq \gamma_1^{2k} n$ vertices. Thus, by Corollary 12, $\mathcal{H}$ contains a perfect matching. □

Note that according to Fact 5 for $\ell \geq k/2$ the Theorem 4 is best possible up to the constant $\gamma$.

## 4. Asymptotic bound for 3-uniform hypergraphs

In this section we prove Theorem 6. The major part is devoted to proving the existence of a matching covering $(1-o(1))n$ vertices in a 3-uniform hypergraph with sufficiently high minimum degree. Together with Corollary 12 it will immediately imply Theorem 6.

4.1. **Auxiliary results.**

**Definition 13.** *Let $M$ be a matching in a 3-uniform hypergraph $\mathcal{H}$. For a vertex $v \in V(\mathcal{H})$ we define the link graph of $v$ with respect to the edges $E_1E_2\dots E_k \in \binom{M}{k}$ to be the graph $L_v(E_1\dots E_k)$ with the vertex set $\bigcup_{i\in[k]} E_i$ and the edge set*

$$\{ab\colon \exists i,j \in [k],\ i \neq j \text{ such that } a \in E_i, b \in E_j \text{ and } vab \in \mathcal{H}\}.$$

Observe that for a large matching $M$ covering all but $o(n)$ vertices of the hypergraph $\mathcal{H}$ we have $e(L_v(M)) \approx \deg(v)$. We will study the link graphs $L_v(M)$ of the vertices $v \in V(\mathcal{H}) \setminus V(M)$ with respect to a largest matching $M$ in $\mathcal{H}$. Our goal is to derive a contradiction by showing that either $M$ can be enlarged or $\mathcal{H}$ must have a rigid structure, which will violate the minimum degree condition of $\mathcal{H}$.

The following statements will be useful for the analysis of the link graph.

**Fact 14.** *Let $B$ be a bipartite graph on six vertices with the partition classes $E = \{e_1, e_2, e_3\}$ and $F = \{f_1, f_2, f_3\}$. Then the following holds:*

(1) *if $e(B) \geq 7$ then $B$ contains a perfect matching,*
(2) *if $e(B) = 6$ then either $B$ contains a perfect matching or is isomorphic to $B_{033}$ (see Figure 1),*
(3) *if $e(B) = 5$ then either $B$ contains a perfect matching or $B$ is isomorphic to a graph in $\{B_{023}, B_{113}\}$ (see Figure 1).*

*Proof.* Suppose $\deg(e_1) \leq \deg(e_2) \leq \deg(e_3)$. Then from $e(B) \geq 7$ we infer $\deg(e_1) \geq 1, \deg(e_2) \geq 2$ and $\deg(e_3) \geq 3$, thus $B$ contains a perfect matching.

For $e(B) = 5$ we consider two cases: $\deg(e_1) = 0$ and $\deg(e_1) = 1$. In the first case we have $\deg(e_2) = 2$ and $\deg(e_3) = 3$ and $B$ is isomorphic to $B_{023}$. If $\deg(e_1) = 1$ then again we distinguish two cases. If $\deg(e_2) = 2$ then $\deg(e_3) = 2$ and $B$ is either isomorphic to $B_{023}$ or contains a perfect matching. Else $\deg(v_2) = 1$ and $\deg(v_3) = 3$ and in this case either $B$ is isomorphic to $B_{113}$ or contains a perfect matching.

Finally we consider $e(B) = 6$. Observe that adding one edge to $B_{113}$ we obtain a graph with a perfect matching since one vertex class has the degree sequence $1, 2, 3$. Adding an edge to $B_{023}$ we see that the resulting graph contains a perfect matching unless it is isomorphic to $B_{033}$. □

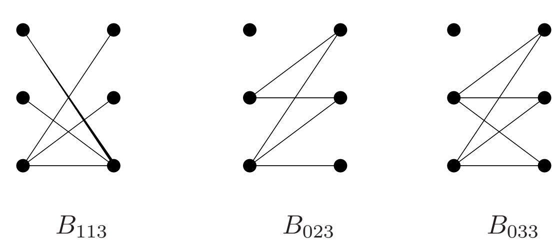


FIGURE 1. The critical graphs: the only balanced bipartite graphs on six vertices and six or five edges without a perfect matching.

We will also need the following result from extremal graph theory which follows from the work of Goodman in [4] (see also [7, 6] ).

**Theorem 15.** *For all $\varepsilon' > 0$ there is a $c = c(\varepsilon') > 0$ and $n_0 = n_0(\varepsilon')$ such that for all $n \geq n_0$ the following holds. Suppose $G$ is a graph on $n$ vertices which contains at least $(1/2 + \varepsilon')\binom{n}{2}$ edges. Then $G$ contains $cn^3$ triangles.* □

The following theorem asserts the existence of a matching covering all but $o(n)$ vertices.

**Theorem 16.** *For all $\gamma > 0$ there is a $n_0$ such that for all $n > n_0$ the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on $n$ vertices with minimum degree $\delta(\mathcal{H}) \geq (5/9 + 4\gamma)\binom{n}{2}$ then $\mathcal{H}$ contains a matching leaving strictly less than $\gamma n$ vertices unmatched.*

*Proof.* For a given $\gamma$ define $\varepsilon = \gamma/150$. Applying Theorem 15 with $\varepsilon' = \min\{\gamma^2, \varepsilon\}$ we obtain $c$ and $n_0'$. Then choose $n_0 = \max\{2^{110}/\varepsilon^5, 2^{50}/c\varepsilon^4, n_0'/\varepsilon\}$.

Next let $M$ be a matching of maximum size in $\mathcal{H}$ and suppose $|M| = \lfloor (1-\gamma)n/3 \rfloor$. (Otherwise we can simply add arbitrary 3-tuples to $M$ to guarantee equality, since we will show that $M$ is not a maximum matching.) Let $X = V(\mathcal{H}) \setminus V(M)$ be the set of the uncovered vertices. Then from the restriction on the minimum degree we infer that the number of edges in the link graph of every vertex $v \in X$ with respect to $M$ is

$$e(L_v(M)) \geq \deg_{\mathcal{H}}(v) - 3|M| - |X|(n - |X|) > \left(\frac{5}{9} + \gamma\right)\binom{n}{2}. \tag{7}$$

To derive a contradiction to (7) it is sufficient to show that there is a vertex $v \in X$ such that the pairs $EF \in \binom{M}{2}$ satisfying $e(L_v(EF)) \geq 6$ contribute at most $30\varepsilon n^2$ edges to $L_v(M)$ in total, since then we would obtain

$$e(L_v(M)) \leq 5\binom{|M|}{2} + 30\varepsilon n^2 < \left(\frac{5}{9} + \gamma\right)\binom{n}{2}. \tag{8}$$

We first prove the following fact.

**Fact 17.** *There are no $v_1v_2v_3 \in \binom{X}{3}$ and $EF \in \binom{M}{2}$ such that*

- $L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ *and*
- $L_{v_1}(EF)$ *contains a perfect matching,*

*Proof.* Let $E = \{a, u, x\}$, $F = \{b, w, y\}$ and let the perfect matching in $L_{v_1}(EF)$ consist of the edges $ab, uw$ and $xy$. Since these edges belong to the link graph of all $v_i, 1 \leq i \leq 3$, we have that $v_1ab, v_2uw, v_3xy \in \mathcal{H}$. Thus, one can replace $E$ and $F$ by these three edges to obtain a larger matching with contradiction to $M$ being the maximum matching. □

**Fact 18.** *Let $Y_1 \subset X$ consist of those vertices $v \in X$ for which there are at least $\varepsilon n^2$ pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains a perfect matching. Then $|Y_1| \leq \varepsilon n$.*

*Proof.* Consider the auxiliary bipartite graph $G_1$ with vertex classes $Y_1$ and $\binom{M}{2}$ and $\{v, EF\}$ being an edge if and only if $L_v(EF)$ contains a perfect matching. Then $G_1$ has at least $|Y_1|\varepsilon n^2$ edges and if $|Y_1|$ exceeds $\varepsilon n$, by averaging, there is a pair $EF \in \binom{M}{2}$ such that $\deg_{G_1}(EF) \geq \varepsilon^2 n$. Since the number of bipartite graphs on six vertices having a perfect matching is at most $2^9$ we conclude from the choice of $n_0$ that there are $\varepsilon^2 n/2^9 \geq 3$ vertices $v_1, v_2, v_3 \in Y_1$ such that $L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ and $L_{v_1}(EF)$ containing a perfect matching. This yields a contradiction to Fact 17. □

Now remove $Y_1$ from $X$ to obtain the set $X_1 \subset X$ of size $|X_1| \geq \gamma n/2$. Note that from Fact 14 each vertex $v \in X_1$ satisfies the following: for all but $\varepsilon n^2$ pairs

$EF \in \binom{M}{2}$ the link graph $L_v(EF)$ either contains at most four edges or is isomorphic to a graph in $\{B_{113}, B_{023}, B_{033}\}$.

Next we introduce some further notations. For a vertex $v \in X$ let

- $\mathcal{A}(v) = \{EF \in \binom{M}{2} \colon\ L_v(EF) \simeq B_{113}\}$,
- $R(v) = \{E \in M \colon$ there are $\varepsilon n$ elements $F \in M$ with $EF \in \mathcal{A}(v)\}$.
- $\mathcal{B}(v) = \{EF \in \binom{M}{2} \colon\ L_v(EF) \simeq B \in \{B_{023}, B_{033}\}\}$.

The remaining part of the proof is now devoted to showing

$$|\mathcal{B}(v)| \le 2\varepsilon n^2 \tag{9}$$

for some vertex $v \in X_1$. This with Fact 18 would imply

$$e(L_v(M)) \le 5|\mathcal{A}(v)| + 6|\mathcal{B}(v)| + 9\varepsilon n^2 + 4\left(\binom{|M|}{2} - |\mathcal{A}(v)| - |\mathcal{B}(v)|\right) \le 5\binom{|M|}{2} + 21\varepsilon n^2$$

thus (8) follows, and by contradiction, we obtain the theorem.

To this end we first argue that there are only few pairs in $\mathcal{B}(v)$ with both elements located in $R(v)$.

**Fact 19.** *There are no $v_1 \dots v_5 \in \binom{X_1}{5}$ and $(E, F, G, H) \in (M)_4$ such that*

(1) $L_{v_i}(EFGH) = L_{v_j}(EFGH)$ *for all* $i, j \in [5]$,
(2) $\{E, F\}, \{G, H\} \in \mathcal{A}(v_1)$, *and* $\{F, G\} \in \mathcal{B}(v_1)$.

*Proof.* It is sufficient to show that there is a matching of size five in $L_{v_i}(EFGH)$. With the five vertices $v_1 \dots v_5$ this yields a matching of size five in $\mathcal{H}$ and using this as replacement of $EFGH$ yields a contradiction to the maximality of $M$.

To this end note first that since $L_{v_1}(EF) \simeq B_{113}$ there is a vertex of degree three in each $E$ and $F$ which we denote by $e_1 \in E$ and $f_1 \in F$. The same holds for $G$ and $H$ and we denote the respective vertices by $g_1 \in G$ and $h_1 \in H$. Note that for a graph $B \in \{B_{023}, B_{033}\}$, $B$ contains two vertices of degree at least two in each partition class. Consequently, since $L_{v_i}(FG) \simeq B \in \{B_{023}, B_{033}\}$ there is a vertex $f_2 \in F$, $f_2 \ne f_1$ which has at least two neighbours in $G$. Thus we can pick the edge $f_2 g_2$ in $L_{v_1}(FG)$ such that $g_2 \ne g_1$. In the graph $L_{v_1}(EF)$ (and $L_{v_1}(GH)$, resp.), by using the vertices $f_1, e_1$ (and $g_1, h_1$, resp.), we now find a matching of size two which does not cover the vertex $f_2$ and $g_2$. This together yields a matching of size five in $L_{v_i}(EFGH)$. □

**Fact 20.** *Let $Y_2 \subset X_1$ consist of those vertices $v \in X_1$ such that there are at least $\varepsilon n^2$ pairs $FG \in \binom{R(v)}{2}$ with $FG \in \mathcal{B}(v)$. Then $|Y_2| \le \varepsilon n$.*

*Proof.* Consider the auxiliary bipartite graph $G_2$ with vertex classes $Y_2$ and $(M)_4$ with $\{v, (E, F, G, H)\}$ being an edge if and only if $EF, GH \in \mathcal{A}(v)$ and $FG \in \mathcal{B}(v)$. Note that for each pair $FG \in \binom{R(v)}{2}$ with $FG \in \mathcal{B}(v)$, by definition of $R(v)$ there are at least $\varepsilon n(\varepsilon n - 1) > (\varepsilon n)^2/2$ pairs $(E, H) \in (M)_2$ such that $\{v, (E, F, G, H)\} \in E(G_2)$. Hence, $v$ has at least $\varepsilon n^2 (\varepsilon n)^2/2$ neighbours and $G_2$ contains at least $|Y_2|\varepsilon^3 n^4/2$ edges.

Suppose $|Y_2| > \varepsilon n$ then, by averaging, there is a $EFGH \in (M)_4$ which has at least $\varepsilon^4 n$ neighbours in $G_2$. Since the number of graphs on twelve vertices does not exceed $2^{66}$ from the choices of $n_0$ we obtain $\varepsilon^4 n/2^{66} \ge 5$ vertices $v_1 \dots v_5 \in \binom{Y_1}{5}$ such that $L_{v_i}(EFGH) = L_{v_j}(EFGH)$ for all $i, j \in [5]$. This contradicts Fact 19. □

Next let $X_2 = X_1 \setminus Y_2$ and $S(v) = M \setminus R(v)$ for $v \in X_2$. Note that $|S(v)| > \varepsilon n$ otherwise from the previous fact we have at most

$$\binom{|S(v)|}{2} + |R(v)||S(v)| + \varepsilon n^2 \leq 2\varepsilon n^2 \tag{10}$$

pairs in $\mathcal{B}(v)$ which by (9) yields the theorem. Now we argue that there are only few pairs of $\mathcal{B}(v)$ containing one element from $R(v)$ and the other from $S(v)$.

**Fact 21.** *There are no $v_1 \dots v_6 \in \binom{X_2}{6}$ and $(E, F, G, H, I) \in (M)_5$ such that*

(1) $L_{v_i}(EFGHI) = L_{v_j}(EFGHI)$ *for all* $i, j \in [5]$,
(2) $\{E, F\}, \{H, I\} \in \mathcal{A}(v_1)$ *and* $\{F, G\}, \{G, H\} \in \mathcal{B}(v_1)$.

*Proof.* Again it is sufficient to prove that one can find a matching of size six in $L_{v_1}(EFGHI)$. To this end first denote the vertices with degree three in $L_{v_1}(EF)$ by $e_1 \in E$, $f_1 \in F$ (and in $L_{v_1}(HI)$ by $h_1 \in H, i_1 \in I$, resp.). Since $FG \in \mathcal{B}(v_1)$ there are two vertices in $G$ having two neighbours in $F$. The same holds for $GH \in \mathcal{B}(v_1)$. Thus there are $g_1, g_2 \in G, g_1 \neq g_2$ such that $g_1$ has two neighbours in $F$ and $g_2$ has two neighbours in $H$. Using them we can pick two matching edges in $L_{v_1}(FGH)$ which avoid $f_1$ and $h_1$. Now the vertices $e_1, f_1$ (and $h_1, i_1$, resp.) can be extended to a matching of size two in $L_{v_1}(EF)$ (and $L_{v_1}(HI)$, resp.) which leaves the chosen neighbours of $g_1$ (and $g_2$, resp.) uncovered. Together this yields a matching of size six. □

**Fact 22.** *Let $Y_3 \subset X_2$ consist of all those vertices $v \in X_2$ such that there are at least $\varepsilon n^2$ pairs $(E, F) \in R(v) \times S(v)$ which satisfy $EF \in \mathcal{B}(v)$. Then $|Y_3| \leq \varepsilon n$.*

*Proof.* For a vertex $v \in Y_3$ and a $G \in S(v)$ let $x_G$ denote the number of those $F \in R(v)$ such that $FG \in \mathcal{B}(v)$. Then there are $x_G(x_G - 1)$ choices $(F, H) \in (R(v))_2$ such that $FG, HG \in \mathcal{B}(v)$. And since $F, H \in R(v)$ we have at least $\varepsilon n(\varepsilon n - 1)$ choices $(E, I) \in (M)_2$ such that $EF, HI \in \mathcal{A}(v)$. Thus $G$ gives rise to at least $x_G^2(\varepsilon n)^2/2$ sets $(E, F, H, I) \in (M)_4$ satisfying $EF, HI \in \mathcal{A}(v)$ and $FG, GH \in \mathcal{B}(v)$. Recall that $s = |S(v)| > \varepsilon n$ according to (10) and that $\sum_{G \in S(v)} x_G \geq \varepsilon n^2$ since $v \in Y_3$. From Jensen's inequality and $s < n/3$ we obtain:

$$\frac{(\varepsilon n)^2}{2} \sum_{G \in S(v)} x_G^2 \quad \geq \quad \frac{(\varepsilon n)^2}{2} s \left( \sum \frac{1}{s} x_G \right)^2 \geq \varepsilon^4 n^5. \tag{11}$$

Thus a vertex $v \in Y_3$ gives rise to at least $\varepsilon^4 n^5$ ordered tuples $(E, F, G, H, I) \in (M)_5$ which satisfy $EF, HI \in \mathcal{A}(v)$ and $FG, GH \in \mathcal{B}(v)$. We consider the auxiliary bipartite graph $G_3$ with vertex classes $Y_3$ and $(M)_5$ and $\{v, (E, F, G, H, I)\}$ being an edge if and only if $(E, F, G, H, I)$ satisfies $EF, HI \in \mathcal{A}(v)$ and $FG, GH \in \mathcal{B}(v)$. If $|Y_3|$ exceeds $\varepsilon n$ then $G_3$ contains at least $\varepsilon^5 n^6$ edges. Then by averaging and the choice of $n_0$ we find $v_1 \dots v_6$ which with $EFGHI$ meet the conditions in Fact 21. This yields a contradiction. □

Let $X_3 = X_2 \setminus Y_3$ and note that $|X_3| \geq \gamma n/4$. Now before deriving the contradiction, we show that the density of $\mathcal{B}(v)$ in $S(v)$ is at most $1/2 + \varepsilon$.

**Fact 23.** *There are no $v_1 \dots v_4$ and $EFG \in \binom{M}{3}$ such that*

(1) $L_{v_1}(EFG) = L_{v_2}(EFG) = L_{v_3}(EFG)$,
(2) $EF, FG, GE \in \mathcal{B}(v_1)$.

*Proof.* Similar to the previous arguments we are looking for a matching of size four in the graph $L_{v_1}(EFG)$. To this end denote the isolated vertex in $L_{v_1}(EF)$ by $x_1$, the one in $L_{v_1}(FG)$ by $x_2$ and the one in $L_{v_1}(GE)$ by $x_3$. Then there are $1 \leq i, j \leq 3$ such that $x_i$ and $x_j$ belong to different edges and without loss of generality let $x_1 \in E$ and $x_2 \in F$. Since in the link graph $L_{v_1}(EF)$ the vertex $x_1$ is not adjacent to any vertex of $F$ there must be a vertex $e_2 \in E$ which has degree three, hence is adjacent to $x_2$. Take $e_2x_2$ as the first matching edge. In the link graph $L_{v_1}(GE)$ there is a vertex $g_1 \in G$ of degree at least two. This we use to match a vertex $e_1 \neq e_2$ in $E$. Note that $e_2$ could equal $x_1$. Lastly in the link graph $L_{v_1}(FG)$ the remaining vertices $f_1 \neq x_2 \neq f_2$ have degree at least two, hence they can be used to create a matching of size two in $L_{v_1}(FG)$ which avoids the vertex $g_1$. Together this yields a matching of size four. □

**Fact 24.** *Let $Y_4 \subset X_3$ contain all those vertices $v \in X_3$ such that there are at least $\left(\frac{1}{2}+\varepsilon\right)\binom{S(v)}{2}$ pairs $EF \in \binom{S(v)}{2}$ such that $EF \in \mathcal{B}(v)$. Then $|Y_4| \leq \varepsilon n$.*

*Proof.* Consider $\mathcal{B}(v) \cap \binom{S(v)}{2}$ as edges on the vertex set $S(v)$. Further note that $|S(v)| \geq \varepsilon n \geq n_0$ and $\varepsilon \geq \varepsilon'$. Applying Theorem 15 we obtain at least $c(\varepsilon n)^3$ triangles in $S(v)$, i.e., $EFG \in \binom{S(v)}{3}$ such that $EF, FG, GE \in \mathcal{B}(v)$.

As before consider the auxiliary bipartite graph $G_4$ on the partition classes $Y_4$ and $\binom{M}{3}$ with the edges $\{v, EFG\}$ if and only if $EFG \in \binom{S(v)}{3}$ and $EF, FG, GE \in \mathcal{B}(v)$. In case $|Y_4| > \varepsilon n$ we find by averaging a set $EFG \in \binom{M}{3}$ which, in $G_4$, is connected to at least $c\varepsilon^4 n$ vertices from $Y_4$. And since $n$ was chosen in such a way that $c\varepsilon^4 n/2^{40} > 3$ there are $v_1v_2v_3 \in \binom{Y_4}{3}$ whose link graphs agree on $EFG$, i.e., $L_{v_1}(EFG) = L_{v_2}(EFG) = L_{v_3}(EFG)$. But by Fact 23 this yields a contradiction. □

From Facts 18, 20, 22, 24 and the choice $\varepsilon = \gamma/150$ we infer that $X \setminus \bigcup_{i\in[4]} Y_i$ is non-empty. For a vertex $v \in X \setminus \bigcup_{i\in[4]} Y_i$ the following properties hold by the definitions of the sets $Y_1, \ldots, Y_4$.

(1) There are at most $\varepsilon n^2$ pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains a perfect matching. So their contribution to $e(L_v(M))$ is at most $9\varepsilon n^2$. (Recalling Fact 14 we note that if $L_v(EF)$ does not contain a perfect matching then $L_v(EF)$ either contains at most four edges or is isomorphic to $B_{113}, B_{023}$ or $B_{033}$.)

(2) There are at most $\varepsilon n^2$ pairs $EF \in \binom{R(v)}{2}$ such that $EF \in \mathcal{B}(v)$, contributing at most $6\varepsilon n^2$ edges to $L_v(M)$. Each of the remaining pairs have a contribution of at most 5.

(3) There are at most $\varepsilon n^2$ pairs $EF \in R(v) \times S(v)$ such that $EF \in \mathcal{B}(v)$ - which yields a contribution of at most $6\varepsilon n^2$. Note that by definition of $S(v)$ all but $\varepsilon n|S(v)|$ of the remaining pairs from $R(v) \times S(v)$ contribute at most 4 edges to $L_v(M)$.

(4) There are at most $\left(\frac{1}{2}+\varepsilon\right)\binom{|S(v)|}{2}$ pairs $EF \in \binom{S(v)}{2}$ such that $EF \in \mathcal{B}(v)$ which yields a contribution of at most $6(1/2+\varepsilon)\binom{|S(v)|}{2}$. For all but at most $\varepsilon n|S(v)|$ of the remaining pairs from $\binom{S(v)}{2}$ we have $e(L_v(EF)) \leq 4$.

Now let $r = |R(v)|$ and $s = |S(v)|$. Counting the edges in the link graph of $v$ with respect to $M = R(v)\dot\cup S(v)$ we obtain from the (1)-(4) and from $s \le |M| < n/3$

$$\begin{aligned} e(L_v(M)) &\le 9\varepsilon n^2 + \left[6\varepsilon n^2 + 5\binom{r}{2}\right] + \left[6\varepsilon n^2 + 5\varepsilon ns + 4rs\right] \\ &\quad + \left[6\left(\frac{1}{2}+\varepsilon\right)\binom{s}{2} + 4\left(\frac{1}{2}-\varepsilon\right)\binom{s}{2} + 5\varepsilon ns\right] \\ &\le 5\binom{r}{2} + 5\binom{s}{2} + 4rs + 30\varepsilon n^2 \\ &< 5\binom{|M|}{2} + 30\varepsilon n^2 < \left(\frac{5}{9}+\gamma\right)\binom{n}{2} \end{aligned}$$

with contradiction to (7). □

As an immediate consequence we obtain Theorem 6.

*Proof of Theorem 6.* Let $\gamma > 0$ be given. Set $\gamma_1 = \gamma/4$ and $\gamma_2 = \gamma_1^6$. Applying Corollary 12 with $k = 3$, $\ell = 1$ and $2\gamma_1$ yields $n_0'$ and applying Theorem 16 with $\gamma_2$ yields $n_0''$. We choose $n_0 = \max\{n_0', 2n_0''\}$. Now let $n > n_0$, $n \in 3\mathbb{Z}$ and suppose $\mathcal{H}$ is a 3-uniform hypergraph on $n$ vertices with $\delta(\mathcal{H}) \ge (5/9+\gamma)\binom{n}{2}$. Then, trivially, $\mathcal{H}$ has minimum degree $\delta(\mathcal{H}) \ge (1/2+2\gamma_1)\binom{n}{2}$ and we would like to apply Corollary 12. To this end note that for all subsets $U \subset V(\mathcal{H})$ of size at most $\gamma_1^3 n$ the remaining hypergraph $\mathcal{H}_U = \mathcal{H}[V \setminus U]$ still has minimum degree

$$\delta(\mathcal{H}_U) \ge \left(\frac{5}{9}+\frac{\gamma}{2}\right)\binom{n}{2} \ge \left(\frac{5}{9}+4\gamma_2\right)\binom{n'}{2}$$

where $n' = |V(\mathcal{H})| - |U|$. Thus, due to Theorem 16 there is a matching in $\mathcal{H}_U$ covering all but $\gamma_2 n' \le \gamma_1^6 n$ vertices. So, we can to apply Corollary 12 and obtain a perfect matching in $\mathcal{H}$. □

## 5. Perfect and nearly perfect matchings with several minimum degrees

In the sequel we are interested in the interplay between several minimum degree parameters of $k$-uniform hypergraphs. Our aim is to give an asymptotic characterisation of the existence of a perfect matching and a nearly perfect matching in terms of several minimum degrees. Recall that a nearly perfect matching in a hypergraph on $n$ vertices is a matching covering all but a constant number of vertices. Here, we mainly focus on the asymptotic behaviour of $k$-uniform hypergraphs.

To be more precise let $k \ge 2$ be fixed integers, $n \in k\mathbb{Z}$ and $\gamma, x_1, \dots, x_{k-1} > 0$ be arbitrary positive reals, then we define the subset $\mathfrak{H}_{k,n}(\gamma, x_1 \dots, x_{k-1})$ of $k$-uniform hypergraphs $\mathcal{H}$ on $n$ vertices to be

$$\mathfrak{H}_{k,n}(\gamma, x_1 \dots, x_{k-1}) = \left\{\mathcal{H}\colon\ \delta_i(\mathcal{H}) \ge (x_i+\gamma)\tbinom{n}{k-i} \text{ for } i = 1, 2, \dots, k-1\right\}.$$

Due to Proposition 9 we have

$$\delta_i(\mathcal{H}) \ge x\binom{n}{k-i} \text{ implies } \delta_{i-1}(\mathcal{H}) \ge x\binom{n}{k-i-1} - O(n^{k-i-2}), \qquad (12)$$

thus, we may assume $x_i \ge x_{i+1}$ for $i = 1, \dots, k-2$.

We say $(x_1, \dots, x_{k-1})$ **asymptotically forces a perfect matching** if for all $\gamma > 0$ there is an $n_0$ such that for all $n > n_0, n \in k\mathbb{Z}$ every $\mathcal{H} \in \mathfrak{H}_{k,n}(\gamma, x_1, \dots, x_{k-1})$

contains a perfect matching. Similarly, we say $(x_1, \dots, x_k)$ **asymptotically forces a nearly perfect matching** if there is a constant $C$ such that for all $\gamma > 0$ there is an $n_0$ such that for all $n > n_0, n \in k\mathbb{Z}$ every $\mathcal{H} \in \mathfrak{H}_{k,n}(\gamma, x_1, \dots, x_{k-1})$ contains a matching covering all but $C$ vertices and there is an $\mathcal{H} \in \mathfrak{H}_{k,n}(\gamma, x_1, \dots, x_{k-1})$ which does not contain a perfect matching.

For arbitrary integers $k \geq 2$ we are interested in the functions

$$s_k : D_{k-1} \to \{0, 1, 2\}$$

on the domain $D_{k-1} = \{(x_1, \dots, x_{k-1}) \in [0,1]^k \colon x_i \geq x_2 \geq \dots x_k\}$ which are defined by

$$s_k(x_1, \dots, x_{k-1}) = \begin{cases} 2 & (x_1, \dots, x_k) \text{ asymptotically forces a perfect matching} \\ 1 & (x_1, \dots, x_k) \text{ asymptotically forces a nearly perfect matching} \\ 0 & \text{otherwise.} \end{cases}$$

First note that $s_k(x_1, \dots, x_{k-1})$ is monotone increasing in each $x_i$. And for $k = 3$ our results determine $s_3(x_1, x_2)$ completely. We know $s_3(5/9, 0) = 2$ by Theorem 6, $s_3(1/2, 1/3) = 2$ by Theorem 3 combined with the Absorbing Lemma, Lemma 10. By Theorem 3 we know $s_3(1/3, 1/3) = 1$ and combined with the lower bounds and the monotonicity we know $s_3(x_1, x_2)$ for all $x_1 \geq x_2$ (see Figure 2). Fact 5 gives examples for $s_3(1/2, 1/2)$ and $s_3(5/9, 1/3)$.

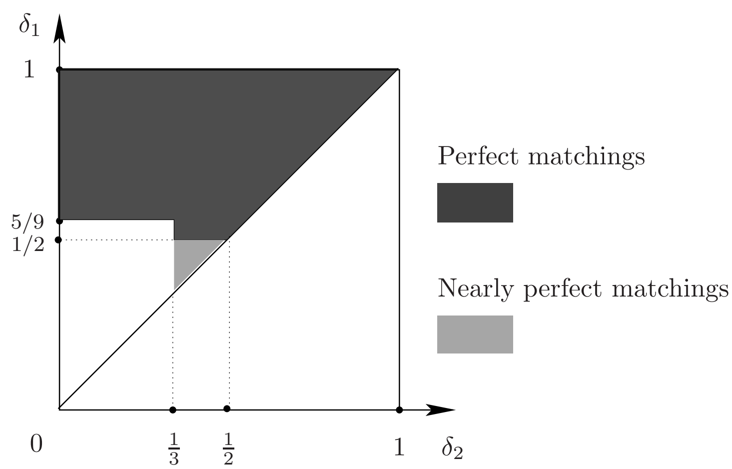


FIGURE 2. The function $s_3(x_1, x_2)$.

## 6. Open Problems

In Theorem 6 we determined the asymptotic value of $t(3, 1, n)$. However, we believe that the error term $\gamma\binom{n}{2}$ in Theorem 6 can be reduced.

For $\ell < k/2$ and $k > 3$ the asymptotic value of $t(k, \ell, n)$ is still not known and the known upper and lower bound are far apart. It would be interesting to close this gap.

Further, we have shown that for $\ell > k/2$ there is a significant difference between perfect and nearly perfect matchings in terms of minimum $\ell$-degrees (compare Theorem 3 and Theorem 4). This phenomenon, however, cannot happen if $\ell = 1$ (due to the Absorbing Lemma, Lemma 10) and, more generally, it cannot happen if $((k-1)/k)^{k-\ell} < 1/2$ (see $\delta_\ell(\mathcal{H}_1)$ in Fact 5) and it would be nice to know for which $\ell = \ell(k)$ the minimum $\ell$-degree for nearly perfect matchings and perfect matchings have the same asymptotics.

More generally, the task of determining the function $s_k(x_1, \ldots, x_{k-1})$ for all $k$ and all $x_i$ remains open.

## References


1. N. Alon and J. H. Spencer, *The probabilistic method*, second ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience [John Wiley & Sons], New York, 2000, With an appendix on the life and work of Paul Erdős. 2.1, 2.2
2. B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998. 1
3. D. E. Daykin and R. Häggkvist, *Degrees giving independent edges in a hypergraph*, Bull. Austral. Math. Soc. **23** (1981), no. 1, 103–109. 1, 1, 1
4. A. W. Goodman, *On sets of acquaintances and strangers at any party*, Amer. Math. Monthly **66** (1959), 778–783. 4.1
5. D. Kühn and D. Osthus, *Matchings in hypergraphs of large minimum degree*, J. Graph Theory **51** (2006), no. 4, 269–280. 1, 1, 1, 1, 2.1
6. J. W. Moon and L. Moser, *On a problem of Turán*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 283–286. 4.1
7. E. A. Nordhaus and B. M. Stewart, *Triangles in an ordinary graph*, Canad. J. Math. **15** (1963), 33–41. 4.1
8. O. Pikhurko, *Perfect matchings and $K_4^3$-tilings in hypergraphs of large codegree*, Graphs Combin. **24** (2008), no. 4, 391–404. 1, 1, 1, 2.1, 2.1
9. V. Rödl, A. Ruciński, M. Schacht, and E. Szemerédi, *A note on perfect matchings in uniform hypergraphs with large minimum collective degree*, Comment. Math. Univ. Carolin. **49** (2008), no. 4, 633–636. 1
10. V. Rödl, A. Ruciński, and E. Szemerédi, *Perfect matchings in large uniform hypergraphs with large minimum collective degree*, submitted. 1, 1, 2.2
11. ______, *Dirac's theorem for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1–2, 229–251. 2.2
12. ______, *Perfect matchings in uniform hypergraphs with large minimum degree*, European J. Combin. **27** (2006), no. 8, 1333–1349. 1, 1


Institut für Informatik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany

*E-mail address*: {hhan | person | schacht}@informatik.hu-berlin.de