Almost all hypergraphs without Fano planes are bipartite*

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Abstract

The hypergraph of the Fano plane is the unique 3-uniform hypergraph with 7 triples on 7 vertices in which every pair of vertices is contained in a unique triple. This hypergraph is not 2-colorable, but becomes so on deleting any hyperedge from it. We show that taking uniformly at random a labeled 3-uniform hypergraph H on n vertices not containing the hypergraph of the Fano plane, H turns out to be 2-colorable with probability at least $1 - 2^{-\Omega(n^2)}$. For the proof of this result we will study structural properties of Fano-free hypergraphs.

1 Introduction

We study monotone properties of the type $\mathrm{Forb}(n, L)$ for a fixed hypergraph L, i.e., the family of all labeled hypergraphs on n vertices, which contain no copy of L as a (not necessarily induced) subgraph. We obtain structural results for an "average" member of $\mathrm{Forb}(n, F)$, where F is the 3-uniform hypergraph of the Fano plane, which is the unique triple system with 7 hyperedges on 7 vertices where every pair of vertices is contained in precisely one hyperedge. The hypergraph of the Fano plane F is not 2-colorable, i.e., for every vertex partition $X\dot{\cup}Y = V(F)$ into two classes there exists an edge of F which is either contained in X or in Y. Consequently, $\mathrm{Forb}(n, F)$ contains any 2-colorable 3-uniform hypergraph on n vertices.

Roughly speaking, we show that almost every hypergraph $H \in \mathrm{Forb}(n, F)$ is 2-colorable. More precisely, let $\mathcal{B}_n$ be the class of all labeled 2-colorable (or bipartite) hypergraphs on n vertices. We prove that

$$\frac{|\mathrm{Forb}(n, F)|}{|\mathcal{B}_n|} \le (1 + 2^{-\Omega(n^2)})$$

(see Theorem 1.1). This shows that choosing uniformly at random a 3-uniform Fano-free hypergraph H, then only with exponentially small probability, $2^{-\Omega(n^2)}$, the hypergraph H turns out not to be bipartite. Moreover, we prove that only with even smaller probability, $2^{-\Omega(n^3)}$, the Fano-free hypergraphs are "far" from being bipartite (see Lemma 4.1 for the precise statement).

Similar questions for graphs (following the work of Kleitman and Rothschild [13] on posets) were first studied by Erdős, Kleitman, and Rothschild [7], who were interested in $\mathrm{Forb}(n, L)$ for a fixed graph L. In particular, those authors proved that almost every triangle-free graph is bipartite. Moreover, they showed for $p \ge 3$

$$\left|\mathrm{Forb}(n, K_{p+1})\right| \le 2^{(1-\frac{1}{p})\binom{n}{2}+o(n^2)}. \tag{1.1}$$

Thus, the dominating term in the exponent turns out to be the extremal number $\mathrm{ex}(n, K_{p+1})$, where by $\mathrm{ex}(n, L)$ we mean the maximum number of edges a graph on n vertices can have without containing L as a subgraph. Later, Erdős, Frankl, and Rödl [6] extended this result from cliques to arbitrary graphs L with chromatic number, $\chi(L)$, at least three, by proving

$$|\mathrm{Forb}(n, L)| \le 2^{(1+o(1))\mathrm{ex}(n,L)}. \tag{1.2}$$

A strengthening of (1.1) was obtained by Kolaitis, Prömel, and Rothschild [15, 16], who showed that almost every K_{p+1}-free graph is p-colorable. This result was further extended by Prömel and Steger [20] (see also [10]) from cliques to such graphs L, which contain a *color-critical edge*, i.e., an edge $e \in E(L)$ such that $\chi(L - e) < \chi(L)$ with $L - e = (V(L), E(L) \setminus \{e\})$. The result of Prömel and Steger states that for graphs L with $\chi(L) = p + 1 \ge 3$ almost every L-free graph is p-colorable if and only if L contains a color-critical edge, which was conjectured earlier by Simonovits [3].

Recently, Balogh, Bollobás, and Simonovits [2] showed a sharper version of (1.2):

$$|\mathrm{Forb}(n, L)| \le 2^{(1-\frac{1}{p})\binom{n}{2}+O(n^{2-\gamma})},$$

where $p = \chi(L) - 1$ and $\gamma = \gamma(L) > 0$ is some constant depending on L (see also [1] for more structural results by the same authors).

In this paper we will study this type of questions for k-uniform hypergraphs, where by a k-uniform hypergraph H we mean a pair (V, E) with $E \subseteq \binom{V}{k}$. Let L

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be a k-uniform hypergraph and denote by $\mathrm{Forb}(n, L)$ the set of all labeled k-uniform hypergraphs H on n vertices not containing a (not necessarily induced) copy of L. Note, that the main term in the exponent in the graph case was $\mathrm{ex}(n, L)$. Recently, in [17, 18] similar results were obtained for k-uniform hypergraphs, i.e,

$$|\mathrm{Forb}(n, L)| \leq 2^{\mathrm{ex}(n,L)+o(n^k)}$$

for arbitrary k-uniform hypergraphs L.

Furthermore, we say a k-uniform hypergraph H contains a $(p + 1)$-color-critical edge, if there exists a hyperedge e in H such that after its deletion one can color the vertices of H with p colors without creating a monochromatic edge in $H' = (V(H), E(H) \setminus \{e\})$, but one still needs $p + 1$ colors to color H.

In this paper we will sharpen the bound on $|\mathrm{Forb}(n, L)|$ in the special case, when L is the 3-uniform hypergraph of the Fano plane. It was shown independently by Füredi and Simonovits [9] and Keevash and Sudakov [12], that the unique extremal Fano-free hypergraph is the balanced, complete, bipartite hypergraph $B_n = (U \dot\cup W, E_{B_n})$, where $|U| = \lfloor n/2 \rfloor$, $|W| = \lceil n/2 \rceil$ and E_{B_n} consists of all hyperedges with at least one vertex in U and one vertex in W. Therefore, for the hypergraph of the Fano plane F we have

$$\begin{aligned}\mathrm{ex}(n, F) = e(B_n) = |E_{B_n}| & \\ &= \binom{n}{3} - \binom{\lceil n/2 \rceil}{3} - \binom{\lfloor n/2 \rfloor}{3} \\ &= \frac{n^3}{8} - \frac{n^2}{4} - O(n) \leq \frac{n^3}{8}\end{aligned}$$

and

$$\delta_1(B_n) = \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \left\lfloor \frac{n}{2} \right\rfloor + \binom{\lfloor n/2 \rfloor}{2} \geq \frac{3}{8} n^2 - n\,, \tag{1.3}$$

where for a hypergraph $H = (V, E)$ we denote by $\delta_1(H)$ the minimum vertex degree, i.e.,

$$\delta_1(H) = \min_{u \in V} \left| \{ \{v, w\} \colon \{u, v, w\} \in E \} \right|.$$

Furthermore, we have

$$e(B_n) = e(B_{n-3}) + \delta_1(B_n) + \delta_1(B_{n-1}) + \delta_1(B_{n-2}).$$

Note that the hypergraph of the Fano plane is a 3-color-critical hypergraph, i.e., the hypergraph of the Fano plane becomes 2-colorable after deletion of *any* edge.

Let $\mathcal{B}_n$ denote the set of all labeled bipartite 3-uniform hypergraphs on n vertices. We know that every labeled bipartite hypergraph does not contain a copy of the hypergraph of the Fano plane. On the other hand, we prove that almost every Fano-free hypergraph is 2-colorable.

THEOREM 1.1. *Let F be the 3-uniform hypergraph of the Fano plane. There exist a real $c > 0$ and an integer n_0, such that for every $n \geq n_0$ we have*

$$|\mathrm{Forb}(n, F)| \leq |\mathcal{B}_n|(1 + 2^{-cn^2}).$$

This result can be seen as a first attempt to derive results for hypergraphs in the spirit of Kolaitis, Prömel and Rothschild [16] and Prömel and Steger [20]. We will prove Theorem 1.1 using techniques developed by Balogh, Bollobás and Simonovits in [2]. In fact, with these methods, one could also reprove the above mentioned theorems for graphs containing color-critical edges [3].

2 Notation and Outline

From now on we will consider only 3-uniform hypergraphs and by a hypergraph we will always mean a 3-uniform hypergraph. For the sake of a simpler notation we set

$$\mathcal{F}_n = \mathrm{Forb}(n, F)\,,$$

where by F we will always denote the hypergraph of the Fano plane. We will refer to hypergraphs not containing a copy of F, as Fano-free hypergraphs.

We will use the following estimates (often without mentioning them explicitly). First of all we note that we can bound $|\mathcal{B}_n|$ by

$$2^{e(B_n)} \leq |\mathcal{B}_n| \leq 2^n \cdot 2^{e(B_n)}, \tag{2.4}$$

as there are at most 2^n partitions of $[n]$ in two disjoint sets and there are at most $e(B_n)$ hyperedges running between those two sets.

For a given hypergraph $H = (V, E)$ a vertex $v \in V$ we define its link $L_H(v) = (V \setminus \{v\}, E_v)$ to be the graph whose edges together with v form hyperedges of H, namely

$$E_v = \{\{u, w\} \colon \{v, u, w\} \in E\}\,.$$

We define the degree of $v \in V$ to be $\deg(v) = \deg_H(v) = |E(L_H(v))|$, and sometimes we omit H when it is clear from the context. For $A \subseteq V(H)$ and $v \in V(H)$ denote by

$$L_A(v) = L_H(x)[A] = (A \setminus \{v\}, E_v \cap \tbinom{A}{2})$$

the link of v induced on A.

By $h(x) = -x \log x - (1 - x) \log(1 - x)$ we denote the entropy function, and by log we always mean $\log_2$. Note further, $h(x) \to 0$ as x tends to 0, and $h(x) \geq x$ for $x \leq \frac{1}{2}$. We will use the entropy function $h(x)$ together with the well-known inequality $\binom{n}{xn} \leq 2^{h(x)n}$, which holds for $0 < x < 1$. Furthermore, we will use that for $n > 3k$ we have $\sum_{j<k} \binom{n}{j} < \binom{n}{k}$. Often we will omit floors and ceilings, as they will have no effect on our asymptotic arguments.

DEFINITION 2.1. *With every hypergraph $H \in \mathcal{F}_n$ we associate a partition $X_H \dot{\cup} Y_H$ of its vertex set $V(H)$ that minimizes $e(X_H) + e(Y_H)$. In case of ambiguity fix one such partition arbitrarily. Furthermore, by X_H and Y_H, we mean this partition of the vertices of H.*

Our proof will be split into several lemmas, and will follow similar steps as in [2]. Namely, we will study several subclasses of $\mathcal{F}_n$ that for appropriately chosen parameters α and $\beta > 0$ form the chains

$$\mathcal{F}_n \supseteq \mathcal{F}'_n(\alpha) \supseteq \mathcal{F}''_n(\alpha, \beta) \supseteq \mathcal{F}'''_n(\alpha, \beta)$$

and

$$\mathcal{F}_n \supseteq \mathcal{B}_n \supseteq \mathcal{F}'''_n(\alpha, \beta).$$

Roughly speaking, we will show that

$$|\mathcal{F}'_n(\alpha)| \geq (1 - o(1))|\mathcal{F}_n|\,,\ |\mathcal{F}''_n(\alpha, \beta)| \geq (1 - o(1))|\mathcal{F}_n| \quad \text{and} \quad |\mathcal{F}'''_n(\alpha, \beta)| \geq (1 - o(1))|\mathcal{F}_n|$$

and due to

$$|\mathcal{F}_n| \leq |\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| + |\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha, \beta)| + |\mathcal{F}''_n(\alpha, \beta) \setminus \mathcal{F}'''_n(\alpha, \beta)| + |\mathcal{B}_n|$$

Theorem 1.1 then follows. Below we informally define all these special subclasses of Fano-free hypergraphs and sketch the main ideas of the proof.

1. $\mathcal{F}'_n(\alpha) \subseteq \mathcal{F}_n$ will be the class of "almost bipartite" hypergraphs, i.e., those Fano-free hypergraphs that admit a partition of its vertices into classes of nearly equal size, such that less than αn^3 edges lie inside the partition classes. Using the weak hypergraph regularity lemma (Theorem 3.2), the key-lemma (Theorem 3.3), and the stability theorem (Theorem 3.1), we will upper bound the number of hypergraphs that are not in $\mathcal{F}'_n(\alpha)$.

2. $\mathcal{F}''_n(\alpha, \beta)$ will denote the set of those hypergraphs that are "dense everywhere" in the sense that whenever we take three disjoint subsets of vertices, say W_1, W_2, W_3, not all of them contained in X_H or Y_H, the number of hyperedges that run between them will be at least $d|W_1||W_2||W_3|$ for some positive constant $d > 0$. The proof of this fact is a straightforward counting argument. Moreover, we will also show that for every $H \in \mathcal{F}''_n(\alpha, \beta)$ the degrees of vertices inside their own partition class, that is X_H or Y_H, are "small".

3. The last class of hypergraphs will be $\mathcal{F}'''_n(\alpha, \beta)$. For members H of this class we demand that the joint link of every set of 3 vertices of any of the two partition classes X_H and Y_H must contain a K_4. Instead of proving $|\mathcal{F}'''_n(\alpha, \beta)| \geq (1 - o(1))|\mathcal{F}_n|$ directly, we will use this class of hypergraphs in order to estimate $\mathcal{F}_n$ inductively.

3 Tools

3.1 Stability theorem for the Fano plane. For a hypergraph $H = (V, E)$ and a set $U \subseteq V$ we write $E_H(U)$ or simply $E(U)$ for the edges completely contained in U, i.e., $E_H(U) = E \cap \binom{U}{3}$. We define the cardinality of $E_H(U)$ by $e_H(U)$ or simply $e(U)$. Similarly, for two disjoint subsets U and W we write

$$\begin{aligned} E(U, W) &= E(U \cup W) \setminus (E(U) \cup E(W)) \\ &= \{e \in E\colon\ e \subset U \dot{\cup} W,\ |e \cap U||e \cap W| \geq 1\} \end{aligned}$$

and $e(U, W) = |E(U, W)|$. For pairwise disjoint sets V_1, V_2, and V_3 denote by $E_H(V_1, V_2, V_3)$ the set of all hyperedges from H that intersect all three sets, further set $e_H(V_1, V_2, V_3) = |E_H(V_1, V_2, V_3)|$

The following stability result for Fano-free hypergraphs was proved by Keevash and Sudakov [12] and Füredi and Simonovits [9].

THEOREM 3.1. *For all $\alpha > 0$ there exists $\lambda > 0$ such that for every Fano-free hypergraph H on n vertices with at least $(\frac{1}{8} - \lambda)n^3$ edges there exists a partition $V(H) = X \dot{\cup} Y$ so that $e(X) + e(Y) < \alpha n^3$.*

3.2 Weak hypergraph regularity lemma. Another tool we use is the so-called *weak hypergraph regularity lemma.* This result is a straightforward extension of Szemerédi's regularity lemma [23] for graphs. Let $H = (V, E)$ be a hypergraph and let W_1, W_2, W_3 be pairwise disjoint non-empty subsets of V. We denote by $d_H(W_1, W_2, W_3) = d(W_1, W_2, W_3)$ the *density* of the 3-partite induced subhypergraph $H[W_1, W_2, W_3]$ of H, defined by

$$d_H(W_1, W_2, W_3) = \frac{e_H(W_1, W_2, W_3)}{|W_1||W_2||W_3|}\,.$$

We say the triple (V_1, V_2, V_3) of pairwise disjoint subsets $V_1, V_2, V_3 \subseteq V$ is (ε, d)*-regular*, for $\varepsilon > 0$ and $d \geq 0$, if

$$|d_H(W_1, W_2, W_3) - d| \leq \varepsilon$$

for all triples of subsets $W_1 \subseteq V_1, W_2 \subseteq V_2, W_3 \subseteq V_3$ satisfying $|W_i| \geq \varepsilon|V_i|$, $i = 1, 2, 3$. We say the triple (V_1, V_2, V_3) is ε*-regular* if it is (ε, d)-regular for some $d \geq 0$. An ε*-regular partition* of a set $V(H)$ has the following properties:

(i) $V = V_1 \dot{\cup} \ldots \dot{\cup} V_t$

(ii) $||V_i| - |V_j|| \leq 1$ for all $1 \leq i, j \leq t$,

(iii) for all but at most $\varepsilon\binom{t}{3}$ sets $\{i_1, i_2, i_3\} \subseteq [t]$, the triple $(V_{i_1}, V_{i_2}, V_{i_3})$ is ε-regular.

The weak hypergraph regularity lemma then states the following.

Theorem 3.2. (Weak regularity lemma) *For every integer $t_0 \geq 1$ and every $\varepsilon > 0$, there exist $T_0 = T_0(t_0, \varepsilon)$ and $n_0 = n_0(t_0, \varepsilon)$ so that for every hypergraph $H = (V, E)$ on $n \geq n_0$ vertices there exists an ε-regular partition $V = V_1 \dot\cup \dots \dot\cup V_t$ with $t_0 \leq t \leq T_0$.*

The proof of Theorem 3.2 follows the lines of the original proof of Szemerédi [23] (for details see e.g. [4, 8, 22]).

Typically, when studying the regular partition of a hypergraph, one defines a new hypergraph of bounded size with the vertex set being the partition classes and the edge set being ε-regular triples with sufficient density. The following definition makes this precise.

Definition 3.1. *For a hypergraph $H = (V, E)$ with an ε-regular partition $V(H) = V_1 \dot\cup \dots \dot\cup V_t$ of its vertex set and a real $\eta > 0$ let $H(\eta) = (V^*, E^*)$ be the cluster-hypergraph with vertex set $V^* = \{1, \dots, t\}$ and edge set E^*, where for $1 \leq i < j < k \leq t$ it is $\{i, j, k\} \in E^*$ if and only if the triple (V_i, V_j, V_k) is ε-regular and the edge-density satisfies $d_H(V_i, V_j, V_k) \geq \eta$.*

In [14] a counting lemma for linear hypergraphs in the context of the weak regularity lemma was proved. A hypergraph is said to be *linear* if no two of its hyperedges intersect in more than one vertex. Below we will give (essentially) the same proof that first appeared in [14] under slightly relaxed conditions. First we will need some more definitions.

Let L be a hypergraph on the vertex set $[\ell]$ and let H be an ℓ-partite hypergraph with vertex partition $V(H) = V_1 \dot\cup \dots \dot\cup V_\ell$. A copy L' of L in H, on the vertices $v_1 \in V_1, \dots, v_\ell \in V_\ell$, is said to be *partite-isomorphic* to L if $i \mapsto v_i$ defines a hypergraph homomorphism.

For the lower bound of the counting lemma it is sufficient to know that the involved triples are "dense enough" on every "small subset", instead of being (ε, d)-regular. More precisely, we say a triple (V_1, V_2, V_3) of pairwise disjoint subsets $V_1, V_2, V_3 \subseteq V$ is *one-sided (ε, d)-regular* for $\varepsilon > 0$ and $d \geq 0$ if

$$d_H(W_1, W_2, W_3) \geq d$$

for all triples of subsets $W_1 \subseteq V_1, W_2 \subseteq V_2, W_3 \subseteq V_3$ satisfying $|W_i| \geq \varepsilon |V_i|$, $i = 1, 2, 3$. Note also, that an (ε, d)-regular triple is one-sided $(\varepsilon, d - \varepsilon)$-regular.

Theorem 3.3. (Key-lemma) *For every $\ell \in \mathbb{N}$ and $d > 0$ there exist $\varepsilon = \varepsilon(\ell, d) > 0$ and a positive integer $m_0 = m_0(\ell, d)$ with the following property.*

If H is an ℓ-partite 3-uniform hypergraph with vertex classes $V_1, \dots, V_\ell$, such that $|V_1| = \dots = |V_\ell| \geq m_0$, and L is a linear hypergraph on ℓ vertices such that for every $e \in E(L)$ the triple $(V_i)_{i \in e}$ is one-sided (ε, d)-regular. Then H contains a copy of L.

Proof. We will prove the following stronger assertion, which gives a lower bound on the number of partite-isomorphic copies of L in H.

Proposition 3.1. *For every $\ell \in \mathbb{N}$ and γ, $d > 0$, there exist $\varepsilon = \varepsilon(\ell, \gamma, d) > 0$ and $m_0 = m_0(\ell, \gamma, d)$ so that the following holds.*

Let $L = ([\ell], E(L))$ be a linear hypergraph and let $H = (V_1 \dot\cup \dots \dot\cup V_\ell, E)$ be an ℓ-partite, 3-uniform hypergraph where $|V_1| = \dots = |V_\ell| \geq m_0$. If for all edges $e = \{i, j, k\} \in E(L)$, the triple (V_i, V_j, V_k) is one-sided (ε, d_e)-regular for some $d_e \geq d$, then the number of partite-isomorphic copies of L in H is at least

$$(1 - \gamma) \prod_{e \in E(L)} d_e \prod_{i \in [\ell]} |V_i| .$$

Let $\ell \in \mathbb{N}$ and $\gamma, d > 0$ be fixed. We shall prove, by induction on $|E(L)|$, that $\varepsilon = \gamma(d/2)^{|E(L)|}$ will suffice to estimate the lower bound on copies of L, provided m_0 is large enough. If $|E(L)| = 0$ or $|E(L)| = 1$, the result is trivial. It is also easy to see that the result holds whenever L consists of pairwise disjoint hyperedges, since then the number of partite-isomorphic copies of L in H is at least $\prod_{e \in E(L)} d_e \prod_{i \in [\ell]} |V_i|$.

For the general case let m_0 be large enough, so that we can apply the induction assumption on $|E(L)| - 1$ edges with precision $\gamma/2$ and d (and note that $\varepsilon = \gamma(d/2)^{|E(L)|} \leq (\gamma/2)(d/2)^{|E(L)|-1}$). All copies of various subhypergraphs discussed below are tacitly assumed to be partite-isomorphic.

Let L have $|E(L)| \geq 2$ edges and let $H = (V, E)$ be a 3-uniform hypergraph satisfying the assumptions of Proposition 3.1. Fix an edge $e_0 \in E(L)$ and let $L_- = ([\ell], E(L) \setminus \{e_0\})$ be the hypergraph obtained from L by removing the edge e_0. Moreover, for a copy L'_- of L_- in H, we denote by $e_0(L'_-)$ the unique triple of vertices which together with L'_- forms a copy of L in H. Furthermore, let $1_E \colon \binom{V}{3} \to \{0, 1\}$ be the indicator function of the edge set E of H. With this notation, a copy L'_- of L_- in H extends to a copy of L if, and only if, $1_E(e_0(L'_-)) = 1$. Consequently, summing over all copies L'_- of L_- in H, we obtain a formula on the number $|\{L \subseteq H\}|$ of copies of L in H:

$$\begin{aligned} |\{L \subseteq H\}| &= \sum_{L'_- \subseteq H} 1_E(e_0(L'_-)) \\ &= \sum_{L'_- \subseteq H} (d_{e_0} + 1_E(e_0(L'_-)) - d_{e_0}) \\ &= d_{e_0} |\{L_- \subseteq H\}| + \sum_{L'_- \subseteq H} (1_E(e_0(L'_-)) - d_{e_0}). \end{aligned}$$

Using the induction assumption for L_- we infer

$$|\{L \subseteq H\}| \geq (1-\tfrac{\gamma}{2}) \prod_{e\in E(L)} d_e \prod_{i\in[\ell]} |V_i| + \sum_{L'_- \subseteq H} (1_E(e_0(L'_-)) - d_{e_0})\,. \tag{3.5}$$

We bound the error term $\sum_{L'_- \subseteq H}(1_E(e_0(L'_-)) - d_{e_0})$ from below. For that, we will appeal to the one-sided regularity of $(V_i)_{i\in e_0}$. Let $L_* = L[[\ell]\setminus e_0]$ be the induced subhypergraph of L obtained by removing the vertices of e_0 and all edges of L intersecting e_0. For a copy L'_* of L_* in H, let $\mathrm{ext}(L'_*)$ be the set of triples $T \in \prod_{i\in e_0} V_i$ such that $V(L'_*)\dot\cup T$ spans a copy of L'_- in H. Hence,

$$\sum_{L'_-\subseteq H} (1_E(e_0(L'_-)) - d_{e_0}) = \sum_{L'_*\subseteq H} \sum_{T\in \mathrm{ext}(L'_*)} (1_E(T) - d_{e_0})$$

and, moreover, since L is a linear hypergraph, we have $|e_0 \cap e| \leq 1$ for every edge e of L_-. Hence, for every fixed copy L'_* of L_* in H and every $i \in e_0$, there exists a subset $W_i^{L'_*} \subseteq V_i$ such that

$$\mathrm{ext}(L'_*) = \prod_{i\in e_0} W_i^{L'_*}\,. \tag{3.6}$$

Indeed, for every $i \in e_0$, the set $W_i^{L'_*}$ consists of those vertices $v \in V_i$ with the property that $V(L'_*)\dot\cup\{v\}$ spans a copy of L induced on $V(L_*)\dot\cup\{i\}$ in H. Therefore, we can bound the error term as follows

$$\begin{aligned}\sum_{L'_-\subseteq H} &(1_E(e_0(L'_-)) - d_{e_0}) \\ &= \sum_{L'_*\subseteq H} \sum \Big\{ 1_E(T) - d_{e_0} \colon\ T \in \prod_{i\in e_0} W_i^{L'_*} \Big\} \\ &\geq -\sum_{L'_*\subseteq H} \varepsilon \prod_{i\in e_0} |V_i| \geq -\frac{\gamma}{2} \prod_{e\in E(L)} d_e \prod_{i\in[\ell]} |V_i|\,,\end{aligned}$$

where the one-sided (ε, d_{e_0})-regularity, the choice of ε, and (3.6) were used for the last two estimates. Now the proposition follows from (3.5), which implies immediately Theorem 3.3.

4 Proof of the main result

4.1 Almost bipartite hypergraphs. Our first step for the proof of Theorem 1.1 is an estimate on the number of those hypergraphs $H \in \mathcal{F}_n$ which are far from being bipartite, namely for which $e(X_H)+e(Y_H) \geq \alpha n^3$ for some $\alpha > 0$ to be specified later. Thus, the remaining hypergraphs will admit a "nice" partition. Moreover, most of these remaining hypergraphs H will have partition classes of nearly same size.

Definition 4.1.

$$\begin{aligned}\mathcal{F}'_n(\alpha) = \big\{ H \in \mathcal{F}_n \colon\ & e(X_H) + e(Y_H) < \alpha n^3 \\ & \textit{and } |X_H|, |Y_H| < n/2 + 2\sqrt{h(6\alpha)}n \big\}\,.\end{aligned}$$

Lemma 4.1. *For every $\alpha \in (0, \frac{1}{12})$ there exist $c' > 0$ and an integer n'_0 such that for all $n \geq n'_0$*

$$|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| < 2^{e(B_n) - c'n^3}\,.$$

Proof. The proof of Lemma 4.1 combines the weak hypergraph regularity lemma with the stability theorem for Fano-free hypergraphs applied to the reduced hypergraph.

Let $\lambda = \lambda(\alpha/2)$ be given by Theorem 3.1. We may assume $\lambda < 16h(6\alpha)$. We set

$$c' = \frac{\lambda}{17}\,.$$

We choose η such that $\lambda > 16h(6\eta)$ and $\eta \leq \alpha/2$. Finally let $\varepsilon = \varepsilon(\eta/2) \leq \eta/2$ be given by Theorem 3.3. Set $t_0 = 1/\varepsilon$ and let n be sufficiently large, in particular, set $n'_0 \gg \max\{T_0, n_0\}$, where T_0 and n_0 are given by the weak regularity lemma, Theorem 3.2. For the main steps of the proof it is sufficient to keep in mind that

$$0 < \varepsilon = t_0^{-1} \leq \eta \ll \lambda \ll \alpha.$$

We may assume in the following that t divides n, and thus $|V_i| = n/t, i = 1, \ldots, t$, as this does not affect our asymptotic considerations.

We will upper bound $|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)|$ in two steps. In the first step we bound the number of hypergraphs H that have $e(X_H) + e(Y_H) \geq \alpha n^3$. In the second step we show that most of the hypergraphs H with $e(X_H) + e(Y_H) < \alpha n^3$ will have nearly equal sizes:

$$\max\{|X_H|, |Y_H|\} < \frac{n}{2} + 2\sqrt{h(6\alpha)}n\,.$$

Step 1. Consider a hypergraph $H \in \mathcal{F}_n$ satisfying $e(X_H) + e(Y_H) \geq \alpha n^3$. We apply the weak regularity lemma, Theorem 3.2, with parameters ε and t_0. Firstly, we estimate the number of hyperedges, which are contained in the "uncontrolled" part of the regular partition:

- the number of hyperedges intersecting at most two of the clusters is at most
$$t\binom{n/t}{2} n < \frac{1}{2t} n^3,$$
- the number of hyperedges contained in irregular triples is at most
$$\varepsilon \binom{t}{3} \left(\frac{n}{t}\right)^3 < \frac{\varepsilon}{6} n^3,$$

- the number of hyperedges that are contained in ε-regular triples of density less than η is at most

$$\eta\left(\frac{n}{t}\right)^3\binom{t}{3}<\frac{\eta}{6}n^3.$$

Thus, the number of discarded edges is less than ηn^3.

Secondly, consider the resulting cluster-hypergraph $H(\eta)$. It must be Fano-free as otherwise Theorem 3.3 would imply that H also contains a copy of the hypergraph of the Fano plane. We assumed that $e(X_H)+e(Y_H)\geq\alpha n^3$, so we can bound the number of hyperedges in $H(\eta)$ from above by $(1-\lambda)t^3/8$. Otherwise, Theorem 3.1 would give us a partition of $V_1,\dots,V_t$ into disjoint sets X and Y with $e_{H(\eta)}(X)+e_{H(\eta)}(Y)<\alpha t^3/2$. Defining a partition of $V(H)$ into the following two sets

$$A=\bigcup_{U\in X}U\quad\text{and}\quad B=\bigcup_{W\in Y}W,$$

with

$$e_H(A)+e_H(B)<\eta n^3+\frac{\alpha}{2}t^3\left(\frac{n}{t}\right)^3\leq\alpha n^3\,,$$

which yields a contradiction to $e(X_H)+e(Y_H)\geq\alpha n^3$.

Now we are able to bound the number of hypergraphs $H\in\mathcal{F}_n$ with $e(X_H)+e(Y_H)\geq\alpha n^3$ from above by calculating the total possible number of ε-regular partitions together with all possible cluster-hypergraphs associated with them and all possible hypergraphs that could give rise to such a particular cluster-hypergraph. This way we get

$$\begin{aligned}
&\left|\mathcal{F}_n\setminus\{H\in\mathcal{F}_n\colon\, e(X_H)+e(Y_H)\geq\alpha n^3\}\right|\\
&\quad\leq\sum_{t=t_0}^{T_0}t^n\cdot 2^{\binom{t}{3}}\cdot 2^{(1-\lambda)\frac{t^3}{8}\left(\frac{n}{t}\right)^3}\cdot\left(\sum_{j=0}^{\eta n^3-1}\binom{\binom{n}{3}}{j}\right)\\
&\quad\leq T_0^{n+1}\cdot 2^{\binom{T_0}{3}}\cdot 2^{(1-\lambda)n^3/8}\cdot\binom{\binom{n}{3}}{\eta n^3}\\
&\quad\leq 2^{(n+1)\log T_0+\binom{T_0}{3}+n^3/8-\lambda n^3/8+h(6\eta)n^3/6}\\
&\quad< 2^{n^3/8-\lambda n^3/16}\,,
\end{aligned}$$

for sufficiently large n, due to the choice of λ.

Step 2**.** We now estimate the number of those hypergraphs H, for which $e(X_H)+e(Y_H)<\alpha n^3$, but $\max\{|X_H|,|Y_H|\}\geq n/2+2\sqrt{h(6\alpha)}n$. First we upper bound $e(X_H,Y_H)$ for such a hypergraph H by

$$\begin{aligned}
e(X_H,Y_H)&\leq|X_H|\binom{|Y_H|}{2}+|Y_H|\binom{|X_H|}{2}\\
&<\frac{n}{2}|X_H||Y_H|<\frac{n^3}{8}-2h(6\alpha)n^3.
\end{aligned}$$

Note that there are at most 2^n possible partitions, and since less than αn^3 hyperedges are completely contained in X_H and Y_H, those hyperedges can be chosen in at most

$$\sum_{i=0}^{\alpha n^3-1}\binom{\binom{n}{3}}{i}\leq\binom{\binom{n}{3}}{\alpha n^3}$$

ways. Finally, as we assumed that our partitions are "unbalanced" we estimate the number of possible choices of hyperedges between X_H and Y_H by $2^{n^3/8-2h(6\alpha)n^3}$. Altogether we get, that there are at most

$$\begin{aligned}
2^n\cdot\binom{\binom{n}{3}}{\alpha n^3}\cdot 2^{n^3/8-2h(6\alpha)n^3}&\leq 2^{n+h(6\alpha)n^3/6+n^3/8-2h(6\alpha)n^3}\\
&\leq 2^{n^3/8-h(6\alpha)n^3}
\end{aligned}$$

hypergraphs with $e(X_H)+e(Y_H)<\alpha n^3$ and

$$\max\{|X_H|,|Y_H|\}\geq\frac{n}{2}+2\sqrt{h(6\alpha)}n.$$

Combining Step 1 and 2 we obtain

$$\begin{aligned}
|\mathcal{F}_n\setminus\mathcal{F}'_n(\alpha)|&\leq 2^{n^3/8-\lambda n^3/16}+2^{n^3/8-h(6\alpha)n^3}\\
&<2^{n^3/8-\lambda n^3/16+1},
\end{aligned}$$

since $h(6\alpha)>\lambda/16$. Due to $n^3/8-e(B_n)\leq n^2/4+O(n)$ and the choice of $c'=\lambda/17$, the lemma follows for sufficiently large n.

4.2 Everywhere dense hypergraphs.

Now we know that almost all Fano-free hypergraphs are nearly bipartite and admit a partition into almost equal classes. We want to restrict our consideration to those hypergraphs that in addition have no sparse "bipartite" spots. Our motivation comes from random bipartite hypergraphs. Namely, we would expect $\frac{1}{2}N\binom{N}{2}$ edges having exactly one end in the first class and two in the second in $\mathcal{H}(N,N,1/2)$, the random bipartite hypergraph with both classes of size N where each edge exists with probability $1/2$. If we would take any three disjoint subsets not all of them in one partition class, each of size, say m, then we would expect there $m^3/2$ hyperedges. Deviations from this value, say only $m^3/4$ edges instead of $m^3/2$, would only happen with very small probability. The following lemma, Lemma 4.2, states that this intuition holds for almost all hypergraphs in $\mathcal{F}'_n(\alpha)$.

Definition 4.2. *Let $\mathcal{F}''_n(\alpha,\beta)$ denote the family of those hypergraphs $H\in\mathcal{F}'_n(\alpha)$, for which the following condition holds.*

For any pairwise disjoint sets $W_1\subset X_H$, $W_2\subset Y_H$ and $W_3\subset Z_H$, where $Z_H\in\{X_H,Y_H\}$, with $|W_i|\geq\beta n$ for $i=1,2,3$ we have

$$e_H(W_1,W_2,W_3)\geq\frac{1}{4}|W_1||W_2||W_3|.$$

The following lemma shows that most hypergraphs in $\mathcal{F}'_n(\alpha)$ belong to $\mathcal{F}''_n(\alpha,\beta)$.

LEMMA 4.2. *For every $\beta > 0$ there exist α, $c'' > 0$ and an integer n''_0 such that for all $n \geq n''_0$*

$$|\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)| < 2^{e(B_n)-c''n^3}.$$

Proof. Choose $\alpha > 0$ such that

$$\beta^3 (1 - h(1/4)) \geq h(6\alpha)/3\,,$$

set $c'' = h(6\alpha)/7$, and let n''_0 be sufficiently large.

Below we bound the number of hypergraphs H with $H \in \mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)$. There are at most 2^n partitions $X_H \dot\cup Y_H = [n]$ of the vertex set and we can choose the edges lying completely within X_H and Y_H in at most

$$\sum_{j=0}^{\alpha n^3 - 1} \binom{\binom{n}{3}}{j} \leq \binom{\binom{n}{3}}{\alpha n^3}$$

possible ways. A simple averaging argument shows that it suffices to consider sets W_i with $|W_i| = \beta n$ and there are at most

$$2\binom{n/2 + 2\sqrt{h(6\alpha)}n}{\beta n}^3 \sum_{0 \leq i < \beta^3 n^3/4} \binom{\beta^3 n^3}{i} < 2^{3n+1}\binom{\beta^3 n^3}{\beta^3 n^3/4}$$

ways to select W_1, W_2, W_3 and the hyperedges in $e_H(W_1, W_2, W_3)$. Finally, there are at most

$$2^{e(B_n)-\beta^3 n^3}$$

ways to choose the remaining edges of H. Multiplying everything together, we obtain

$$\begin{aligned} &|\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)| \\ &\qquad \leq 2^{4n+1}\binom{\binom{n}{3}}{\alpha n^3}\binom{\beta^3 n^3}{\beta^3 n^3/4} 2^{e(B_n)-\beta^3 n^3} \\ &\qquad \leq 2^{4n+1+h(6\alpha)n^3/6 + h(1/4)\beta^3 n^3 + e(B_n) - \beta^3 n^3} \\ &\qquad \leq 2^{e(B_n)-c''n^3}, \end{aligned}$$

for sufficiently large n.

We will also need the following useful observation, that for suitably chosen α and β, every $H \in \mathcal{F}''_n(\alpha,\beta)$ has no vertex of high degree in its own partition class.

LEMMA 4.3. *For every $\gamma > 0$ there exist α, $\beta > 0$ and an integer n_0, such that for every $H \in \mathcal{F}''_n(\alpha,\beta)$ we have*

$$\max\{\Delta(H[X_H]), \Delta(H[Y_H])\} < \gamma n^2,$$

for all $n \geq n_0$.

For the proof of Lemma 4.3, we will use a simple consequence of the regularity lemma for graphs. For that we need the definition of an ε-regular pair in graphs.

For a bipartite graph $G = (V_1, V_2, E)$ we define the density of G as $d(V_1, V_2) = \frac{|E|}{|V_1||V_2|}$. The density of a subpair (U_1, U_2), where $U_i \subseteq V_i$ for $i = 1, 2$ is defined as $d(U_1, U_2) = \frac{e(U_1,U_2)}{|U_1||U_2|}$. Here $e(U_1, U_2)$ denotes the number of edges with one vertex in U_1 and one vertex in U_2. We say, G, or simply (V_1, V_2), is ε-regular if all subsets $U_i \subseteq V_i$, with $|U_i| \geq \varepsilon|V_i|$, $i = 1, 2$, satisfy

$$|d(V_1, V_2) - d(U_1, U_2)| \leq \varepsilon.$$

Now we state the theorem, which one could easily obtain from the usual regularity lemma [23] (for a better dependency of the constants one also could also use [19, Theorem 1.1]).

THEOREM 4.1. *For every $\gamma > 0$ and $\varepsilon \in (0, \gamma/3)$ there exist T_0, N_0 such that the following holds.*

For all vertex disjoint graphs G_X and G_Y on $|V(G_X)| + |V(G_Y)| = n \geq N_0$ vertices with $e(G_X)$, $e(G_Y) \geq \gamma n^2$ there exist $t \leq T_0$ and pairwise disjoint sets $X_1, X_2, Y_1, Y_2, Y_3, Y_4$, each of size n/t, and $X_1, X_2 \subset V(G_X)$ and $Y_i \subset V(G_Y), i \in [4]$, so that $G_X[X_1, X_2]$, $G_Y[Y_1, Y_2]$ and $G_Y[Y_3, Y_4]$ are ε-regular with density at least $\gamma/3$.

With this result at hand we can give the proof of Lemma 4.3.

Proof. Let $\varepsilon = \min\{\frac{1}{2}\varepsilon(\gamma/6), \gamma/6\}$, where $\varepsilon(\gamma/6)$ is given by Theorem 3.3. Set $\beta = \varepsilon/(2T_0)$, with $T_0 = T_0(\gamma, \varepsilon)$ given by Theorem 4.1. Let $\alpha = \alpha(\beta)$ be given by Lemma 4.2 and let n_0 be sufficiently large. Again, it is sufficient to keep in mind:

$$0 \ll \alpha \ll \beta \ll T_0^{-1} \ll \varepsilon \ll \gamma.$$

We prove our lemma by contradiction. More precisely, we will assume that there exists a hypergraph $H \in \mathcal{F}''_n(\alpha,\beta)$ with $\max\{\Delta(H[X_H]), \Delta(H[Y_H])\} \geq \gamma n^2$, and we will show that H contains a copy of the hypergraph of the Fano plane.

Without loss of generality assume that there exists $H \in \mathcal{F}''_n(\alpha,\beta)$ and a vertex $x \in X_H$ with $\deg_{H[X]}(x) \geq \gamma n^2$. Thus, $e(L_Y(x)) \geq e(L_X(x)) \geq \gamma n^2$, as otherwise this violates the minimality condition of the partition $X_H \dot\cup Y_H = V(H)$. We consider the graphs

$$G_X = L_X(x) = (X_H \setminus \{x\}, E_x \cap \tbinom{X}{2})$$

and

$$G_Y = L_Y(x) = (Y_H, E_x \cap \tbinom{Y}{2})$$

and apply Theorem 4.1 to $G_X \dot\cup G_Y$. This way we obtain ε-regular pairs $(X_1, X_2) \subset G_X$ and $(Y_1, Y_2), (Y_3, Y_4) \subset G_Y$, with $|X_i| = |Y_j| \geq (n-1)/T_0$ and $i \in [2]$, $j \in [4]$, each of density at least $\gamma/3$.

Consider the following 7-partite subhypergraph L with vertex classes $\{x\}$, X_1, X_2, Y_1, Y_2, Y_3, and Y_4. Denote L^x to be the hypergraph obtained from L by blowing up its first vertex class $\{x\}$ to the size of X_1 (all other partition classes are equal), and denote this blown-up class by $\tilde{X}$. More precisely, $L^x = (W^x, E^x)$, where

$$W^x = \tilde{X} \dot\cup X_1 \dot\cup X_2 \dot\cup Y_1 \dot\cup Y_2 \dot\cup Y_3 \dot\cup Y_4$$

and

$$\begin{aligned} &\{a,b,c\} \in E^x \\ &\qquad \Leftrightarrow \begin{cases} \{a,b,c\} \in E(L)\,, & \text{if } \{a,b,c\} \cap \tilde{X} = \emptyset\,, \\ \{x,b,c\} \in E(L)\,, & \text{if } a \in \tilde{X} \text{ and } b,\, c \notin \tilde{X}\,. \end{cases} \end{aligned}$$

Note, that L contains a copy of the hypergraph of the Fano plane if, and only if, L^x contains one. Now we apply Theorem 3.3 to L^x, as L^x contains now 7 one-sided $(\varepsilon, \gamma/6)$-regular triples and these triples form a Fano plane. This is true since the triples $(\tilde{X}, X_1, X_2)$, $(\tilde{X}, Y_1, Y_2)$ and $(\tilde{X}, Y_3, Y_4)$ "inherit" the ε-regularity from the ε-regular pairs of (X_1, X_2), (Y_1, Y_2), and (Y_3, Y_4), while the other triples are one-sided $(\varepsilon, \gamma/6)$-regular due to the choice of β and the properties of $H \in \mathcal{F}''_n(\alpha, \beta)$. This yields a contradiction and Lemma 4.3 follows.

4.3 Proof of Theorem 1.1. We will need the following consequence from Janson's inequality [11].

Lemma 4.4. *The probability that the binomial random graph $\mathcal{G}(m, \frac{1}{8})$ with $m \geq 253$ vertices and edge probability $1/8$ does not contain a copy of K_4 is bounded from above by* $\exp(-2^{-11}m^2)$.

Proof. Let $t_1, \ldots, t_{\binom{m}{2}}$ be jointly independent Boolean random variables representing the edges of $\mathcal{G}(m, \frac{1}{8})$. Let the collection of those 6-sets of the set $\binom{[m]}{2}$ that correspond to K_4's be denoted by $\mathcal{A}$. Therefore, the random variable $X = \sum_{A \in \mathcal{A}} \prod_{j \in A} t_j$ counts the number of K_4's in $\mathcal{G}(m, \frac{1}{8})$. Applying Janson's inequality [11] we bound the probability of the event $X = 0$ by

$$\Pr(X = 0) \leq \exp\left(-\frac{\mathbb{E}(X)^2}{2\Delta}\right),$$

where $\mathbb{E}(X) = (\frac{1}{8})^6 \binom{m}{4}$ is the expectation and

$$\begin{aligned} \Delta &= \sum_{A, B \in \mathcal{A} : A \cap B \neq \emptyset} \mathbb{E}\left(\prod_{j \in A \cup B} t_j\right) \\ &\leq \mathbb{E}(X) \sup_{A \in \mathcal{A}} \sum_{B \in \mathcal{A} : A \cap B \neq \emptyset} \mathbb{E}\left(\prod t_j\right) \\ &\leq \mathbb{E}(X) \left(\binom{m}{2} \cdot 6 \cdot \left(\frac{1}{8}\right)^5 + m \cdot 4 \cdot \left(\frac{1}{8}\right)^3\right) \\ &\leq \mathbb{E}(X) \left(\frac{3m^2 + 253m}{2^{15}}\right) \leq \mathbb{E}(X) \frac{m^2}{2^{13}}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \Pr(X = 0) \leq \exp\left(-\frac{2^{12}\mathbb{E}(X)}{m^2}\right) &= \exp\left(-\frac{\binom{m}{4}}{2^6 m^2}\right) \\ &\leq \exp\left(-2^{-11} m^2\right). \end{aligned}$$

We finally define the last subclass of Fano-free hypergraphs.

Definition 4.3. *Let $\mathcal{F}'''_n(\alpha, \beta)$ denote the family of those hypergraphs $H \in \mathcal{F}''_n(\alpha, \beta)$, for which the following condition holds.*

For all triples $z_1, z_2, z_3 \in Z$ of vertices with $Z \in \{X_H, Y_H\}$ we have $L_Q(z_1) \cap L_Q(z_2) \cap L_Q(z_3) \supseteq K_4$, where $\{Q, Z\} = \{X_H, Y_H\}$. In other words, we require that the common link of any triple from X_H or Y_H contains a copy of K_4 in the other vertex class.

It follows directly from the definition, that every $H \in \mathcal{F}'''_n(\alpha, \beta)$ is bipartite, i.e., $\mathcal{F}'''_n(\alpha, \beta) \subseteq \mathcal{B}_n$. Otherwise, any hyperedge e, say in X_H, together with the K_4 in Y_H, which lies in the common link of the vertices of e would span a copy of the hypergraph of the Fano plane. We also note that we could replace K_4 in the definition of $\mathcal{F}'''_n(\alpha, \beta)$ by a 1-factor of K_4 that is created by the union of the links of any three vertices.

We are now going to prove our main theorem, Theorem 1.1, by induction. The proof is based on the lemmas from the previous sections.

Proof. (Theorem 1.1) We set

$$\vartheta = 2^{-17} \quad \text{and} \quad c = \frac{\vartheta}{3} \tag{4.7}$$

and choose $\gamma > 0$ such that

$$3h(2\gamma) < \vartheta\,. \tag{4.8}$$

Let α and $\beta > 0$ be given by Lemma 4.3. We may also assume that

$$3\sqrt{h(6\alpha)} + 6h(6\alpha) < \vartheta/2\,, \tag{4.9}$$

as choosing α smaller we will only have to eventually increase n_0. Again, it is sufficient to keep in mind that

$$0 < \alpha \ll \beta \ll \gamma \ll \vartheta = 2^{-17}\,.$$

Let c' and c'' be given by Lemma 4.1 and Lemma 4.2. Finally, let

$$n_0 \geq \max\{2^{20}, 14/\vartheta, 1/c', 1/c''\}$$

be sufficiently large so that Lemma 4.1, Lemma 4.2 and Lemma 4.3 hold.

By induction on n we will verify the following statement, which implies Theorem 1.1

$$|\mathcal{F}_n| \leq |\mathcal{B}_n|(1 + 2^{n_0^2 n - cn^2})\,. \tag{4.10}$$

For $n \leq n_0$ the statement is trivial, since then $2^{n_0^2 n - cn^2}$ is bigger than the number of all hypergraphs on n vertices and we now proceed with the induction step and verify (4.10) for $n > n_0$.

The proof is based on the following chains

$$\mathcal{F}_n \supseteq \mathcal{F}'_n(\alpha) \supseteq \mathcal{F}''_n(\alpha,\beta) \supseteq \mathcal{F}'''_n(\alpha,\beta),$$

and

$$\mathcal{F}_n \supseteq \mathcal{B}_n \supseteq \mathcal{F}'''_n(\alpha,\beta).$$

Consequently,

$$\begin{aligned}|\mathcal{F}_n| \leq |\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| + |\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)|& \\ + |\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)| + |\mathcal{B}_n|\,.&\end{aligned}$$

Lemma 4.1 bounds $|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)|$ and Lemma 4.2 bounds $|\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)|$. Hence, it remains to estimate $|\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)|$.

For that we will use the induction assumption and proceed as follows. Let $H \in \mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)$ and $X_H \dot\cup Y_H$ be its minimal partition. Consider a subset $S \in \binom{X_H}{3} \dot\cup \binom{Y_H}{3}$. Deleting S from H, we obtain a Fano-free hypergraph H' on $n-3$ vertices, where $V(H') = [n] \setminus S$. On the other hand, for every $H \in \mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)$ there exists a hypergraph $H' \in \mathcal{F}_{n-3}$ such that H can be reconstructed from H' in the following way. For $H' \in \mathcal{F}_{n-3}$ we choose a set S of 3 vertices, which we "connect" in an appropriate manner, so that the resulting hypergraph is in $\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)$.

We can choose the set S, the partition of H' and the set which contains S in at most

$$\binom{n}{3} 2^{n-3}$$

ways. Since $H \in \mathcal{F}''_n(\alpha,\beta)$ and Lemma 4.3 holds, we also know that every vertex in S has at most γn^2 neighbors in its own partition class. This again bounds the number of ways for choosing these hyperedges by

$$\left(\sum_{j=0}^{\gamma n^2 - 1} \binom{\binom{n}{2}}{j}\right)^3 \leq \binom{\binom{n}{2}}{\gamma n^2}^3.$$

For every vertex in S we have at most $2^{n^2/4}$ possibilities for choosing edges with one more end in the same partition as S and the other end in the other partition class, this gives us at most

$$2^{3n^2/4}$$

ways to choose that type of hyperedges. The last estimate concerns the number of ways we can connect our triple S to the other partition class, say Y, without creating any single copy of K_4, which is contained in the joint link of the vertices from S. Here we use Lemma 4.4. For every vertex v in S, say $S \subset X$, we can choose its link graph $L_Y(v)$ in at most $2^{\binom{|Y|}{2}}$ ways. However, since the joint link of three vertices in S contains no K_4, we infer from Lemma 4.4, that there are at most

$$2^{3\binom{|Y|}{2}} \exp(-2^{-11}|Y|^2) < 2^{3\binom{|Y|}{2} - |Y|^2/2^{11}}$$

ways to choose all three link graphs such that no K_4 appears in the joint link.

Combining the above estimates and

$$n/4 \leq |Y| \leq n/2 + 2\sqrt{h(6\alpha)}n\,,$$

we obtain

$$\begin{aligned}
&|\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)| \\
&\quad\leq \binom{n}{3} 2^{n-3} \cdot \binom{\binom{n}{2}}{\gamma n^2}^3 2^{3n^2/4} 2^{3\binom{|Y|}{2} - |Y|^2/2^{11}} |\mathcal{F}_{n-3}| \\
&\quad\overset{(4.9)}{\leq} 2^{3\log n + n + 3h(2\gamma)n^2/2 + 9n^2/8 + \vartheta n^2/2 - n^2/2^{15}} |\mathcal{F}_{n-3}| \\
&\quad\overset{(4.8)}{\leq} 2^{\delta_1(B_{n-2}) + \delta_1(B_{n-1}) + \delta_1(B_n) + \vartheta n^2 - n^2/2^{16}} |\mathcal{F}_{n-3}| \\
&\quad\overset{(4.7)}{=} 2^{\delta_1(B_{n-2}) + \delta_1(B_{n-1}) + \delta_1(B_n) - \vartheta n^2} |\mathcal{F}_{n-3}| \\
&\quad\overset{(4.10)}{\leq} 2^{-\vartheta n^2} \cdot 2^{n-3} \cdot |\mathcal{B}_n| \cdot (1 + 2^{n_0^2(n-3) - c(n-3)^2}) \\
&\quad\leq |\mathcal{B}_n|(2^{-\vartheta n^2/2} + 2^{n_0^2 n - cn^2 - \vartheta n^2/2})\,,
\end{aligned}$$

where we used (2.4) for the penultimate inequality and $n_0 \geq 14/\vartheta$ and $c \leq 1$ for the last inequality. Finally, we derive the required upper bound on $|\mathcal{F}_n|$

$$\begin{aligned}
|\mathcal{F}_n| &\leq |\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| + |\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)| \\
&\qquad + |\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)| + |\mathcal{B}_n| \\
&\leq 2^{e(B_n) - c'n^3} + 2^{e(B_n) - c''n^3} \\
&\qquad + |\mathcal{B}_n|(2^{-\vartheta n^2/2} + 2^{n_0^2 n - cn^2 - \vartheta n^2/2}) + |\mathcal{B}_n| \\
&\leq |\mathcal{B}_n|(1 + 4 \cdot 2^{n_0^2 n - \vartheta n^2/2}) \leq |\mathcal{B}_n|(1 + 2^{n_0^2 n - cn^2}).
\end{aligned}$$

5 Concluding remarks

The structural results leading to the proof of Theorem 1.1 might be useful for the average case analysis of the running time of coloring algorithms on restricted hypergraph classes. For example, after proving a similar result to ours for K_{p+1}-free graphs ($p \geq 2$), Prömel and Steger [21] developed an algorithm which colors a randomly chosen K_{p+1}-free graph G on n vertices with $\chi(G)$ colors in $O(n^2)$ expected time, regardless of the value of $\chi(G)$. Together with the fact that almost all K_{p+1}-free graphs are p-colorable [16], this result in turn extended the work of Dyer and Frieze [5] who developed an algorithm which colored every p-colorable graph on n vertices properly (with p colors) in $O(n^2)$ expected time (see also [24]). We intend to come back to this problem for 3-uniform hypergraphs in $\mathcal{F}_n$ in the near future.

We believe that with essentially the same methods one could prove results similar to Theorem 1.1 for other linear k-uniform hypergraphs (instead of the hypergraph of the Fano plane), which contain at least one color-critical hyperedge and which admit a stability result similar to Theorem 3.1.

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