EXTREMAL RESULTS IN RANDOM GRAPHS

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Dedicated to the memory of Paul Erdős on the occasion of his 100*th birthday*

ABSTRACT. According to Paul Erdős [*Some notes on Turán's mathematical work*, J. Approx. Theory **29** (1980), page 4] it was Paul Turán who "created the area of extremal problems in graph theory". However, without a doubt, Paul Erdős popularized *extremal combinatorics*, by his many contributions to the field, his numerous questions and conjectures, and his influence on discrete mathematicians in Hungary and all over the world. In fact, most of the early contributions in this field can be traced back to Paul Erdős, Paul Turán, as well as their collaborators and students. Paul Erdős also established the *probabilistic method* in discrete mathematics, and in collaboration with Alfréd Rényi, he started the systematic study of *random graphs*. We shall survey recent developments at the interface of extremal combinatorics and random graph theory.

CONTENTS

1. Extremal Graph Theory

1.1. **Introduction.** We first discuss a few classical results in extremal graph theory. Since by no means we can give a full account here, we restrict ourselves to some well known results in the area and highlight some of the pivotal questions. For a thorough introduction to the area we refer to the standard textbook of Bollobás [12].

A large part of extremal graph theory concerns the study of graphs G which do not contain a given subgraph F. The first classical problem is to maximize the number of edges of such a graph G with n vertices. An instance of this question was addressed already in 1938 by Erdős. In [28] he proved bounds for an extremal problem in combinatorial number theory, and in his proof he asserts a lemma that every n-vertex graph without a cycle of length four can have at most $cn^{3/2}$ edges (see Figure 1 below).

The argument was really based upon the following theorem for graphs. Let $2\,k$ points be given. We split them into two classes each containing k of them. The points of the two classes are connected by segments such that the segments form no closed quadrilateral. Then the number of segments is less than $3\,k^{3/2}$. Putting $k = n^{1/2}$ we obtain our result.

FIGURE 1. Quote from [28, page 78]

Turán initiated the systematic study of such questions, and in Section 1.3 we give a short account of Turán's theorem [114] in graph theory and some important results related to it. In fact, we will restrict ourselves only to extremal questions in graph theory here. However, even within extremal graph theory we can only discuss a few selected results and are bound to neglect not only many important topics, but also many beautiful generalizations and improvements of those classical results. Our certainly biased selection of results presented here is guided by the recent generalizations, which were obtained for subgraphs of random graphs. First we introduce the necessary notation.

1.2. **Notation.** Below we recall some notation from graph theory, which will be used here. For notation not defined here we refer to the standard text books [13, 16, 27].

All graphs considered here are finite, simple and have no loops. For a graph $G = (V, E)$ we denote by $V(G) = V$ and $E(G) = E$ its *vertex set* and its *edge set*, respectively. We denote by $e(G) = |E(G)|$ the number of edges of G and by $d(G) = e(G)/\binom{|V(G)|}{2}$ its *edge density*. Moreover, for a subset $U \subseteq V$ let $e_G(U)$ be the number of edges of G contained in U. By $\omega(G)$, $\alpha(G)$, and $\chi(G)$ we denote the standard graph parameters known as *clique number*, *independence number*, and *chromatic number* of G, respectively. We say that a graph G contains a *copy* of a graph F if there is an injective map $\varphi \colon V(F) \to V(G)$ such that $\{\varphi(u), \varphi(v)\} \in E(G)$, whenever $\{u, v\} \in E(F)$. If G contains no such copy, then we say G is *F-free*. Also, G and F are *isomorphic* if there exists a bijection $\varphi \colon G \to F$ such that $\{\varphi(u), \varphi(v)\} \in E(G)$ if, and only if, $\{u, v\} \in E(F)$. In this case we often write $G = F$. A graph H is a *subgraph* of $G = (V, E)$, if $V(H) \subseteq V$ and $E(H) \subseteq E$, which we denote by $H \subseteq G$.

The complete graph on t vertices with $\binom{t}{2}$ edges is denoted by K_t, and a *clique* is some complete graph. A graph G is *t-partite* or *t-colorable*, if there is a partition of its vertex set into t classes (some of them might be empty) such that every edge of G has its vertices in two different partition classes. We denote by $\mathrm{Col}_n(t)$ the set of all t-colorable graphs on n vertices, i.e.,

$$\mathrm{Col}_n(t) = \{H \subseteq K_n \colon \chi(H) \leq t\}.$$

A t-partite graph $G = (V, E)$ with vertex classes $V_1 \cup \ldots \cup V_t = V$ is *complete* if for every $1 \leq i < j \leq t$ and every $u \in V_i$ and $v \in V_j$ we have $\{u, v\} \in E$. We denote by $T_{n,t}$ the complete t-partite graph on n vertices with the maximum number of edges. It is easy to show that $T_{n,t}$ is unique up to isomorphism and that it is the complete t-partite graph with every vertex class having cardinality either $\lfloor n/t \rfloor$ or $\lceil n/t \rceil$.

For a graph F with at least one edge and an integer n, we denote by $\mathrm{Forb}_n(F)$ the set of F-free subgraphs of K_n, i.e.,

$$\mathrm{Forb}_n(F) = \{H \subseteq K_n \colon H \text{ is } F\text{-free}\},$$

and we recall the *extremal function* $\mathrm{ex}_n(F)$ defined by

$$\mathrm{ex}_n(F) = \max\{e(H) \colon H \in \mathrm{Forb}_n(F)\}.$$

Note that the set $\mathrm{Forb}_n(F)$ is closed under taking subgraphs, i.e., if $H \in \mathrm{Forb}_n(F)$ and $H' \subseteq H$, then $H' \in \mathrm{Forb}_n(F)$. In general such sets of graphs are called *monotone*. In fact, any monotone property $\mathcal{P}_n$ of subgraphs of K_n can be expressed by a family of *forbidden* subgraphs, and many results discussed below allow generalizations in this direction (and even more generally towards hereditary properties). However, we will concentrate on generalizations for subgraphs of random graphs and restrict the discussion to a forbidden set of graphs consisting of only one graph.

1.3. **Turán's Theorem and Related Results.** Generalizing a result of Mantel [81] for $F = K_3$, Turán [114] determined $\mathrm{ex}_n(F)$ when F is a complete graph.

Theorem 1.1 (Turán 1941). *For all integers $t \geq 2$ and $n \geq 1$ we have*

$$\mathrm{ex}_n(K_{t+1}) = e(T_{n,t}).$$

Moreover, $T_{n,t}$ is, up to isomorphism, the unique K_{t+1}-free graph on n vertices with $\mathrm{ex}_n(K_{t+1})$ edges.

Theorem 1.1 determines the maximum number of edges of a K_{t+1}-free graph on n vertices. Moreover, it characterizes the *extremal graphs*, i.e., those K_{t+1}-free graphs on n vertices having the maximum number of edges. In fact, these are instances of two very typical questions in extremal combinatorics. The questions below are stated more generally and could be applied in other contexts like hypergraphs, multigraphs, subsets of the integers, etc. However, we shall mostly restrict ourselves to questions in graph theory here.

- (*Q1*) Given a monotone property of discrete structures, like the monotone set $\mathrm{Forb}_n(K_{t+1})$ of subgraphs of K_n, what maximum density can its members attain?
- (*Q2*) What are the extremal discrete structures, e.g., like $T_{n,t}$ is the extremal subgraph of K_n for $\mathrm{Forb}_n(K_{t+1})$?

Theorem 1.1 answers $(Q1)$ and $(Q2)$ in a precise way. In fact, it not only determines the maximum density, as required for $(Q1)$, but actually gives a full description of the function $\text{ex}_n(K_{t+1})$. Often only the density question can be addressed.

To this end for a given graph F we recall the definition of the *Turán denisty* $\pi(F)$, which is given by

$$\pi(F) = \lim_{n \to \infty} \frac{\text{ex}_n(F)}{\binom{n}{2}}.$$

Note that the limit indeed exists since one can show that $\text{ex}_n(F)/\binom{n}{2}$ is non-increasing in n. Erdős and Stone [40] determined $\pi(F)$ for every graph F.

Theorem 1.2 (Erdős & Stone, 1946). *For every graph F with at least one edge we have*

$$\pi(F) = 1 - \frac{1}{\chi(F) - 1}.$$

In particular, $\pi(F) = 0$ for every bipartite graph F (see also [74] for stronger estimates for this problem). On the other hand, for a graph F of chromatic number at least three the lower bound in Theorem 1.2 is established by the Turán graph $T_{n,\chi(F)-1}$.

Refining Theorem 1.2 by determining $\text{ex}_n(F)$ for arbitrary F is a very hard problem (see, e.g., [39, 105, 106] for some partial results in this direction). Consequently, a precise solution for question $(Q2)$ is still unknown for most graphs F. Owing to the *stability theorem*, which was independently obtained by Erdős [34] and Simonovits [104], we however have an approximate answer for question $(Q2)$. In fact, the stability theorem determines an approximate structure of the extremal, as well as the *almost extremal*, graphs up to $o(n^2)$ edges.

Theorem 1.3 (Erdős 1967, Simonovits 1968). *For every $\varepsilon > 0$ and every graph F with $\chi(F) = t + 1 \geq 3$ there exist $\delta > 0$ and n_0 such that the following holds. If H is an F-free graph on $n \geq n_0$ vertices satisfying*

$$e(H) \geq \text{ex}_n(F) - \delta n^2,$$

then there exists a copy T of $T_{n,t}$ on $V(G)$ such that

$$|E(H) \triangle E(T)| \leq \varepsilon n^2,$$

where $\triangle$ denote the symmetric difference of sets.

In other words, H can be obtained from the graph $T_{n,t}$ by adding and deleting up to at most εn^2 edges.

In particular, H can be made t-partite by removing at most εn^2 edges from it.

Note that Theorem 1.3 holds trivially for bipartite graphs F as well, since in this case $\text{ex}_n(F) = o(n^2)$, and $T_{n,1}$ corresponds to an independent set.

Next we state two more commonly asked questions in extremal combinatorics, which we shall discuss in the context of being F-free.

$(Q3)$ How many discrete structures of given size have the monotone property? E.g., how large is the set $\text{Forb}_n(F)$?

$(Q4)$ Do the *typical* (drawn uniform at random) discrete structures with this property have some common features,? E.g., are there any common features of almost all graphs in $\text{Forb}_n(F)$?

For K_{t+1}-free graphs both of these questions were addressed in the work of Erdős, Kleitman, and Rothschild [37] and Kolaitis, Prömel, and Rothschild [70, 71]. In particular, it was shown that *almost all K_{t+1}-free graph on n vertices are t-colorable* subgraphs of K_n.

Theorem 1.4 (Kolaitis, Prömel & Rothschild, 1985)**.** *For every integer $t \geq 2$ the limit* $\lim_{n\to\infty} |\operatorname{Forb}_n(K_{t+1})|/|\operatorname{Col}_n(t)|$ *exists and*

$$\lim_{n\to\infty} \frac{|\operatorname{Forb}_n(K_{t+1})|}{|\operatorname{Col}_n(t)|} = 1\,.$$

Similarly to the extension of Turán's theorem in [105], Theorem 1.4 was extended by Prömel and Steger [87] from cliques K_{t+1} to graphs *containing a color-critical edge*, i.e., $(t+1)$-chromatic graphs F with the property that $\chi(F - f) = t$ for some edge $f \in E(F)$ (see also [6, 7] for more recent extensions of Theorem 1.4).

Regarding question $(Q3)$, for arbitrary graphs F, the size of $\operatorname{Forb}_n(F)$ was studied by Erdős, Frankl, and Rödl [36], and those authors arrived at the following estimate (see also [5] for a more recent improvement).

Theorem 1.5 (Erdős, Frankl & Rödl, 1986)**.** *For every $\varepsilon > 0$ and every graph F there exists n_0 such that for every $n \geq n_0$ we have*

$$|\operatorname{Forb}_n(F)| \leq 2^{\operatorname{ex}_n(F) + \varepsilon n^2}\,.$$

Note that $|\operatorname{Forb}_n(F)| \geq 2^{\operatorname{ex}_n(F)}$ holds trivially, since every subgraph of an extremal graph on n vertices is F-free. Therefore, Theorem 1.5 implies for every graph F that

$$\lim_{n\to\infty} \frac{\log_2 |\operatorname{Forb}_n(F)|}{\binom{n}{2}} = \pi(F)\,.$$

The extremal results stated above were motivated by Turán's theorem, and the problems addressed by those results allow natural generalizations for subgraphs of random graphs. In the next section we consider such extensions, where the complete graph K_n (in the definition of $\operatorname{ex}_n(F)$ and $\operatorname{Forb}_n(F)$) is replaced by a *random graph* with vanishing edge density. We will discuss some further extremal results, including the *removal lemma* and the *clique density theorem* in Section 4.

2. EXTREMAL PROBLEMS FOR RANDOM GRAPHS

Motivated by questions in Ramsey theory (also known as *Folkman-type problems*), in 1983, at the first *Random Structures and Algorithms* conference in Poznań, Erdős and Nešetřil (see [35]) posed the following extremal problem: Is it true that for every $\varepsilon > 0$ there exists a K_4-free graph G such that any subgraph $H \subseteq G$ containing at least $(1/2 + \varepsilon)e(G)$ edges must contain a triangle? In other words, Erdős and Nešetřil asked whether for $F = K_3$ one may replace K_n in the Erdős–Stone theorem by a graph which contains no larger cliques than the triangle itself. This question was answered positively by Frankl and Rödl [43] by a random construction. Those authors considered the *binomial random graph* $G(n, p)$ with vertex set $[n] = \{1, \ldots, n\}$, in which the edges are chosen independently, each with, probability p (see, e.g., [14, 58] for standard textbooks on the topic). More precisely, it was shown that for $p = n^{-1/2+o(1)}$ a.a.s. one may remove $o(pn^2)$ edges from $G \in G(n, p)$ (one from every copy of K_4 in G) such that the remaining graph has the desired property. In particular, a.a.s. the largest triangle-free subgraph of $G(n, p)$ contains at most $(\pi(K_3) + o(1))p\binom{n}{2}$ edges (see Theorem 2.1 below).

It will be convenient to extend the definitions $\mathrm{Col}_n(t)$, $\mathrm{Forb}_n(F)$ and $\mathrm{ex}_n(F)$ from Section 1.2 to a more general setting. For a graph G and an integer t let

$$\mathrm{Col}_G(t) = \{H \subseteq G \colon \chi(H) \le t\}$$

be the set of t-colorable subgraphs of G. Similarly, for a graph F with at least one edge we denote by $\mathrm{Forb}_G(F)$ the set of all subgraphs of G not containing a copy of F, i.e.,

$$\mathrm{Forb}_G(F) = \{H \subseteq G \colon H \text{ is } F\text{-free}\},$$

and we define the *generalized extremal function* $\mathrm{ex}_G(F)$ as the maximum number of edges of the elements of $\mathrm{Forb}_G(F)$, i.e.,

$$\mathrm{ex}_G(F) = \max\{e(H) \colon H \in \mathrm{Forb}_G(F)\}.$$

The following was proved by Frankl and Rödl in [43].

Theorem 2.1. *Let $\varepsilon > 0$ and $p \ge n^{-1/2+\xi}$ for some $\xi > 0$. Then a.a.s. for $G \in G(n,p)$ we have $\mathrm{ex}_G(K_3) \le (\pi(K_3) + \varepsilon)e(G)$.*

In view of Theorem 2.1 several questions arise (see below). The systematic study of these questions was initiated by the work of Kohayakawa and his collaborators in [56, 57, 63, 65, 78]. In particular, Kohayakawa, Łuczak, and Rödl formulated conjectures in [65], which led to the subsequent work discussed here.

- (*R1*) What is the smallest p such that Theorem 2.1 holds?
- (*R2*) For which p can Theorem 2.1 be extended to other graphs F instead of the triangle?
- (*R3*) Are there stability versions for those results?
- (*R4*) Is there a strengthening of Theorem 2.1 which, similarly as Mantel's theorem (Theorem 1.1 for $t = 2$), establishes the equality between the maximum size of a bipartite subgraph and that of a triangle-free subgraph? More precisely, what is the smallest p such that a.a.s. $G \in G(n,p)$ has the following property: every $H \in \mathrm{Forb}_G(K_3)$ with $e(H) = \mathrm{ex}_G(K_3)$ is bipartite?
- (*R5*) What can be said about extensions of Theorems 1.4 and 1.5, where instead of K_{t+1}-free subgraphs of K_n, one studies K_{t+1}-free subgraphs of $G(n,p)$ for appropriate p? Are almost all of those t-partite or "close" to being t-partite?

We will address questions (*R1*)–(*R3*) in the next section, Section 2.1. In Section 2.2 we address a generalization of the question of Erdős–Nešetřil that led to Theorem 2.1. Results addressing question (*R4*) will be discussed in Section 2.3 and then we turn to question (*R5*) in Section 2.4.

2.1. Threshold for the Erdős–Stone Theorem.

A common theme in the theory of random graphs is the *threshold phenomenon*. For example, it was already observed by Erdős and Whitney (unpublished) and Erdős and Rényi [38] that within a "small range" of p (around $\ln n/n$) the random graph $G(n,p)$ quickly changes its behavior from being a.a.s. disconnected to being a.a.s. connected. In other words, $\hat{p} = \ln n/n$ is the *threshold* for $G(n,p)$ being connected. In more generality, for a *graph property* $\mathcal{P}$, i.e, a set of graphs closed under isomorphism, we say $0 \le \hat{p} = \hat{p}(n) \le 1$ is a *threshold function* for $\mathcal{P}$, if

$$\lim_{n \to \infty} \mathbb{P}(G(n,p) \in \mathcal{P}) = \begin{cases} 0, & \text{if } p \ll \hat{p}, \\ 1, & \text{if } p \gg \hat{p}. \end{cases} \tag{1}$$

We refer to the two statements involved in this definition as the 0-*statement* and the 1-*statement* of the threshold. It is well known (see, e.g., [15]) that every *monotone property* $\mathcal{P}$ has a threshold.

In Theorem 2.1 the following property is studied for $F = K_3$. For given $\varepsilon > 0$, a graph F with at least one edge, and an integer n, consider

$$\mathcal{G}_n(F, \varepsilon) = \{G = (V, E) \colon V = [n] \text{ and } \mathrm{ex}_G(F) \leq (\pi(F) + \varepsilon)e(G)\} \,.$$

We note that $\mathcal{G}_n(F, \varepsilon)$ is not monotone. Consider, for example, the case when $F = K_3$, and let $G \subseteq G'$ be graphs with vertex set $[n]$, where G consists of a clique on $n^{1/3}$ vertices all other vertices isolated and G' consists of the union of G and a perfect matching.

Since $\mathcal{G}_n(F, \varepsilon)$ is not monotone, the threshold is not guaranteed to exist by the aforementioned result from [15]. On the other hand, $\mathcal{G}_n(F, \varepsilon)$ is "probabilistically monotone" (see, e.g., [58, Proposition 8.6]), and from this it follows that indeed it has a threshold for all non-trivial F and $\varepsilon > 0$. In view of this, questions $(R1)$ and $(R2)$ ask to determine the threshold for $\mathcal{G}_n(K_3, \varepsilon)$ and, more generally, for $\mathcal{G}_n(F, \varepsilon)$ for general F.

Concerning the threshold for $\mathcal{G}_n(K_3, \varepsilon)$, it follows from Theorem 2.1 that for every $\varepsilon > 0$ this threshold is at most $\hat{p} < n^{-1/2+\varepsilon}$. However, a more careful analysis of the proof presented in [43] yields $O(n^{-1/2})$ as an upper bound (see, e.g., [58, Section 8.2]). For the lower bound on the threshold, we note that the expected number of triangles in $G(n, p)$ for $p = o(n^{-1/2})$ is $o(pn^2)$. Hence by removing from $G(n, p)$ one edge from every triangle, we expect to be left with a triangle-free subgraph of $G(n, p)$ containing $1 - o(1)$ proportion of the edges of $G(n, p)$. In fact, this argument can be made precise, and it follows that $\hat{p} = n^{-1/2}$ is a threshold for $G(n, p) \in \mathcal{G}_n(K_3, \varepsilon)$ for every $\varepsilon > 0$, which answers question $(R1)$. (We remark that, in particular, the threshold function $\hat{p}$ is independent of ε).

Regarding question $(R2)$, we note that the lower bound for the threshold discussed above can be extended to arbitrary graphs and leads to the definition of the 2-*density* $m_2(F)$ of a graph F with at least one edge given by

$$m_2(F) = \max\{d_2(F') \colon F' \subseteq F \text{ with } e(F') \geq 1\} \,, \tag{2}$$

where

$$d_2(F') = \begin{cases} \frac{e(F')-1}{|V(F')|-2}, & \text{if } |V(F')| > 2 \,, \\ 1/2, & \text{if } F' = K_2 \,. \end{cases}$$

We say a graph F *is* 2-*balanced* if $d_2(F) = m_2(F)$ and it is *strictly* 2-*balanced* if $d_2(F') < d_2(F) = m_2(F)$ for all subgraphs $F' \subsetneq F$ with at least one edge.

It follows from the definition of the 2-density that $p = \Omega(n^{-1/m_2(F)})$ if, and only if the expected numbers of copies of F or any of its subgraphs in $G(n, p)$ is at least of order $\Omega(pn^2)$ – the order of the expected number of edges in $G(n, p)$. Similarly as above, one can deduce that for every $\varepsilon > 0$ and every graph F with at least one edge, $n^{-1/m_2(F)}$ is a lower bound for the threshold for $\mathcal{G}_n(F, \varepsilon)$. Moreover, Kohayakawa, Łuczak, and Rödl [65, Conjecture 1(i)] conjectured that this heuristic gives the right bound, and that a matching upper bound for the threshold can be proved. Until recently this conjecture was only proved for cliques of size at most six [65, 52, 48] and for cycles [56, 57]. In 2009 the conjecture was confirmed independently by Conlon and Gowers [22] for strictly 2-balanced graphs F and by

Schacht [102] for all graphs F. This work yields the following probabilistic version of the Erdős–Stone theorem for the random graph $G(n, p)$.

Theorem 2.2. *For every graph F with $\delta(F) \geq 2$ and every $\varepsilon > 0$ the function $\hat{p} = n^{-1/m_2(F)}$ is a threshold for $\mathcal{G}_n(F, \varepsilon)$.*

Next we discuss research addressing question $(R3)$. Recall that every graph G contains a t-partite subgraph with at least $(1 - 1/t)e(G)$ edges, which is clearly F-free for every F with chromatic number $t + 1$. On the other hand, the 1-statement (see (1)) of Theorem 2.2 implies that a.a.s. the F-free subgraph of $G \in G(n, p)$ with the maximum number of edges has at most $(1 - 1/t + o(1))e(G)$ edges. The question that arises is whether those two subgraphs of $G(n, p)$, the maximum t-partite subgraph and the maximum F-free subgraph, have similar structure. It was conjectured by Kohayakawa, Łuczak, and Rödl [65, Conjecture $1(ii)$] that such a statement is true as long as p is of the order of magnitude given in the 1-statement of the threshold in Theorem 2.2. Conlon and Gowers [22] verified this conjecture for strictly 2-balanced graphs F, and Samotij [100] adapted and simplified the approach of Schacht [102] to obtain such a result for all graphs F. This led to the following probabilistic version of the Erdős–Simonovits stability theorem.

Theorem 2.3. *For every $\varepsilon > 0$ and every graph F with $\chi(F) = t + 1 \geq 3$ there exist constants C and $\delta > 0$ such that for $p > Cn^{-1/m_2(F)}$ the following holds a.a.s. for $G \in G(n, p)$. If H is an F-free subgraph of G satisfying*

$$e(H) \geq \mathrm{ex}_G(F) - \delta p n^2 \,,$$

then H can be made t-partite by removing at most $\varepsilon p n^2$ edges from it.

We recall that Theorems 2.2 and 2.3 were conjectured (together with Conjecture 3.6 stated in Section 3.2) in [65]. These conjectures played a central rôle in the area. In particular, partial results towards these conjectures were made by the authors of the conjecture and their collaborators [49, 66, 67, 68], by Gerke and Steger and their collaborators [48, 50, 51, 52, 54] (see also the survey [53]), and by Szabó and Vu [109].

2.2. **General Erdős–Nešetřil Problem.** Before we continue with the discussion of extremal results for sparse random graphs, we generalize the problem of Erdős and Nešetřil. Based on Theorem 2.2, one now can prove the following generalization of the Erdős–Nešetřil problem, which extends the results of [43] from forbidding triangles to forbidding cliques of arbitrary fixed size.

Corollary 2.4. *For every integer $k \geq 3$ and $\varepsilon \in (0, 1 - \pi(K_k))$ the following holds:*
  - *(i) there exists a K_{k+1}-free graph G such that $\mathrm{ex}_G(K_k) \leq (\pi(K_k) + \varepsilon)e(G)$;*
  - *(ii) for every fixed $d > 0$ there exists an n_0 such that there is no graph G on $n \geq n_0$ vertices with $e(G) = d\binom{n}{2}$ having the properties from part (i).*

While the first statement of Corollary 2.4 asserts the existence of a K_{k+1}-free graph with the property that every $(\pi(K_k) + \varepsilon)$ proportion of its edges contains a K_k, the second statement asserts that such a graph must have vanishing density.

In the proof of part (i) we consider $G(n, p)$ for $p = Cn^{-1/m_2(K_k)}$. Owing to Theorem 2.2 we know that a.a.s. $G \in G(n, p)$ satisfies $\mathrm{ex}_G(K_k) \leq (\pi(K_k) + o(1))e(G)$. On the other hand, since $n^{-1/m_2(K_k)} \ll n^{-1/m_2(K_{k+1})}$ for this choice of p, the number of copies of K_{k+1} in G will be of order $o(p n^2)$. Consequently, we may remove

$o(pn^2)$ edges from G and the resulting graph is K_{k+1}-free and satisfies the properties of part (i) of Corollary 2.4. In fact, one may check that the same proof works for all values of p with $Cn^{-1/m_2(K_k)} \leq p \leq cn^{-1/m_2(K_{k+1})}$ for appropriate constants C and $c > 0$. We give the details of this proof after the following remark.

Remark 2.5. One can show that statement (ii) is best possible. Indeed, given $d = d(n) = o(1)$, let $(G_m)_{m \in \mathbb{N}}$ be a sequence of m vertex graphs with the properties of part (i) and with density $\varrho = \varrho(m) \ll d(m)$. Since $d = o(1)$, we can find infinitely many values for which $d(n) \sim \varrho(m)$. For such an m we "blow-up" G_m by replacing each vertex by an independent set of size n/m and every edge by a complete bipartite graph with vertex classes of size n/m. The resulting graph G has n vertices, density approximately $d(n)$, and it "inherits" the properties of G_m with respect to statement (i).

Finally, we remark that in Section 4.5 we will generalize part (ii) and show that no relatively dense subgraph of $G(n,p)$ for $p \gg n^{-1/m_2(K_{k+1})}$ satisfies the properties of part (ii) (see Theorem 4.10).

Proof of Corollary 2.4 part (i). Part (i) follows directly from Theorem 2.2 combined with an alteration argument (similar to the one carried out by Erdős in [31], see also [3, Section 3]).

Let $\varepsilon > 0$ and $k \geq 3$ be given. Applying Theorem 2.2 for $\varepsilon/2$ and $F = K_k$ implies that there exists a constant $C > 0$ such that for $p = p(n) = Cn^{-1/m_2(K_k)}$ a.a.s. for $G \in G(n,p)$ we have

$$\mathrm{ex}_G(K_k) \leq \left(\pi(K_k) + \frac{\varepsilon}{2} \right) e(G). \tag{3}$$

Since $k \geq 3$, we have

$$m_2(K_k) = \frac{\binom{k}{2} - 1}{k - 2} = \frac{k+1}{2} \geq 2,$$

and thus also

$$p = Cn^{-\frac{2}{k+1}} \geq \frac{C}{\sqrt{n}}.$$

It follows that $pn^2 \geq Cn^{3/2}$. Chebyshev's inequality easily yields that a.a.s.

$$e(G) \geq \frac{1}{2} p \binom{n}{2}. \tag{4}$$

Finally, we note that the expected number of copies of K_{k+1} in G is at most

$$p^{\binom{k+1}{2}} n^{k+1} = C^{\binom{k+1}{2}} n \leq \frac{\varepsilon}{4} p \binom{n}{2},$$

for sufficiently large n. Hence it follows from Markov's inequality that, with probability at least $1/2$, the graph G contains at most $(\varepsilon/2)p\binom{n}{2}$ copies of K_{k+1}. Consequently, for sufficiently large n there exists a graph G containing at most $(\varepsilon/2)p\binom{n}{2}$ copies of K_{k+1} and for which (3) and (4) also hold. Let G' be the graph obtained from G by removing one edge from every copy of K_{k+1} in G. Obviously, the graph G' is K_{k+1}-free,

$$e(G') \geq e(G) - \frac{\varepsilon}{4} p \binom{n}{2},$$

and owing to

$$(\pi(K_k) + \varepsilon)e(G') > (\pi(K_k) + \varepsilon)e(G) - \frac{\varepsilon}{4}p\binom{n}{2}$$

$$= \left(\pi(K_k) + \frac{\varepsilon}{2}\right)e(G) + \frac{\varepsilon}{2}e(G) - \frac{\varepsilon}{4}p\binom{n}{2}$$

$$\overset{(4)}{\geq} \left(\pi(K_k) + \frac{\varepsilon}{2}\right)e(G),$$

it follows from (3) that $\mathrm{ex}_{G'}(K_k) \leq (\pi(K_k) + \varepsilon)e(G')$, which concludes the proof of assertion (i) in Corollary 2.4. $\qquad\square$

Next we prove assertion (ii). The proof follows the main ideas of [43, Theorem 4].

Definition 2.6. *For a graph $G = (V, E)$ we call a partition $V_1 \dot\cup \ldots \dot\cup V_t = V$ a t-cut. We denote by $E_G(V_1, \ldots, V_t)$ the edges of the t-cut, i.e., those edges of G with its vertices in two different sets of the partition and we denote by $e_G(V_1, \ldots, V_t) = |E_G(V_1, \ldots, V_t)|$ the size of the t-cut. Moreover, we say a t-cut is balanced, if $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$.*

A simple averaging argument shows that there always exists a balanced t-cut of G of size at least $(1 - 1/t)e(G)$. The following lemma, which implies assertion (ii) of Corollary 2.4, shows that if on the other hand all balanced t-cuts have size at most $(1 - 1/t + o(1))e(G)$, then G contains cliques of arbitrary size.

Lemma 2.7. *For all integers s, $t \geq 2$ and every $d > 0$ there exist $\varepsilon > 0$ and n_0 such that the following holds. Let $G = (V, E)$ be a graph on $|V| = n \geq n_0$ vertices, with $|E| = d\binom{n}{2}$ edges, and with the property that every balanced t-cut has size at most $(1 - 1/t + \varepsilon)d\binom{n}{2}$. Then G contains a copy of K_s.*

Before we prove Lemma 2.7, we deduce assertion (ii) of Corollary 2.4 from it.

Proof of Corollary 2.4 part (ii). Suppose that part (ii) of Corollary 2.4 fails to be true. We assume that there is a K_{k+1}-free graph G on n vertices with $\mathrm{ex}_n(K_k) \leq (\pi(K_k) + \varepsilon)e(G)$ and with $d\binom{n}{2}$ edges for some constant $d > 0$. We apply Lemma 2.7 with $s = k + 1$ and $t = k - 1$. Since the edges of every $(k-1)$-cut span no copy of K_k the assumption of Corollary 2.4 part (ii) guarantees that the size of every $(k-1)$-cut in G is bounded from above by

$$(\pi(K_k) + \varepsilon)e(G) = \left(1 - \frac{1}{k-1} + \varepsilon\right)d\binom{n}{2},$$

and it follows from Lemma 2.7 that G contains a K_{k+1}, which contradicts the assumption on G. $\qquad\square$

The proof of Lemma 2.7 draws on some ideas from the theory of quasi-random graphs [20]. In particular, it is based on the following well known fact (see, e.g., [94, Theorem 2]).

Lemma 2.8. *For all integers s, $t \geq 2$ and every $d > 0$ there exist $\delta > 0$ and n_0 such that the following holds. Let $G = (V, E)$ be a graph on $|V| = n \geq n_0$ vertices such that $e_G(U) = (d \pm \delta)\binom{\lfloor n/t \rfloor}{2}$ for every $U \subseteq V$ with $|U| = \lfloor n/t \rfloor$. Then G contains a copy of K_s.*

Proof of Lemma 2.7. Let integers s and $t \geq 2$ be fixed. Suppose for a contradiction that the lemma fails to be true with this choice of s and t. This means that there is a density $d > 0$ for which the statement fails, so we fix "the largest such d". More precisely, let $d > 0$ be chosen in such a way that the statement fails for s, t, and d, but it holds for s, t and any $d' > d$ provided $\varepsilon' > 0$ is sufficiently small and n is sufficiently large. We remark that such a choice is possible, since for fixed s and t the validity of the statement for d implies it for every $d' \geq d$.

Our choice of δ' will be given by Lemma 2.8. First let $\delta > 0$ be the constant guaranteed by Lemma 2.8 for the already fixed s, t, and d and set

$$\delta' = \frac{\delta}{2(t-1)} \quad \text{and} \quad \varepsilon = \min\left\{ \frac{\delta}{4t^2}, \frac{\varepsilon'(d+\delta')}{4t^2} \right\}, \tag{5}$$

where $\varepsilon' > 0$ is given by Lemma 2.7 applied with $d' \geq d + \delta'$ (which holds by our assumption). Finally, let n_0 be sufficiently large (for example, so that we can appeal to Lemma 2.8 with s, t, d, and δ and to the validity of Lemma 2.7 for $d' \geq d + \delta'$ and $\varepsilon' > 0$). Let $G = (V, E)$ with $|V| = n \geq n_0$ be a counterexample for those choices. Without loss of generality we assume that t^2 divides n.

Since G contains no copy of K_s, Lemma 2.8 implies that there exists a set V_1 of size n/t such that either $e_G(V_1) < (d - \delta)\binom{n/t}{2}$ or $e_G(V_1) > (d + \delta)\binom{n/t}{2}$. Fix some balanced t-cut $V_1 \cup \ldots \cup V_t = V$ which contains V_1. We will infer that G induces a denser graph on one of the sets of the partition. This is obvious if $e_G(V_1) > (d + \delta)\binom{n/t}{2}$. However, if $e_G(V_1) < (d - \delta)\binom{n/t}{2}$, then we will show that there also is a partition class that induces a denser graph. In fact, using the assumption on G for the sizes of the balanced t-cuts, an averaging argument shows that there exists some $i = 2, \ldots, t$ such that

$$\begin{aligned}
e_G(V_i) &\geq \frac{e(G) - e_G(V_1, \ldots, V_t) - e_G(V_1)}{t-1} \\
&= \frac{d\binom{n}{2} - \left(1 - \frac{1}{t} + \varepsilon\right)d\binom{n}{2} - (d-\delta)\binom{n/t}{2}}{t-1} \\
&\geq \frac{(1/t - \varepsilon)d\binom{n}{2} - d\binom{n/t}{2} + \delta\binom{n/t}{2}}{t-1} \\
&\geq \frac{(t - 1 - 2\varepsilon t^2)d\binom{n/t}{2} + \delta\binom{n/t}{2}}{t-1} \\
&\overset{(5)}{\geq} (d + \delta')\binom{n/t}{2}.
\end{aligned}$$

Summarizing, we can fix some $i \in [t]$ such that for $W = V_i$ we have $e_G(W) = d'\binom{n/t}{2}$ for some $d' \geq d + \delta'$.

Since G (and hence also the induced subgraph $G[W]$) contains no copy of K_s, by our assumptions $G[W]$ fails to satisfy the assumptions of Lemma 2.7. Consequently, there exists a balanced t-cut $W_1 \cup \ldots \cup W_t = W$ with

$$e_{G[W]}(W_1, \ldots, W_t) > \left(1 - \frac{1}{t} + \varepsilon'\right)d'\binom{n/t}{2} = \left(1 - \frac{1}{t} + \varepsilon'\right)e_G(W). \tag{6}$$

We will extend this balanced t-cut of $G[W]$ to a balanced t-cut of G with size bigger than

$$\left(1 - \frac{1}{t} + \varepsilon\right)d\binom{n}{2}, \tag{7}$$

which will then contradict the assumptions on G.

For that we consider a random balanced t-cut $U_1 \uplus \ldots \uplus U_t$ of $U = V \setminus W$. A standard application of Chernoff's inequality for the hypergeometric distribution (see, e.g., [58, Theorem 2.10]) shows that with probability close to one, we have

$$e_G(W_i, U_j) = \left(\frac{1}{t} \pm o(1)\right) e_G(W_i, U) \quad \text{for all } i, j \in [t] \tag{8}$$

and

$$e_G(U_1, \ldots, U_t) = \left(1 - \frac{1}{t} \pm o(1)\right) e_G(U). \tag{9}$$

Let such a t-cut be fixed. Since both t-cuts were balanced, the t-cut $V_1' \uplus \ldots \uplus V_t'$ of V given by $V_i' = W_i \uplus U_i$ is also balanced. We estimate the size of this cut as follows:

$$e_G(V_1', \ldots, V_t') = e_G(W_1, \ldots, W_t) + \sum_{i=1}^{t} \sum_{j \neq i} e_G(W_i, U_j) + e_G(U_1, \ldots, U_t). \tag{10}$$

By (8), we have

$$\sum_{i=1}^{t} \sum_{j \neq i} e_G(W_i, U_j) = \sum_{i=1}^{t} \sum_{j \neq i} \left(\frac{1}{t} \pm o(1)\right) e_G(W_i, U)$$

$$= \left(\frac{t-1}{t} \pm o(1)\right) \sum_{i=1}^{t} e_G(W_i, U) = \left(1 - \frac{1}{t} \pm o(1)\right) e_G(W, U)$$

and combined with (6) and (9), from (10) we get

$$e_G(V_1', \ldots, V_t') \geq \left(1 - \frac{1}{t} - o(1)\right) e(G) + \varepsilon' e_G(W).$$

Hence, we obtain (7) from

$$\varepsilon' e_G(W) = \varepsilon' d' \binom{n/t}{2} \geq \frac{\varepsilon' d'}{2t^2} \binom{n}{2} \overset{(5)}{\geq} 2\varepsilon \binom{n}{2}.$$

$\square$

2.3. Turan's Theorem for Random Graphs.

Turán's theorem not only determines the extremal function $\mathrm{ex}_n(K_{t+1})$ precisely, but also asserts that the complete balanced t-partite graph on n vertices is the unique extremal graph. The extremal results for $G(n, p)$ discussed in Section 2.1 do not fully address this question (see also $(R4)$). For example, Theorem 2.2 applied for $F = K_{t+1}$ gives no information about the structure of extremal K_{t+1}-free subgraphs of $G(n, p)$. In this section, we discuss results motivated by this question.

For an integer $t \geq 2$ and a graph G let $\mathrm{col}_G(t)$ be the maximum number of edges of a t-colorable subgraph of G, i.e., the size of the maximum t-cut in G. For simplicity we write $\mathrm{col}_n(t)$ for $\mathrm{col}_{K_n}(t)$. Turán's theorem establishes

$$\mathrm{ex}_n(K_{t+1}) = \mathrm{col}_n(t).$$

Babai, Simonovits, and Spencer [4] were the first to investigate the extent to which such an identity can be extended to random graphs. In particular, those authors

showed that it holds for $G(n, 1/2)$ in the case of triangles ($t = 2$), by showing that a.a.s. $G \in G(n, 1/2)$ satisfies

$$\mathrm{ex}_G(K_3) = \mathrm{col}_G(2). \tag{11}$$

Answering a question from [4], it was shown by Brightwell, Panagiotou, and Steger [17] that $p = 1/2$ can be replaced by $p = n^{-\eta}$ for some $\eta > 0$. Moreover, their proof extends to cliques of arbitrary fixed size and establishes that the identity $ex_G(K_{t+1}) = \mathrm{col}_G(t)$ holds a.a.s. for $G \in G(n, p)$ as long as $p > n^{-\eta_t}$ for some sufficiently small $\eta_t > 0$. Those authors conjectured that this result can be extended to smaller values of p. Note that (11) holds trivially for very small p, when a.a.s. the random graph itself is bipartite. However, here and below we shall exclude this range of p. It was noted in [17] that (with the exception of small p) in order for (11) to hold, $p > c(\log n/n)^{1/2}$ is a necessary condition for some sufficiently small $c > 0$. The reason for this is that for $p < c(\log n/n)^{1/2}$, cycles of length five appear in $G(n, p)$ which have the additional property that none of its edges is contained in a triangle. Recently, DeMarco and Kahn [26] obtained a matching upper bound by proving the following probabilistic version of Mantel's theorem (Theorem 1.1 for $t = 2$).

Theorem 2.9. *There exists a constant $C > 0$ such that for $p > C(\log n/n)^{1/2}$ a.a.s. $G \in G(n, p)$ satisfies $\mathrm{ex}_G(K_3) = \mathrm{col}_G(2)$. Moreover, every triangle-free subgraph of G with the maximum number of edges is bipartite.*

It would be interesting to generalize this results to larger cliques. It seems plausible that a necessary condition on p for such a generalization should come from the requirement that *all* edges of $G(n, p)$ are contained in a cliques of size $t + 1$. In particular, the edges not contained in a copy of K_{t+1} should not form a high chromatic subgraph. For this we require on average $\Omega(\log n)$ such cliques per edge, instead of a constant number of cliques per edge, which gave rise to the 2-density. For K_{t+1} we get

$$p^{\binom{t+1}{2}} n^{t+1} = \Theta(pn^2 \log n).$$

Solving this for p leads to the following conjecture, which was stated by DeMarco and Kahn [26].

Conjecture 2.10. *For every integer $t \geq 2$ there exists a $C > 0$ such that for $p \geq C((\log n)^{\frac{1}{t-1}}/n)^{\frac{2}{t+2}}$ a.a.s. $G \in G(n, p)$ satisfies $\mathrm{ex}_G(K_{t+1}) = \mathrm{col}_G(t)$.*

It would be also of interest to prove similar results for graphs F containing a color-critical edge. Partial results in this direction can be found in [4] (see also [17]).

2.4. **Triangle-free Graphs with Given Number of Vertices and Edges.** In this section we discuss extensions of Theorems 1.4 and 1.5. Most of the work studied $\mathrm{Forb}_{n,M}(K_3)$, the set of triangle-free graphs with n vertices and M edges. The first result in this direction is due to Prömel and Steger [88], who proved a strengthening of the Erdős–Kleitman–Rothschild theorem (Theorem 1.4 for $t = 2$). It was shown that for $M > Cn^{7/4} \log n$, almost every graph $H \in \mathrm{Forb}_{n,M}(K_3)$ is bipartite. Similarly to the case of Turán's theorem for random graphs discussed in the last section, such an assertion holds also for very small values of M, but not in the medium range (see Theorem 2.11 below). It was also noted in [88] that the statement fails to be true if $M = cn^{3/2}$ for some $c > 0$. The gap between $cn^{3/2}$ and $Cn^{7/4} \log n$ was closed by Osthus, Prömel, and Taraz [86] (see also Steger [108]

for a bit weaker result). In particular, the following result was shown in [86]. For positive integers n, M, and t, we denote by $\text{Col}_{n,M}(t)$ the set of t-colorable graphs with n vertices and M edges.

Theorem 2.11. *For every $\varepsilon > 0$ the following holds*

$$\lim_{n \to \infty} \frac{|\text{Forb}_{n,M}(K_3)|}{|\text{Col}_{n,M}(2)|} = \begin{cases} 1, & \text{if } M = M(n) = o(n), \\ 0, & \text{if } n/2 \le M = M(n) \le (1-\varepsilon)\frac{\sqrt{3}}{4}n^{3/2}\sqrt{\ln n}, \\ 1, & \text{if } M = M(n) \ge (1+\varepsilon)\frac{\sqrt{3}}{4}n^{3/2}\sqrt{\ln n}. \end{cases}$$

Note that similarly to Theorem 2.9 the "critical window" in Theorem 2.11 concerns graphs with $\Theta(n^{3/2}\sqrt{\log n})$ edges. That might not be a coincidence, since having the property that every pair is covered by a path of length two seems to be a necessary condition for both problems. Generalizing this to the property that adding an edge for any pair of vertices would close a copy of K_{t+1} suggests a joint generalization of the Kolaitis–Prömel–Rothschild theorem, Theorem 1.4, and of Theorem 2.11, which was recently obtained by Balogh, Morris, Samotij, and Warnke [9].

A closely related result was proved by Łuczak. In [78] he studied slightly sparser triangle-free graphs and showed that for $M = M(n) \gg n^{3/2}$, almost every graph $H \in \text{Forb}_{n,M}(K_3)$ is "close" to a bipartite graphs, i.e., it can be made bipartite by removing at most $o(M)$ edges. In fact, he also proved that this result generalizes for larger cliques, provided Conjecture 3.6 (stated below), which we discuss in the next section, holds. Recently Balogh, Morris, and Samotij [8] and Saxton and Thomason [101] developed an approach which, among other results, allowed them to prove Conjecture 3.6 and it could be used to verify Łuczak's statement directly.

Theorem 2.12. *For every $\delta > 0$ and $t \ge 2$ there exists a $C > 0$ and n_0 such that for $M = M(n) \ge Cn^{2-1/m_2(K_{t+1})}$ almost every graph H drawn uniformly at random from $\text{Forb}_{n,M}(K_{t+1})$ can be made t-colorable by removing at most δM edges.*

It is known that, up to the constant C, this result is best possible, and we also remark that the smallest $M = M(n)$ in Theorem 2.12 coincides in order of magnitude with the expected number of edges in $G(n, p)$ around the thresholds from Theorem 2.2.

3. Regularity Method

One of the most important tools in extremal graph theory is Szemerédi's regularity lemma [111], and for a thorough discussion of its history and many of its applications we refer to [72, 73]. In fact, there were some applications of this lemma addressing extremal and Ramsey-type questions of random graphs (see, e.g., [4, 97]). However, for the systematic study of extremal problems of $G(n, p)$ for $p = o(1)$, a variant of the lemma discovered independently by Kohayakawa [61] and Rödl (unpublished) seemed to be an appropriate tool. We begin the discussion with Szemerédi's regularity lemma.

3.1. **Szemerédi's Regularity Lemma.** We first introduce the necessary definitions. Let $H = (V, E)$ be a graph, and let $X, Y \subseteq V$ be a pair of non-empty and

disjoint subsets of the vertices. We denote by $e_H(X, Y)$ the number of edges in the bipartite subgraph induced by X and Y, i.e.,

$$e_H(X, Y) = \left| \left\{ \{x, y\} \in E \colon x \in X \text{ and } y \in Y \right\} \right|.$$

We also define the *density of the pair* (X, Y) by setting

$$d_H(X, Y) = \frac{e_H(X, Y)}{|X||Y|}.$$

Moreover, we say a the pair (X, Y) is *ε-regular* for some $\varepsilon > 0$, if

$$|d_H(X, Y) - d_H(X', Y')| < \varepsilon$$

for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon |X|$ and $|Y'| \geq \varepsilon |Y|$. With this notation we can formulate Szemerédi's regularity lemma from [111].

Theorem 3.1 (Regularity lemma)**.** *For every $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exist integers T_0 and n_0 such that every graph $H = (V, E)$ with $|V| = n \geq n_0$ vertices admits a partition $V = V_1 \uplus \ldots \uplus V_t$ satisfying*

(i) $t_0 \leq t \leq T_0$,
(ii) $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$, and
(iii) all but at most εt^2 pairs (V_i, V_j) with $i \neq j$ are ε-regular.

Note that most applications of Theorem 3.1 involve dense graphs (i.e., n-vertex graphs with $\Omega(n^2)$ edges). For each graph the lemma allows us to decompose the graph into bipartite "blocks," the majority of which have a uniform edge distribution. If such a graph has only $o(n^2)$ edges, it may not provide such control, since all edges may be contained in exceptional pairs (see property *(iii)* in Theorem 3.1). Moreover, even for ε-regular pairs, we do not gain any information if the density of that pair is $o(1)$.

The following well known fact is used in many applications of the regularity lemma (see, e.g., [73, 98]). For future reference, we state both the *embedding lemma* and the *counting lemma*, even though the latter clearly implies the former.

Fact 3.2 (Embedding and counting lemma for dense graphs)**.** *For every graph F with $V(F) = [\ell]$ and every $d > 0$, there exist $\varepsilon > 0$ and m_0 such that the following holds.*

Let $H = (V_1 \uplus \ldots \uplus V_\ell, E_H)$ be an ℓ-partite graph with $|V_1| = \cdots = |V_\ell| = m \geq m_0$ and with the property that for every edge $\{i, j\} \in E(F)$ the pair (V_i, V_j) is ε-regular in H with density $d_H(V_i, V_j) \geq d$.

Embedding Lemma: *Then H contains a partite copy of F, i.e., there exists a graph homomorphism $\varphi \colon F \to H$ with $\varphi(i) \in V_i$.*

Counting Lemma: *The number of partite copies satisfies*

$$\left| \left\{ \varphi \colon F \to H \colon \varphi \text{ is a graph homomorphism with } \varphi(i) \in V_i \right\} \right|$$

$$= (1 \pm f(\varepsilon)) \prod_{\{i,j\} \in E(F)} d(V_i, V_j) \prod_{i=1}^{\ell} |V_i|, \quad (12)$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

As mentioned, the counting lemma implies the embedding lemma from Fact 3.2. However, for quite a few applications the existence of one copy is sufficient.

3.2. **Sparse Regularity Lemma for Subgraphs of Random Graphs.** In this section we state a modified version of Szemerédi's regularity lemma, which allows applications to sparse graphs. Though more general lemmas are known, we restrict ourselves to a version which applies a.a.s. to all subgraphs of a random graph $G \in G(n, p)$. For that we first strengthen the notion of an ε-regular pair.

Definition 3.3 ((ε, p)-regular pair)**.** *Let $\varepsilon > 0$, let $p \in (0, 1]$, let $H = (V, E)$ be a graph, and let $X, Y \subseteq V$ be non-empty and disjoint. We say the pair (X, Y) is (ε, p)-regular if*

$$|d_H(X, Y) - d_H(X', Y')| < \varepsilon p$$

for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon |X|$ and $|Y'| \geq \varepsilon |Y|$.

Note that ε-regularity coincides with the case $p = 1$ in the definition above. However, for $p = p(n) = o(1)$ and graphs of density $\Omega(p)$ the notion of (ε, p)-regularity gives additional control and addresses the second concern discussed after Theorem 3.1. The sparse regularity lemma for subgraphs of $G(n, p)$ stated below asserts that, for those graphs ε-regularity in Theorem 3.1 can be replaced by (ε, p)-regularity. In fact, besides the restriction to subgraphs of $G(n, p)$, this is the only difference between the following version of the sparse regularity lemma from [61] and Theorem 3.1.

Theorem 3.4 (Sparse regularity lemma for subgraphs of $G(n, p)$)**.** *For every $\varepsilon > 0$, $t_0 \in \mathbb{N}$, and every function $p = p(n) \gg 1/n$ there exist integers T_0 such that a.a.s. $G \in G(n, p)$ has the following property. Every subgraph graph $H = (V, E)$ of G with $|V| = n$ vertices admits a partition $V = V_1 \dot\cup \ldots \dot\cup V_t$ satisfying*

   (i) $t_0 \leq t \leq T_0$,
   (ii) $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$, and
   (iii) all but at most εt^2 pairs (V_i, V_j) with $i \neq j$ are (ε, p)-regular.

In order to make Theorem 3.4 applicable in a similar way to Szemerédi's regularity lemma, one needs extensions of Fact 3.2. Theorem 3.4 can be proved like the original regularity lemma with fairly straightforward adjustments. To prove a corresponding form of Fact 3.2 turns out to be a challenging problem, which was resolved only recently in [8, 23, 101]. In particular, in the work of Balogh, Morris, and Samotij [8] and of Saxton and Thomason [101], a conjecture of Kohayakawa, Łuczak, and Rödl [65] was addressed. This conjecture implies a version of the embedding lemma of Fact 3.2 appropriate for applications of Theorem 3.4. In [23] only such a version was derived (see Theorem 3.8 below). For the formulation of the conjecture from [65], we require some more notation.

Definition 3.5. *Let $\varepsilon > 0$, $p \in (0, 1]$, $d > 0$ and let ℓ, m, M be integers. Let F be a graph with vertex set $V(F) = [\ell]$. We denote by $\mathcal{G}(F, m, M, \varepsilon, p, d)$ the set of all ℓ-partite graphs $H = (V_1 \dot\cup \ldots \dot\cup V_\ell, E_H)$ with*

   (i) $|V_1| = \cdots = |V_\ell| = m$,
   (ii) $e_H(V_i, V_j) = M \geq dpm^2$ for all $\{i, j\} \in E(F)$, and
   (iii) (V_i, V_j) is (ε, p)-regular for all $\{i, j\} \in E(F)$.

We denote by $\mathcal{B}(F, m, M, \varepsilon, p, d)$ the set of all those graphs from $\mathcal{G}(F, m, M, \varepsilon, p, d)$, which contain no (partite) copy of F, i.e.,

$$\mathcal{B}(F, m, M, \varepsilon, p, d) = \{H \in \mathcal{G}(F, m, M, \varepsilon, p, d) \colon \text{there is no}$$
$$\text{graph homomorphism } \varphi \colon F \to H \text{ with } \varphi(i) \in V_i\}.$$

The first part of Fact 3.2 asserts that for $p = 1$, sufficiently small $\varepsilon = \varepsilon(F, d) > 0$ and sufficiently large $m = m(F, d)$, the set $\mathcal{B}(F, m, M, \varepsilon, p, d)$ is empty. However, if $p = o(1)$ then $\mathcal{B}(F, m, M, \varepsilon, p, d)$ is not empty for graphs F containing a cycle. In other words, if $p = o(1)$, then the regularity condition does not ensure the occurrences of copies of F. This prohibits a straightforward extension of Fact 3.2 for the sparse regularity lemma. For example, as noted earlier for $p \ll n^{-1/m_2(F)}$ a.a.s. the random graph $G(n, p)$ contains only $o(pn^2)$ copies of some subgraph $F' \subseteq F$. Therefore, a.a.s. $G(n, p)$ contains an F-free subgraph with $(p - o(1))\binom{n}{2}$ edges. This can be used to construct many F-free graphs $H \in \mathcal{G}(F, m, M, \varepsilon, p, d)$ for any $p = o(1)$ and appropriate choices of m, M, and d. (For details see the discussion below Conjecture 3.6.) On the other hand, for $p \geq Cm^{-1/m_2(F)}$ for sufficiently large $C > 0$, it was conjectured by Kohayakawa, Łuczak, and Rödl in [65] that $\mathcal{B}(F, m, M, \varepsilon, d)$ contains only "very few" graphs.

Conjecture 3.6 (Kohayakawa, Łuczak & Rödl 1997). *For every $\alpha > 0$, $d > 0$, and every graph F with vertex set $V(F) = [\ell]$, there are $\varepsilon > 0$, $C > 0$ and m_0 such that for every $m \geq m_0$, $p \geq Cm^{-1/m_2(F)}$ and $M \geq dpm^2$ we have*

$$|\mathcal{B}(F, m, M, \varepsilon, p, d)| \leq \alpha^M |\mathcal{G}(F, m, M, \varepsilon, p, d)|. \tag{13}$$

Next we show that the lower bound on p in Conjecture 3.6 is necessary. For this, let $p = \delta m^{-1/m_2(F)}$ for some δ tending to 0 with m. We consider the family of graphs $\widetilde{\mathcal{G}}(F, m, p, d)$ satisfying only properties (i) and (ii) of Definition 3.5, with $M = dpm^2$. It is not hard to show that for every $\varepsilon > 0$ almost every $H \in \widetilde{\mathcal{G}}(F, m, M, p, d)$ is also contained in $\mathcal{G}(F, m, M, \varepsilon, p, d)$, i.e.,

$$|\mathcal{G}(F, m, M, \varepsilon, p, d)| \geq (1 - o(1))|\widetilde{\mathcal{G}}(F, m, p, d)|. \tag{14}$$

Moreover, let $F' \subseteq F$ be the subgraph with $d_2(F') = m_2(F')$ (see (2)), and let e and v denote its number of edges and vertices, respectively. The expected number of partite copies of F' in a graph H chosen uniformly at random from $\widetilde{\mathcal{G}}(F, m, p, d)$ is

$$O((dp)^e m^v) = O((\delta d)^e pm^2) = o(pm^2).$$

Hence, all but $o(|\widetilde{\mathcal{G}}(F, m, p, d)|)$ graphs $H \in \widetilde{\mathcal{G}}(F, m, p, d)$ have the property that, only $o(pm^2)$ edges of H are contained in a copy of F', and consequently also in a copy of F. Delete from each such $H \in \mathcal{G}(F, m, M, \varepsilon, p, d)$ the edges contained in copies of F and possibly a few more from each pair (V_i, V_j), so that the resulting graph has precisely $M' = d'pm^2 = (1 - o(1)M$ edges for each such pair. This way we obtain a graph $H' \in \mathcal{G}(F, m, M', \varepsilon', p, d')$ with $\varepsilon' = \varepsilon + o(1)$, which is F-free,

i.e., H' is contained in $\mathcal{B}(F, m, M', \varepsilon', p, d')$. Consequently, we have

$$
\begin{aligned}
|\mathcal{B}(F, m, M', \varepsilon', p, d')| \ &\geq\ (1 - o(1)) \frac{|\widetilde{\mathcal{G}}(F, m, p, d)|}{\binom{M}{o(M)}^{e(F)}} \\
&=\ (1 - o(1)) \left(\frac{\binom{m^2}{M}}{\binom{M}{o(M)}} \right)^{e(F)} \\
&\geq\ (1 - o(1))^{M'} \binom{m^2}{M'}^{e(F)} \\
&=\ (1 - o(1))^{M'} |\widetilde{\mathcal{G}}(F, m, p, d')| \\
&\geq\ (1 - o(1))^{M'} |\mathcal{G}(F, m, M', \varepsilon', p, d')| \,,
\end{aligned}
$$

which shows that (13) fails for $p = \delta n^{-1/m_2(F)}$ for sufficiently small $\delta > 0$.

3.3. **Sparse Embedding and Counting Lemma.** Conjecture 3.6 is obvious, if F is a matching. For all other graphs F, we have $m_2(F) \geq 1$, and the conjecture holds trivially for forests. More interestingly, the conjecture was shown for cliques on at most six vertices [51, 52, 64] and (with an additional technical assumption) for cycles [62] (see also [46] for an earlier related results for $F = C_4$).

Recently Conjecture 3.6 was verified by Balogh, Morris, and Samotij [8] for 2-balanced graphs F and by Saxton and Thomason [101] for all graphs F.

Theorem 3.7. *Conjecture 3.6 holds for all graphs F.*

One of the main motivations for the conjectured bound on the cardinality of $\mathcal{B}(F, m, M, \varepsilon, p, d)$ in (13) was that it easily implies that such "bad" graphs do not appear as subgraph of the random graph $G(n, p)$. In particular, we obtain an appropriate generalization of the embedding lemma from Fact 3.2, for subgraphs of $G(n, p)$ (see Theorem 3.8). This result was also shown by Conlon, Gowers, Samotij, and Schacht [23] directly (without proving Conjecture 3.6).

Theorem 3.8 (Embedding lemma for subgraphs of random graphs). *For every graph F with vertex set $V(F) = [\ell]$ and every $d > 0$ there exists $\varepsilon > 0$ such that for every $\eta > 0$ there exists $C > 0$ such that for $p > C n^{-1/m_2(F)}$ a.a.s. $G \in G(n, p)$ satisfies the following.*

If $H = (V_1 \cup \ldots \cup V_\ell, E_H)$ is an ℓ-partite (not necessarily induced) subgraph of G with $|V_1| = \cdots = |V_\ell| \geq \eta n$ and with the property that for every edge $\{i, j\} \in E(F)$ the pair (V_i, V_j) in H is (ε, p)-regular and satisfies $d_H(V_i, V_j) \geq dp$, then H contains a partite copy of F, i.e., there exists a graph homomorphism $\varphi \colon F \to H$ with $\varphi(i) \in V_i$.

Proof. We deduce Theorem 3.8 from Theorem 3.7. In fact, it will follow by a standard first moment argument. Since the result is trivial for matchings F we may assume that $m_2(F) \geq 1$.

For given F and d we set

$$
\alpha = \left(\frac{d}{2\mathrm{e}} \right)^{e(F)} ,
$$

where $\mathrm{e} = 2.7182\ldots$ is the base of the natural logarithm. Let $\varepsilon' > 0$ be given by the statement of Conjecture 3.6 applied with F, d, and α and set $\varepsilon = \varepsilon'/2$. Following

the quantification of Theorem 3.8, we are given η. Finally, let $C' > 0$ be given by Conjecture 3.6 and set

$$b = d\eta^2 \quad \text{and} \quad C = \max\left\{\frac{C'}{\eta^{1/m_2(F)}}, \frac{\ell}{b}\right\}.$$

Consider a graph $H' \subseteq G \in G(n,p)$ satisfying the assumptions of Theorem 3.8. Let $m \geq \eta n$ be the size of the vertex classes, $V_1, \ldots, V_\ell$, and set $M = dpm^2$. A straightforward application of Chernoff's inequality asserts that H' contains a spanning subgraph H such that, for every $\{i,j\} \in E(F)$, the pair (V_i, V_j) is $(2\varepsilon, p)$-regular, and $e_H(V_i, V_j) = M$. In other words, $H \in \mathcal{G}(F, m, M, 2\varepsilon, p, d)$ and it suffices to show that a.a.s. $G \in G(n,p)$ contains no graph H from $\mathcal{B}(F, m, M, 2\varepsilon, p, d)$.

For that we consider the expected number of subgraphs in G, which belong to $\mathcal{B}(F, m, M, 2\varepsilon, p, d)$ for some $m \geq \eta n$. For $m \geq \eta n$ fixed, our choice of constants allows us to appeal to the conclusion of Theorem 3.7, and we obtain the following upper bound for the expected number of such graphs:

$$p^{Me(F)} \cdot |\mathcal{B}(F, m, M, 2\varepsilon, p, d)| \cdot \binom{n}{m}^\ell \leq p^{Me(F)} \cdot \alpha^M |\mathcal{G}(F, m, M, 2\varepsilon, p, d)| \cdot \binom{n}{m}^\ell$$

$$\leq p^{Me(F)} \left(\frac{d}{2e}\right)^{Me(F)} \binom{m^2}{M}^{e(F)} 2^{\ell n}$$

$$\leq \left(p \cdot \frac{d}{2e} \cdot \frac{e}{pd}\right)^{Me(F)} 2^{\ell n}$$

$$= 2^{\ell n - Me(F)}$$

$$\leq 2^{-bpn^2},$$

where we used for the last estimate $M \geq dp(\eta n)^2$, $e(F) \geq 2$, and $b = d\eta^2$ combined with $\ell n \leq bpn^2$ (which follows from $m_2(F) \geq 1$ and $C \geq \ell/b$).

Summing the obtained bound over all possible values of m shows that the expected number of bad graphs in G is at most $n2^{-bpn^2}$, and hence, Markov's inequality implies that a.a.s. $G \in G(n,p)$ contains no such graph. $\qquad\square$

Also the counting lemma of Fact 3.2 was partly extended to subgraphs in $G(n,p)$ in [23]. We state these results below.

Theorem 3.9 (Counting lemma for subgraphs of random graphs)**.** *For every graph F with vertex set $V(F) = [\ell]$ and every $d > 0$ there exist $\varepsilon > 0$ and $\xi > 0$ such that for every $\eta > 0$ there exists $C > 0$ such that for $p > Cn^{-1/m_2(F)}$ a.a.s. $G \in G(n,p)$ satisfies the following holds.*

Let $H = (V_1 \cup \ldots \cup V_\ell, E_H)$ be an ℓ-partite (not necessarily induced) subgraph of G with $|V_1| = \cdots = |V_\ell| \geq \eta n$ and with the property that for every edge $\{i,j\} \in E(F)$ the pair (V_i, V_j) in H is (ε, p)-regular with density $d_H(V_i, V_j) \geq dp$.

(i) Then the number of partite copies of F in H is at least

$$\xi p^{e(F)} \prod_{i=1}^{\ell} |V_i|. \tag{15}$$

(ii) *If in addition F is strictly 2-balanced, then the number of partite copies of F in H satisfies*

$$(1 \pm f(\varepsilon))p^{e(F)} \prod_{\{i,j\} \in E(F)} d(V_i, V_j) \prod_{i=1}^{\ell} |V_i|, \qquad (16)$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let us briefly compare Theorems 3.7–3.9. Theorem 3.7, which was proved in [101], gives an affirmative answer to Conjecture 3.6 for all graphs F, and as we showed above, it implies Theorem 3.8. Also part (i) of Theorem 3.9 is a stronger version of Theorem 3.8. While Theorem 3.8 ensures only one copy of the given graph F in an appropriate (ε, p)-regular environment, part (i) of Theorem 3.9 guarantees a constant fraction of the "expected number" of copies of F. For strictly 2-balanced graphs F, part (ii) of Theorem 3.9 guarantees the expected number of copies of F, which can be viewed as the generalization of the counting lemma of Fact 3.2 for such graphs F.

Although Theorem 3.8 is the weakest result in this direction, it turns out to be sufficient for many natural applications of the regularity lemma or subgraphs of sparse random graphs (Theorem 3.4). For example, it allows new and conceptually simple proofs of Theorems 2.2 and 2.3 (see, e.g., Section 4.2 for such a proof of Theorem 2.3).

However, there are a few applications, where the full strength of Theorem 3.7 was needed. For example, following the proof from [62] (see also [82]), one can use the positive resolution of Conjecture 3.6 to prove the 1-statement of the threshold for the asymmetric Ramsey properties of random graphs (see Section 4.1), but Theorems 3.8 and 3.9 seem to be insufficient for this application. In Section 4 we will also mention some applications, which require the quantitative estimates of Theorem 3.9 (see Section 4.3 and 4.4).

Finally, we remark that $G(n, p)$ has the properties of Theorem 3.8 and of part (i) of Theorem 3.9 with probability $1 - 2^{-\Omega(pn^2)}$, while part (ii) of Theorem 3.9 holds with probability at least $1 - n^{-k}$ for any constant k and sufficiently large n (see [23]). Also we note that, due to the upper bound on the number of copies of F given in part (ii) of Theorem 3.9, an error probability of the form $2^{-\Omega(pn^2)}$ can not hold. This is because, for $o(1) = p \gg 1/n$, the upper tail for the number of copies of a graph F (with at least as many edges as vertices) in $G(n, p)$ fails to have such a sharp concentration. In fact, the probability that $G(n, p)$ contains a clique of size $2pn$ is at least $p^{\binom{2pn}{2}} = 2^{-O(p^2 \log(1/p)n^2)} \gg 2^{-\Omega(pn^2)}$, and such a clique gives rise to $(2pn)^{|V(F)|} > 2p^{e(F)}n^{|V(F)|}$ copies of F.

4. Applications of the Regularity Method for Random Graphs

In this section we show some examples how the regularity lemma and its counting and embedding lemmas for subgraphs of random graphs can be applied.

In Section 4.1 we briefly review thresholds for asymmetric Ramsey properties of random graphs. In particular, Theorem 3.7 can be used to establish the 1-statement for such properties. We remark that, even though this is a statement about $G(n, p)$, in the proof suggested by Kohayakawa and Kreuter [62] one applies the sparse regularity lemma to an auxiliary subgraph of $G(n, p)$ with density $o(p)$.

As a result Theorems 3.8 and 3.9 cannot be applied anymore and an application of Theorem 3.7 is pivotal here.

In Section 4.2, we transfer the Erdős–Simonovits theorem (Theorem 1.3) to subgraphs of random graphs, i.e., we deduce Theorem 2.3. The proof given here is based on the sparse regularity lemma, and Theorem 3.8 suffices for this application. It also utilizes the Erdős–Simonovits stability theorem, which will be applied to the so-called *reduced graph*.

In Section 4.3 we discuss another application and extend the *removal lemma* (see Theorem 4.3 for the special case of triangles). The standard proof of the removal lemma is based on Szemerédi's regularity lemma and the counting lemma of Fact 3.2. In fact, the embedding lemma seems not be sufficient for such a proof. The probabilistic version of the removal lemma for subgraphs of random graphs, Theorem 4.4, can be obtained by following the lines of the standard proof, where Szemerédi's regularity lemma and the counting lemma of Fact 3.2 are replaced by the sparse regularity lemma (Theorem 3.4) and part (i) of Theorem 3.9.

In Section 4.4 we state the recent *clique density theorem* of Reiher [93] (see Theorem 4.5 below) and its probabilistic version for random graphs. In the proof of the probabilistic version the "right" counting lemma (part (ii) of Theorem 3.9), giving the expected number of copies of cliques in an appropriate regular environment is an essential tool. Moreover, the clique density theorem itself will be applied to the *weighted* reduced graph.

Finally in Section 4.5 we briefly discuss some connection between the theory of quasi-random graphs and the regularity lemma. In particular, we will mention a generalization of a result of Simonovits and Sós [107] for subgraphs of random graphs and a strengthening of part (ii) of Corollary 2.4.

4.1. **Ramsey Properties of Random Graphs.** Ramsey theory is another important field in discrete mathematics, which was influenced and shaped by Paul Erdős. His seminal work with Szekeres [41] laid the ground for a lot of the research in Ramsey theory. For example, Graham, Spencer, and Rothschild [55, page 26] stated that, *"It is difficult to overestimate the effect of this paper."*

For an integer $r \geq 2$ and graphs $F_1, \ldots, F_r$, we denote by $\mathcal{R}_n(F_1, \ldots, F_r)$ the set of all n-vertex graphs G with the Ramsey property, i.e., the n-vertex graphs G with the property that for every r-coloring of the edges of G with colors $1, \ldots, r$ there exists a color s such that G contains a copy of F_s with all edges colored with color s. Ramsey's theorem [89] implies that $\mathcal{R}_n(F_1, \ldots, F_r)$ is not empty for any r and all graphs $F_1, \ldots, F_r$ for sufficiently large n.

While probabilistic techniques in Ramsey theory were introduced by Erdős [29] in 1947, the investigation of Ramsey properties of the random graph $G(n, p)$ was initiated only in early 90's by Łuczak, Ruciński, and Voigt [80]. In particular, one was interested in the threshold of $\mathcal{R}_n(F_1, \ldots, F_r)$ for the symmetric case, i.e., $F_1 = \cdots = F_r = F$, for which we use the short hand notation $\mathcal{R}_n(F; r)$. This question was addressed by Rödl and Ruciński [95, 96, 97]. There it was shown that $n^{-1/m_2(F)}$ is the threshold for $\mathcal{R}_n(F; r)$ for all graphs F containing a cycle and all integers $r \geq 2$. Note that the threshold is independent of the number of colors r. The proof of the 1-statement was based on an application of Szemerédi's regularity lemma (Theorem 3.1) for dense graphs, even though the result appeals to sparse random graphs. Based on the recent embedding lemma for subgraphs of random

graphs (Theorem 3.8) and a standard application of the sparse regularity lemma (Theorem 3.4) a conceptually simpler proof is now possible.

Below we discuss the asymmetric Ramsey properties, i.e., the case when not all F_i are the same graph. Here we restrict ourselves the the two-color case. Thresholds for asymmetric Ramsey properties involving cycles were obtained by Kohayakawa and Kreuter [62]. Furthermore, these authors put forward a conjecture for the threshold of $\mathcal{R}_n(F_1, F_2)$ for graphs F_1 and F_2 containing a cycle.

Conjecture 4.1. *Let F_1 and F_2 be graphs containing a cycle and $m_2(F_1) \leq m_2(F_2)$. Then $\hat{p} = n^{-1/m_2(F_1,F_2)}$ is a threshold for $\mathcal{R}_n(F_1, F_2)$, where*

$$m_2(F_1, F_2) = \max \left\{ \frac{e(F')}{|V(F')| - 2 + 1/m_2(F_1)} : \ F' \subseteq F_2 \ and \ e(F') \geq 1 \right\}.$$

There is an intuition behind the definition of $m_2(F_1, F_2)$, which has some analogy to the definition of $m_2(F)$ in (2). One can first observe that

$$m_2(F, F) = m_2(F) \quad \text{and} \quad m_2(F_1) \leq m_2(F_1, F_2) \leq m_2(F_2).$$

Moreover, for $p \geq n^{-1/m_2(F_1,F_2)}$ the expected number of copies of F_2 (and all its subgraphs) in $G(n, p)$ is of the same order of magnitude as the expected number of edges $G(n, n^{-1/m_2(F_1)})$. Assuming that there is a two-coloring of $G(n, p)$ with no copy of F_2 with edges in color two, one may hope that picking an edge of color one in every copy of F_2 may result in a graph with "similar properties" as $G(n, n^{-1/m_2(F_1)})$. In particular, those edges should form a copy of F_1 in color one.

In [62] the 1-statement of Conjecture 4.1 for $\mathcal{R}_n(C, F)$ for any cycle C and any 2-balanced graph F with $m_2(C) \geq m_2(F)$ was verified. Moreover, the 0-statement was shown for the case when F_1 and F_2 are cliques [82], and the 1-statement was shown for graphs F_1 and F_2 with $m_2(F_1, F_2) > m_2(F_1, F')$ for every $F' \subsetneq F_2$ with $e(F') \geq 1$ appeared in [69]. In particular, those results yield the threshold for $\mathcal{R}(K_k, K_\ell)$.

It was also known that the resolution of Conjecture 3.6 for the (sparser) graph F_1 allows us to generalize the proof from [62] to verify the 1-statement of Conjecture 4.1 when F_2 is strictly 2-balanced (see, e.g., [82]). Therefore, Theorem 3.7 has the following consequence.

Theorem 4.2. *Let F_1 and F_2 be graphs with $1 \leq m_2(F_1) \leq m_2(F_2)$ and let F_2 be strictly 2-balanced. There exists a constant $C > 0$ such that for $p \geq C n^{-1/m_2(F_1,F_2)}$ a.a.s. $G \in G(n, p)$ satisfies $G \in \mathcal{R}_n(F_1, F_2)$.*

4.2. **Stability Theorem for Subgraphs of Random Graphs.** Below we deduce a probabilistic version of the Erdős–Simonovits theorem from the classical stability theorem, based on the regularity method for subgraphs of random graphs.

Proof of Theorem 2.3. Let a graph F with chromatic number $\chi(F) \geq 3$ and $\varepsilon > 0$ be given. In order to deliver the promised constants C and δ, we have to fix some auxiliary constants. First we appeal to the Erdős–Simonovits stability theorem, Theorem 1.3, with F and $\varepsilon/8$ and obtain constants $\delta' > 0$ and n'_0. Set

$$\delta = \delta'/3.$$

Moreover, set $d = \min\{\delta/4, \varepsilon/4\}$ and set $\varepsilon_{\mathrm{RL}} = \min\{\delta/8, \varepsilon/8, \varepsilon_{\mathrm{EMB}}\}$, where $\varepsilon_{\mathrm{EMB}}$ is given by Theorem 3.8 applied with F and d. Then apply the sparse regularity

lemma, Theorem 3.4, with $\varepsilon_{\mathrm{RL}}$ and $t_0 = \max\{n'_0, 4/\delta, 8/\varepsilon\}$ and obtain the constant T_0. This gives us a lower bound of n/T_0 on the size of the partition classes after an application of Theorem 3.4. To a suitable collection of those classes, we will want to apply Theorem 3.8. Therefore, we set $\eta = 1/T_0$. Due to our choice of $\varepsilon_{\mathrm{RL}} \leq \varepsilon_{\mathrm{EMB}}$ Theorem 3.8 guarantees a constant $C = C(F, d, \varepsilon_{\mathrm{RL}}, \eta)$ and we let $p \geq C n^{-1/m_2(F)}$.

For later reference we observe that, due this choice of constants above, for every $t \geq t_0$ we have

$$\frac{t}{2} + d\binom{t}{2} + \varepsilon_{\mathrm{RL}} t^2 < \delta\binom{t}{2} \tag{17}$$

and

$$\frac{1}{t} + \frac{d}{2} + \varepsilon_{\mathrm{RL}} + \frac{\varepsilon}{8} \leq \frac{\varepsilon}{2}. \tag{18}$$

We split the argument below into a probabilistic and a deterministic part. First, in the probabilistic part, we single out a few properties (see (a)–(c) below), which the random graph $G \in G(n, p)$ has a.a.s. In the second, deterministic part, we deduce the stability result for all graphs G satisfying those properties.

In the probabilistic part we note that a.a.s. $G \in G(n, p)$ satisfies the following:

(a) for all sets $X, Y \subseteq V(G)$ we have $e_G(X, Y) \leq (1 + o(1))p|X||Y|$, where the edges contained in $X \cap Y$ are counted twice,

(b) G satisfies the conclusion of Theorem 3.4 for $\varepsilon_{\mathrm{RL}}$, t_0, and T_0,

(c) G satisfies the conclusion of Theorem 3.8 for F, d, $\varepsilon_{\mathrm{RL}}$, η and C.

Property (a) follows a.a.s. by a standard application of Chernoff's inequality, and properties (b) and (c) hold a.a.s. due to Theorems 3.4 and 3.8.

In the deterministic part we deduce the conclusion of Theorem 2.3 for all graphs satisfying properties (a)–(c). To this end, let $G = (V, E)$ be a graph with these properties. Consider an F-free subgraph $H \subseteq G$ with

$$e(H) \geq \mathrm{ex}_G(F) - \delta p n^2.$$

We will show that we can remove at most $\varepsilon p n^2$ edges from H, so that the remaining graph is $(\chi(F) - 1)$-colorable.

Since every graph G contains a $(\chi(F) - 1)$-cut (see Definition 2.6) of size at least

$$\left(1 - \frac{1}{\chi(F) - 1}\right) e(G) = \pi(F) e(G),$$

it follows from property (a) that

$$e(H) \geq \pi(F) p \binom{n}{2} - 2\delta p n^2. \tag{19}$$

We appeal to property (b), which ensures the existence of a partition $V_1 \dot\cup \ldots \dot\cup V_t = V$ having properties (i)–(iii) of Theorem 3.4 for $\varepsilon_{\mathrm{RL}}$, t_0, and T_0. Without loss of generality, we may assume that t divides n since removing at most t vertices from H affects only $O(tn) = o(pn^2)$ edges.

For the given partition, we consider the so-called *reduced graph* $R = R(H, \varepsilon_{\mathrm{RL}}, d)$ with vertex set $[t]$. The pair $\{i, j\}$ is an edge in R if, and only if the pair (V_i, V_j) is $(\varepsilon_{\mathrm{RL}}, p)$-regular and $d_H(V_i, V_j) \geq dp$. Note that R does not represent the following edges of H:

(I) edges which are contained in some V_i,

(II) edges which are contained in a pair (V_i, V_j) which is not (ε, p)-regular, and

(III) edges which are contained in a pair (V_i, V_j) with $d_H(V_i, V_i) < dp$.

Owing to property (a) we infer, that there are at most

$$t \cdot (1 + o(1))p\binom{n/t}{2} \tag{20}$$

edges described in (I) and at most

$$\varepsilon_{\mathrm{RL}} t^2 \cdot (1 + o(1))p\left(\frac{n}{t}\right)^2 \tag{21}$$

edges described in (II). By definition at most

$$\binom{t}{2} \cdot dp\left(\frac{n}{t}\right)^2 \tag{22}$$

edges of H are contained in pairs described in (III).

Moreover, since (again because of property (a))

$$e_H(V_i, V_j) \le e_G(V_i, V_j) \le (1 + o(1))p\left(\frac{n}{t}\right)$$

it follows from the definition of R, that the number of edges in R satisfies

$$e(R) \ge \frac{e(H) - t \cdot (1 + o(1))p\binom{n/t}{2} - \varepsilon_{\mathrm{RL}} t^2 \cdot (1 + o(1))p(n/t)^2 - \binom{t}{2} \cdot dp(n/t)^2}{(1 + o(1))p(n/t)^2}$$

$$\overset{(17),(19)}{\ge} (\pi(F) - 3\delta)\binom{t}{2} = (\pi(F) - \delta')\binom{t}{2}.$$

Moreover, property (c) implies that R is F-free, since otherwise a copy of F in R would lead to a copy of F in H. In particular, R satisfies the assumptions of the classical Erdős–Simonovits stability theorem, Theorem 1.3. Recall, that $\delta' > 0$ was given by an application of Theorem 1.3 applied with F and $\varepsilon/8$. We will only need the weaker assertion of Theorem 1.3, which concerns the deletion of edges rather than the symmetric difference. Consequently, we may remove up to at most $(\varepsilon/8)t^2$ edges from R, so that the resulting graph R' is $(\chi(F) - 1)$-colorable. Let $f\colon [t] \to [\chi(F) - 1]$ be such a coloring of R' and consider the corresponding partition $W_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} W_{\chi(F)-1} = V$ of H given by

$$W_i = \bigcup \left\{ V_j \colon j \in f^{-1}(i) \right\}.$$

It is left to show that

$$\sum_{i=1}^{\chi(F)-1} e_H(W_i) \le \varepsilon p n^2.$$

Note that there besides the three types of edges described in (I)–(III) the following type of edges of H could be contained in $E_H(W_i)$ for some $i \in [\chi(F) - 1]$

(IV) edges which are contained in a pair (V_i, V_j) for some $\{i, j\} \in E(R) \setminus E(R')$.

Again property (a) combined with $|E(R) \setminus E(R')| \le (\varepsilon/8)t^2$ implies that there are at most

$$\frac{\varepsilon}{8} t^2 \cdot (1 + o(1))p\left(\frac{n}{t}\right)^2 \tag{23}$$

edges described in (IV). Finally, the desired bound follows from (20)–(23)

$$\sum_{i=1}^{\chi(F)-1} e_H(W_i) \;\leq\; t \cdot (1 + o(1))p \binom{n/t}{2}$$

$$+\, \varepsilon_{\mathrm{RL}} t^2 \cdot (1 + o(1))p \left(\frac{n}{t}\right)^2$$

$$+\, \binom{t}{2} \cdot dp \left(\frac{n}{t}\right)^2$$

$$+\, \frac{\varepsilon}{8} t^2 \cdot (1 + o(1))p \left(\frac{n}{t}\right)^2$$

$$\leq\; (1 + o(1))(1/t + d/2 + \varepsilon_{\mathrm{RL}} + \varepsilon/8)pn^2$$

$$\overset{(18)}{\leq}\; \varepsilon pn^2 \,.$$

This concludes the proof of Theorem 2.3. $\hfill\square$

We remark that there are several other classical results involving forbidden subgraphs F, which can be transferred to subgraphs of random graphs, using a very similar approach, i.e., by applying the classical result to a suitably chosen reduced graph R. For example, the 1-statement of Theorem 2.2 or the 1-statement of the Ramsey threshold from [97] can be reproved by such an approach. In the next section, we discuss an example, where one can obtain the probabilistic result by "repeating" the original proof with the sparse regularity lemma and a matching, embedding or counting lemma replacing Szemerédi's regularity lemma and Fact 3.2.

4.3. **Removal Lemma for Subgraphs of Random Graphs.** In one of the first applications of an earlier variant of the regularity lemma, Ruzsa and Szemerédi [99] answered a question of Brown, Sós, and Erdős [18] and essentially established the following removal lemma for triangles.

Theorem 4.3 (Ruzsa & Szemerédi, 1978). *For every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that every graph $G = (V, E)$ with $|V| = n \geq n_0$ containing at most δn^3 copies of K_3 can be made K_3-free by omission of at most εn^2 edges.*

In fact, the same statement holds, when K_3 is replaced by any graph F and δn^3 is replaced by $\delta n^{|V(F)|}$, as was shown by Füredi [47] (see also [2] for the case when F is a clique and [36] for related results). This result is now known as the *removal lemma* for graphs (we refer to the recent survey of Conlon and Fox [21] for a thorough discussion of its importance and its generalizations).

The following probabilistic version for subgraphs of random graphs was suggested by Łuczak [79] and first proved for strictly 2-balanced graphs F by Conlon and Gowers [22]. The general statement for all F follows from the work in [23].

Theorem 4.4. *For every graph F with ℓ vertices and $\varepsilon > 0$ there exist $\delta > 0$ and $C > 0$ such that for $p \geq Cn^{-1/m_2(F)}$ a.a.s. for $G \in G(n, p)$ the following holds. If $H \subseteq G$ contains at most $\delta p^{e(F)} n^\ell$ copies of F, then H can be made F-free by omission of at most εpn^2 edges.*

Proof. Let a graph F with $V(F) = [\ell]$ vertices and $\varepsilon > 0$ be given. Since the result is trivial for matchings F, we may assume that $m_2(F) \geq 1$. We will apply the counting lemma for subgraphs of $G(n, p)$ given by part (i) of Theorem 3.9. We prepare for

such an application with F by setting $d = \varepsilon/6$ and choosing $\varepsilon_{\mathrm{RL}} = \min\{\varepsilon/6, \varepsilon_{\mathrm{CL}}/\ell\}$, where $\varepsilon_{\mathrm{CL}}$ is given by Theorem 3.9. Moreover, we set $t_0 = 3/\varepsilon$ and let T_0 be given by the sparse regularity lemma, Theorem 3.4, applied with $\varepsilon_{\mathrm{RL}}$ and t_0. We then follow the quantification of Theorem 3.9. For that we set $\eta = (T_0\ell)^{-1}$ and let $C > 0$ be given by Theorem 3.9. Finally, we set

$$\delta = \frac{1}{2}\xi p^{e(F)}\eta^{\ell}\,. \tag{24}$$

For later reference we observe that due this choice of constants above, for every $t \geq t_0$ and sufficiently large n we have

$$t\binom{n/t}{2} + 2d\binom{t}{2}\left(\frac{n}{t}\right)^2 + \varepsilon_{\mathrm{RL}}n^2 \leq \frac{\varepsilon}{2}n^2\,. \tag{25}$$

Similarly, as in the proof given in Section 4.2, we split the argument in a probabilistic and a deterministic part. For the probabilistic part we note that a.a.s. $G \in G(n, p)$ satisfies the following:

(A) for all sets $X, Y \subseteq V(G)$ we have $e_G(X, Y) \leq (1 + o(1))p|X||Y|$, where the edges contained in $X \cap Y$ are counted twice,

(B) G satisfies the conclusion of Theorem 3.4 for $\varepsilon_{\mathrm{RL}}$, t_0, and T_0,

(C) G satisfies the conclusion of part (i) of Theorem 3.9 for F, $d - \varepsilon_{\mathrm{RL}}$, $\varepsilon_{\mathrm{RL}}$, ξ, η, and C.

Again property (A) follows a.a.s. by a standard application of Chernoff's inequality and properties (B) and (C) hold a.a.s. due to Theorems 3.4 and 3.9.

It is left to deduce the conclusion of Theorem 4.4 for any graph $G = (V, E)$ satisfying properties (A)–(C) and with sufficiently large $n = |V|$. Let $H \subseteq G$ containing at most $\delta p^{e(F)}n^{\ell}$ copies of F.

Next we appeal to property (B), which ensures the existence of a partition $V_1 \cup \ldots \cup V_t = V$ having properties (i)–(iii) of Theorem 3.4 for $\varepsilon_{\mathrm{RL}}$, t_0, and T_0. Without loss of generality we may assume that $t\ell$ divides n since removing at most $t\ell$ vertices from H affects only $O(t\ell n) = o(pn^2)$ edges.

We remove the following edges from H:

- edges which are contained in some V_i,
- edges which are contained in a pair (V_i, V_j) with $d_H(V_i, V_i) < 2dp$, and
- edges which are contained in a pair (V_i, V_j) which is not (ε, p)-regular.

Let H' be the resulting subgraph. Owing to property (A) we obtain

$$e(H) \setminus e(H') \leq t \cdot (1 + o(1))p\binom{n/t}{2} + \binom{t}{2} \cdot 2dp\left(\frac{n}{t}\right)^2 + \varepsilon_{\mathrm{RL}}t^2 \cdot (1 + o(1))p\left(\frac{n}{t}\right)^2$$

$$\overset{(25)}{\leq} (1 + o(1))\frac{\varepsilon}{2}pn^2 \leq \varepsilon pn^2\,.$$

It is left to show that H' is F-free. Suppose for a contradiction that H' contains a copy of F. Let $V_{i_1}, \ldots, V_{i_k}$ be the vertex classes, that contain a vertex from this copy.

Note that if $k = \ell$, i.e., each class contains exactly one vertex from F, then the ℓ-partite induced subgraph $H'[V_{i_1}, \ldots, V_{i_\ell}]$ meets the assumptions of part (i) of Theorem 3.9 for the constants $2d > d$, $\varepsilon_{\mathrm{RL}} < \varepsilon_{\mathrm{CL}}$, ξ, η, and C fixed above. Consequently, it follows from property (C) that H', and hence also H, contains at

least

$$\xi p^{e(F)} \left(\frac{n}{t}\right)^{\ell} \geq \xi p^{e(F)} (\eta \ell)^{\ell} \cdot n^{\ell} \overset{(24)}{>} \delta p^{e(F)} n^{\ell}$$

copies of F, which contradicts the assumptions on H.

If $k < \ell$, then subdivide every V_{i_j} for $1 \leq j \leq k$ into ℓ disjoint sets of size $|V_{i_j}|/\ell$ in such a way that every subclass created this way contains at most one vertex of the given copy of F in H. Let $W_1, \dots, W_\ell$ be the classes containing one vertex of the copy of F and we may assume that W_i contains the copy of vertex i of F. Note that for every $i \in V(F)$ the set W_i has size at least $n/(t\ell) \geq \eta n$. Moreover, if $\{i, j\} \in E(F)$, then $H'[W_i, W_j]$ contains at least one edge. In particular, this edge is contained in H' and, hence, it signifies that (W_i, W_j) is contained in some pair $(V_{i'}, V_{j'})$, which has density at least dp and which is $(\varepsilon_{\mathrm{RL}}, p)$-regular. Moreover, it follows from the definition of $(\varepsilon_{\mathrm{RL}}, p)$-regularity (see Definition 3.3), that (W_i, W_j) is still $(\ell\varepsilon_{\mathrm{RL}}, p)$-regular and has density at least $(d - \varepsilon_{\mathrm{RL}})p$. In other words, $H'[W_1, \dots, W_\ell]$ is ready for an application of of Theorem 3.9 for the constants $2d - \varepsilon_{\mathrm{RL}} \geq d$, $\ell\varepsilon_{\mathrm{RL}} \leq \varepsilon_{\mathrm{CL}}$, ξ, η, and C fixed above. Consequently, it follows from property (C) that H', and hence also H, contains at least

$$\xi p^{e(F)} \left(\frac{n}{t\ell}\right)^{\ell} \geq \xi p^{e(F)} \eta^{\ell} \cdot n^{\ell} \overset{(24)}{>} \delta p^{e(F)} n^{\ell}$$

copies of F, which also in this case contradicts the assumptions on H and concludes the proof of Theorem 4.4. $\qquad\square$

4.4. Clique Density Theorem for Subgraphs of Random Graphs.
Turán's theorem establishes the minimum number $\mathrm{ex}_n(K_k) + 1$ of edges in an n-vertex graph that implies the existence of a copy of K_k. For the triangle case, it was proved by Hans Rademacher (unpublished) that every n-vertex graph with $\mathrm{ex}_n(K_3) + 1$ edges contains not only one, but at least $n/2$ triangles. More generally, Erdős suggested to study the minimum number of triangles in n-vertex graphs with $\mathrm{ex}_n(K_3) + s$ edges [30, 32]. He conjectured that for $s < n/4$ there are at least $s\lfloor n/2 \rfloor$ triangles, which is best possible due to the graph obtained by balanced, complete bipartite graph with s independent edges in the vertex with $\lceil n/2 \rceil$ vertices. This conjecture was proved by Lovász and Simonovits [76] (see also [60]). For larger values of s and k this problem was studied by Erdős [33], Moon and Moser [83], Nordhaus and Stewart [85], Bollobás [10, 11], and Khadzhiivanov and Nikiforov [59].

In particular, in [77] Lovász and Simonovits formulated a conjecture which relates the minimum density of K_k with a given edges density. More precisely, for an integer $k \geq 3$ and a graph H, let $\mathcal{K}_k(H)$ be the number of (unlabeled) copies of K_k in H. We denote by $\mathcal{K}_k(n, M)$ the minimum over all graphs with n vertices and M edges, i.e.,

$$\mathcal{K}_k(n, M) = \min\{\mathcal{K}_k(H) \colon |V(H)| = n \text{ and } |E(H)| = M\}.$$

In [76] Lovász and Simonovits conjectured that the extremal graph for $\mathcal{K}_k(n, M)$ is obtained from complete t-partite graph (for some appropriate t) by adding a matching to one of the vertex classes. In [77] those authors proposed an approximate version of this conjecture by considering densities of cliques and edges instead relating the number of cliques with the number of edges. For that we define for $\alpha \in [0, 1)$

$$\kappa_k(\alpha) = \liminf_{n \to \infty} \frac{\mathcal{K}_k(n, \lceil \alpha \binom{n}{2} \rceil)}{\binom{n}{k}},$$

i.e., n-vertex graphs with $\alpha\binom{n}{2}$ edges contain at least $(\kappa_k(\alpha) - o(1))\binom{n}{k}$ copies of K_k and $\kappa_k(\alpha)$ is the largest clique density which can be guaranteed. Clearly, $\kappa_k(\cdot)$ is non-decreasing and for $\alpha \in [0, \pi(K_k))$ we have $\kappa_k(\alpha) = 0$. Lovász and Simonovits suggested that the graphs described below attain the infimum of $\kappa_k(\alpha)$: For a given $\alpha > 0$, let t be the integer with the property

$$\alpha \in \left(1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right] \tag{26}$$

and set $\varrho \in \mathbb{R}$ to the smaller root of the quadratic equation

$$2\varrho(1 - \varrho) + \left(1 - \frac{1}{t}\right)(1 - \varrho)^2 = \alpha\,.$$

One can check that (26) implies $0 < \varrho \leq \frac{1}{t+1}$. We then define the graphs $T_{n,\alpha}$ to be the complete $(t+1)$-partite graph with vertex classes $V_1 \cup \ldots \cup V_{t+1} = V(T_{n,\alpha})$ satisfying

$$|V_{t+1}| = \lceil \varrho n \rceil \quad \text{and} \quad |V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1\,.$$

Maybe, a more intuitive description of these graphs is the following. For edge densities α of the form $1 - 1/t$ the graph $T_{n,\alpha}$ is the Turán graph $T_{n,t}$ with t classes. For $\alpha \in (1 - \frac{1}{t}, 1 - \frac{1}{t+1})$ a "small" $(t+1)$st class of size ϱn appears and all other classes have size $(1 - \varrho)n/t$. With α tending to $1 - \frac{1}{t+1}$ the difference in size between the $(t+1)$st class and the other classes becomes smaller. Finally, for $\alpha = 1 - \frac{1}{t+1}$ we get $\varrho = 1/(t+1)$ and $T_{n,\alpha}$ becomes the Turán graph with $t + 1$ classes.

Lovász and Simonovits conjectured that for every k and α

$$\kappa_k(\alpha) = \lim_{n \to \infty} \frac{\mathcal{K}_k(T_{n,\alpha})}{\binom{n}{k}}\,. \tag{27}$$

We remark that the conjectured extremal graph $T_{n,\alpha}$ is independent of the size of clique K_k. This conjecture was known to be true in the "symmetric case," i.e., for densities $\alpha \in \{1 - 1/t : t \in \mathbb{N}\}$, due to the work of Moon and Moser [83] (see [85] for the triangle case). Fisher addressed (27) for $k = 3$ and $1/2 \leq \alpha \leq 2/3$.

A few years ago Razborov introduced the so-called *flag algebra method* in extremal combinatorics [91] (see [90] for a survey on the topic) and based on this calculus he solved the triangle case for every $\alpha \in (0, 1)$ in [92]. This work was followed by Nikiforov [84], which led to the solution of the case $k = 4$ and finally Reiher [93] verfied the conjecture for every k.

Theorem 4.5 (Clique density theorem). *For every integer $k \geq 3$ and for every $\alpha \in (0, 1)$ we have*

$$\kappa_k(\alpha) = \lim_{n \to \infty} \frac{\mathcal{K}_k(T_{n,\alpha})}{\binom{n}{k}}\,.$$

Based on the counting lemma for subgraphs of random graphs, part *(ii)* of Theorem 3.9, one can use the sparse regularity lemma to transfer this result to subgraphs of random graphs. The following appears in [23].

Theorem 4.6. *For every graph $k \geq 3$ and $\delta > 0$ there exists $C > 0$ such that for $p \geq Cn^{-1/m_2(K_k)}$ the following holds a.a.s. for $G \in G(n, p)$. If $H \subseteq G$ contains at least $(\alpha + \delta)e(G)$ edges, then*

$$\mathcal{K}_k(H) \geq \kappa_k(\alpha)\mathcal{K}_k(G)\,.$$

As mentioned above, the proof of Theorem 4.6 is based on the regularity method for subgraphs of random graphs and relies on the counting lemma giving the "expected number" of copies of K_k in an appropriate (ε, p)-regular environment. Moreover, in the proof a *weighted version* of the clique density theorem, Theorem 4.5 is applied to the weighted reduced graph (see [23] for details).

4.5. **Quasi-random Subgraphs of Random Graphs.** In this section we discuss scaled versions of the Chung-Graham-Wilson theorem [20] on quasi-random graphs for subgraphs a random graphs. The systematic study of quasi-random graphs was initiated by Thomason [112, 113] and Chung, Graham, and Wilson [20] (see also [1, 44, 94] for partial earlier results and [75] for a recent survey on the topic). In [20] several properties of dense random graphs, i.e., properties a.a.s. satisfied by $G(n, p)$ for $p > 0$ independent of n, were shown to be equivalent in a deterministic sense. This phenomenon fails to be true for $p = o(1)$ (see, e.g., [19, 66]). For relatively dense subgraphs of sparse random graphs however several deterministic equivalences among (appropriately scaled) quasi-random properties remain valid. Below we will discuss one such equivalence (see Theorem 4.9 below), whose analog for dense graphs was obtained by Simonovits and Sós [107].

Before we mention the result of Simonovits and Sós we begin with the following quasi-random properties of graphs, concerning the edge distribution (see DISC below) and the number of copies (or embeddings) of given graph F (see EMB below).

Definition 4.7. *Let F be a graph on ℓ vertices and let $d > 0$.*

DISC: *We say a graph $H = (V, E)$ with $|V| = n$ satisfies DISC(d), if for every subset $U \subseteq V$ we have*

$$e_H(U) = d \binom{|U|}{2} \pm o(n^2) \,.$$

EMB: *We say a graph $H = (V, E)$ with $|V| = n$ satisfies EMB(F, d), if the number $N_F(H)$ of labeled copies of F in H satisfies*

$$N_F(H) = d^{e(F)} n^\ell \pm o(n^\ell) \,.$$

It is well known that the property DISC(d) implies the property EMB(F, d) for every graph F. By this we mean that for every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that every n-vertex ($n \geq n_0$) graph H satisfying DISC(d) with $o(n^2)$ replaced by δn^2 also satisfies property EMB(F, d) with $o(n^\ell)$ replaced by εn^ℓ.

The opposite implication is known to be false. For example, for $F = C_\ell$ being a cycle of length ℓ we may consider n-vertex graphs H consisting of a clique of size dn and isolated vertices. Such a graph H satisfies EMB(C_ℓ, d), but fails to have DISC(d). However, note that such a graph H fails to have density d. If we add this as an additional condition, then for even ℓ one of the main implications of the Chung-Graham-Wilson theorem asserts the implication EMB(C_ℓ, d) implies DISC(d).

For odd cycles imposing global density d does not suffice, as the following interesting example from [20] shows: Partition the vertex set $V(H) = V_1 \cup V_2 \cup V_3 \cup V_4$ as equal as possible into four sets and add the edges of the complete graph on V_1 and on V_2, add edges of the complete bipartite graph between V_3 and V_4, and add edges of a random bipartite graph with edge probability $1/2$ between $V_1 \cup V_2$ and $V_3 \cup V_4$.

One may check that a.a.s. such a graph H has density $d(H) \geq 1/2 - o(1)$ and it satisfies $\mathrm{EMB}(K_3, 1/2)$, but clearly it fails to have $\mathrm{DISC}(1/2)$.

Summarizing the discussion above, while $\mathrm{DISC}(d)$ implies $\mathrm{EMB}(F, d)$ is known to be true for every graph F, $\mathrm{EMB}(C_{2\ell+1}, d)$ does not imply $\mathrm{DISC}(d)$. In fact, $\mathrm{EMB}(C_{2\ell+1}, d) \not\Rightarrow \mathrm{DISC}(d)$ even when restricting to graphs H with density d.

Note that the property $\mathrm{DISC}(d)$ is *hereditary* in the sense that for subsets $U \subseteq V$ the induced subgraph $H[U]$ must also satisfy $\mathrm{DISC}(d)$, if H has $\mathrm{DISC}(d)$. As a result the implication $\mathrm{DISC}(d) \Rightarrow \mathrm{EMB}(F, d)$ extends to the following hereditary strengthening of $\mathrm{EMB}(F, d)$.

HEMB: We say a graph $H = (V, E)$ with $|V| = n$ satisfies $\mathrm{HEMB}(F, d)$, if for every $U \subseteq V$ the number $N_F(H[U])$ of labeled copies of F in the induced subgraph $H[U]$ satisfies

$$N_F(H[U]) = d^{e(F)}|U|^{\ell} \pm o(n^{\ell}) \,. \tag{28}$$

It was shown by Simonovits and Sós [107] (see also [115] for a recent strengthening) that HEMB indeed is a quasi-random property, i.e., those authors showed that for every graph F with at least one edge and for every $d > 0$ the properties $\mathrm{DISC}(d)$ and $\mathrm{HEMB}(F, d)$ are equivalent, i.e., $\mathrm{DISC}(d) \Rightarrow \mathrm{HEMB}(d)$ and $\mathrm{HEMB}(d) \Rightarrow \mathrm{DISC}(d)$.

Based on the sparse regularity lemma (Theorem 3.4) and its appropriate counting lemma (part (ii) of Theorem 3.9) a generalization of the Simonovits–Sós theorem for subgraphs of random graphs $G \in G(n, p)$ can be derived. First we introduce the appropriate sparse versions of DISC and HEMB for this context.

Definition 4.8. *Let $G = (V, E)$ be a graph with $|V| = n$, let F be a graph on ℓ vertices, and let $d > 0$ and $\varepsilon > 0$.*

DISC_G: *We say a subgraph $H \subseteq G$ satisfies $\mathrm{DISC}_G(d)$, if for every subset $U \subseteq V$ we have*

$$e_H(U) = d|E(G[U])| \pm o(|E|) \,,$$

i.e., the relative density of $H[U]$ with respect to $G[U]$ is close to d for all sets U of linear size. Furthermore, we say H satisfies $\mathrm{DISC}_G(d, \varepsilon)$ if $e_H(U) = d|E(G[U])| \pm \varepsilon|E|$.

HEMB_G: *We say a subgraph $H \subseteq G$ satisfies $\mathrm{HEMB}_G(F, d)$, if for every $U \subseteq V$ the number $N_F(H[U])$ of labeled copies of F in the induced subgraph $H[U]$ satisfies*

$$N_F(H[U]) = d^{e(F)}N_F(G[U]) \pm o(|N_F(G)|) \,,$$

i.e., approximately a $d^{e(F)}$ proportion of the copies of F in $G[U]$ is contained in $H[U]$ for sets U spanning a constant proportion of copies of F in G.

Furthermore, we say a subgraph $H \subseteq G$ satisfies $\mathrm{HEMB}_G(F, d, \varepsilon)$, if for every $U \subseteq V$ we have $N_F(H[U]) = d^{e(F)}N_F(G[U]) \pm \varepsilon|N_F(G)|$.

For those properties one can prove an equivalence in the sense described above, when G is a random graph.

Theorem 4.9. *Let F be a strictly 2-balanced graph with at least one edge and let $d > 0$. For every $\varepsilon > 0$ there exist $\delta > 0$ and $C > 0$ such that for $p \geq Cn^{-1/m_2(F)}$, a.a.s. for $G \in G(n, p)$ the following holds.*

(i) *If $H \subseteq G$ satisfies $\mathrm{DISC}_G(d, \delta)$, then H satisfies $\mathrm{HEMB}_G(F, d, \varepsilon)$.*

(ii) *If $H \subseteq G$ satisfies $\mathrm{HEMB}_G(F, d, \delta)$, then H satisfies $\mathrm{DISC}_G(d, \varepsilon)$.*

Consequently, for $p \gg n^{-1/m_2(F)}$ a.a.s. $G \in G(n,p)$ has the property that $\mathrm{DISC}_G(d)$ and $\mathrm{HEMB}_G(F,d)$ are equivalent.

We will briefly sketch some ideas from the proofs of both implications of the Simonovits-Sós theorem and indicate its adjustments for the proof of Theorem 4.9.

The implication $\mathrm{DISC}(d) \Rightarrow \mathrm{HEMB}(F,d)$ (for dense graphs) easily follows from the counting lemma in Fact 3.2. Indeed, given $U \subseteq V(H)$ for a graph H satisfying $\mathrm{DISC}(d)$, we consider a partition of U into $\ell = |V(F)|$ classes $U_1 \uplus \ldots \uplus U_\ell$ with sizes as equal as possible. Based on the identity

$$e_H(U_i, U_j) = e_H(U_i \uplus U_j) - e_H(U_i) - e_H(U_j)$$

we infer from $\mathrm{DISC}(d)$ that (U_i, U_j) is $g(\varepsilon)$-regular and $d_H(U_i, U_j) = d \pm o(1)$, where $g(\varepsilon)$ tends to 0 with the error parameter ε from the property $\mathrm{DISC}(d)$. In particular, for sufficiently small $\varepsilon > 0$ the assumptions of the counting lemma of Fact 3.2 are met for F and d. Using the upper and lower bound on the number of partite copies of F in $U_1 \uplus \ldots \uplus U_\ell$ provided by the counting lemma and a simple averaging argument over all possible partitions $U_1 \uplus \ldots \uplus U_\ell$ yields (28). This simple argument with part (ii) of Theorem 3.9 replacing Fact 3.2 can be transferred to $G(n,p)$ without any further adjustments.

The proof of the opposite implication, $\mathrm{HEMB}(F,d) \Rightarrow \mathrm{DISC}(d)$, is more involved. All known proofs for dense graphs are based on Szemerédi's regularity lemma (Theorem 3.1) and the counting lemma in Fact 3.2. The proof of Simonovits and Sós requires not only an applications of Fact 3.2 for F, but also for a graph obtained from F by taking two copies of F and identifying one of their edges. Note that this "double-F" is not strictly 2-balanced, since it contains F as a proper subgraph, which has the same 2-density as double-F. Consequently, we run into some difficulties, if we want to extend this proof to subgraphs of random graphs based on part (ii) of Theorem 3.9. In some recent generalizations of the Simonovits–Sós theorem applications of Fact 3.2 for double-F could be avoided (see, e.g., [103, 24, 25]). In particular, the proof presented in [25, pages 174-175] extends to subgraphs of random graphs, by replacing Szemerédi's regularity lemma and the counting lemma of Fact 3.2 by its counterparts for sparse random graphs.

4.5.1. *Problem of Erdős and Nešetřil revisited.* We close this section by returning to the question of Erdős and Nešetřil from Section 2.2, which perhaps led to one of the first extremal results for random graphs.

Here we want to focus on generalizations of part (ii) of Corollary 2.4. That statement asserts that any K_{k+1}-free graph H with the additional property

$$\mathrm{ex}_H(K_k) \leq (\pi(K_k) + o(1))e(H) \tag{29}$$

must have vanishing density $d(H) = o(1)$.

Based on Theorem 3.8 the following generalization can be proved. Consider the random graph $G \in G(n,p)$ for $p \gg n^{-1/m_2(K_{k+1})}$. We will show that a.a.s. G has the property that any K_{k+1}-free graph $H \subseteq G$ satisfying (29) must have vanishing *relative* density (w.r.t. the density of G), i.e., $d(H) = o(p)$.

Theorem 4.10 (Generalization of Corollary 2.4(ii) for subgraphs of $G(n,p)$)**.** *For every integer $k \geq 3$, every $d > 0$, and every $\varepsilon \in (0, 1 - \pi(K_k))$ there exists some $C > 0$ such that for $p > Cn^{-1/m_2(K_{k+1})}$ the following holds a.a.s. for $G \in G(n,p)$. If $H \subseteq G$ satisfies $e(H) = d|E(G)|$ and $\mathrm{ex}_H(K_k) \leq (\pi(K_k) + \varepsilon)e(H)$, then H contains a K_{k+1}.*

The proof of Theorem 4.10 follows the lines of the proof of Corollary 2.4(ii) given in Section 2.2 and we briefly sketch the main adjustments needed. Recall that the proof given in Section 2.2 relied on Lemma 2.8 from [94], which is based on the embedding lemma of Fact 3.2. Replacing the embedding lemma for dense graphs by the appropriate version for subgraphs of random graphs, i.e., by Theorem 3.8, yields the following.

Lemma 4.11. *For all integers s, $t \geq 2$ and every $d > 0$ there exist $\delta > 0$ and $C > 0$ such that for $p > Cn^{-1/m_2(K_s)}$ the following holds a.a.s. for $G \in G(n, p)$.*

If $H \subseteq G$ satisfying $e_H(U) = (d \pm \delta)e_G(U)$ for every $U \subseteq V$ with $|U| = \lfloor n/t \rfloor$. Then H contains a copy of K_s.

Equipped with Lemma 4.11 one can repeat the proof of Lemma 2.7 and the following appropriate version for subgraphs of $G(n, p)$ can be verified.

Lemma 4.12. *For all integers s, $t \geq 2$ and every $d > 0$ there exist $\varepsilon > 0$ and $C > 0$ such that for $p > Cn^{-1/m_2(K_s)}$ the following holds a.a.s. for $G \in G(n, p)$.*

If $H \subseteq G$ with $e(H) = d|E(G)|$ and with the property that every balanced t-cut has size at most $(1 - 1/t + \varepsilon)de(G)$, then H contains a copy of K_s.

Finally, a standard application of Lemma 4.12 with $s = k + 1$ and $t = k - 1$ yields Theorem 4.10. We omit the details here.

5. Concluding Remarks

We close with a few comments of related results and open problems.

Related Results. We restricted the discussion to extremal question in random graphs. However, the results of Conlon and Gowers [22] and Schacht [102] and also the subsequent work of Samotij [100], Balogh, Morris, Samotij [8], and Saxton and Thomason [101] applied in a more general context and led to extremal results for random hypergraphs and random subsets of the integers. Here we state a probabilistic version of Szemerédi's theorem on arithemtic progressions [110] (see Theorem 5.1 below).

For integers $k \geq 3$ and $n \in \mathbb{N}$, and a set $A \subseteq \mathbb{Z}/n\mathbb{Z}$, let $r_k(A)$ denote the cardinality of a maximum subset of A, which contains no arithmetic progression of length k, i.e.,

$$r_k(A) = \{|B| \colon B \subseteq A \text{ and } B \text{ contains no arithmetic progresion of length } k\}$$

Answering a well known conjecture of Erdős and Turán [42], Szemerédi's theorem asserts that

$$r_k(\mathbb{Z}/n\mathbb{Z}) = o(n)$$

for every integer $k \geq 3$. The following probabilistic version of Szemerédi's theorem was obtained for $k = 3$ by Kohayakawa, Łuczak, and Rödl [64] and for all k in [22, 102].

Theorem 5.1. *For every integer $k \geq 3$ and every $\varepsilon > 0$ the function $\hat{p} = n^{-1/(k-1)}$ is a threshold for $\mathcal{S}_n(k, \varepsilon) = \{A \subseteq \mathbb{Z}/n\mathbb{Z} \colon r_k(A) \leq \varepsilon|A|\}$.*

Note that similarly as for the threshold for the Erdős–Stone theorem for random graphs, the threshold for Szemerédi's theorem coincides with that p for which a random subset of $\mathbb{Z}/n\mathbb{Z}$ has in expectation the same number of elements and number of arithmetic progressions of length k.

Let us remark that the methods from [22, 102] also can be used to derive thresholds for Ramsey properties for random hypergraphs and random subsets of the integers (see [22, 45] for details).

Open Problems. Besides these recents advances, several important questions are still unresolved. For example, it would be very interesting if the result of DeMarco and Kahn [26] (Theorem 2.9) could be extended to cliques of arbitrary size (see Conjecture 2.10). Finally, we would like to point out that for some applications (see, e.g., Theorem 4.9) a generalization of part (*ii*) of Theorem 3.9 for all graphs F would be useful.

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