Ramsey numbers for bipartite graphs with small bandwidth

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Abstract

We determine asymptotically the two color Ramsey numbers for bipartite graphs with small bandwidth and constant maximum degree and the three color Ramsey numbers for *balanced* bipartite graphs with small bandwidth and constant maximum degree. In particular, we determine asymptotically the two and three color Ramsey numbers for grid graphs.

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1 Introduction and Results

For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one H_i $(1 \leq i \leq r)$ contains a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical theorem. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. Determining $R(G_1, G_2, \ldots, G_r)$ for general graphs appears to be a difficult problem (see e.g. [13]). For

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$r = 2$, a well-known theorem of Gerencsér and Gyárfás [7] states that $R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor$, where P_n denotes the path with $n \geq 2$ vertices. In [9] more general trees were considered. For a tree T, we write t_1 and t_2, $t_2 \geq t_1$, for the sizes of the vertex classes of T as a bipartite graph. Note that if $2t_1 \geq t_2$, then $R(T, T) \geq 2t_1 + t_2 - 1$, since the following edge-coloring of $K_{2t_1+t_2-2}$ has no monochromatic copy of T. Partition the vertices into two classes V_1 and V_2 such that $|V_1| = t_1 - 1$ and $|V_2| = t_1+t_2-1$, then use color "red" for all edges inside the classes and use color "blue" for all edges between the classes. On the other hand, if $2t_1 < t_2$, a similar edge-coloring of K_{2t_2-2} with two classes both of size $t_2 - 1$ shows that $R(T, T) \geq 2t_2$. Thus, $R(T, T) \geq \max\{2t_1 + t_2, 2t_2\} - 1$. Haxell, Łuczak and Tingley proved in [9] that for a tree T with maximum degree $o(t_2)$, this lower bound is the asymptotically correct value of $R(T, T)$.
We try to extend this to bipartite graphs with small bandwidth (although with a more restrictive maximum degree condition). A graph is said to have *bandwidth* at most b, if there exists a labelling of the vertices by numbers $1, \ldots, n$ such that for every edge $\{i, j\}$ of the graph we have $|i - j| \leq b$. We will focus on the following class of bipartite graphs.

Definition 1.1. *A bipartite graph H is called a (β, Δ)-graph if it has bandwidth at most $\beta|V(H)|$ and maximum degree at most Δ. Furthermore, we say that H is a balanced (β, Δ)-graph if it has a legal 2-coloring $\chi : V(H) \to [2]$ such that $1 - \beta \leq |\chi^{-1}(1)|/|\chi^{-1}(2)| \leq 1 + \beta$.*

For example, all bounded degree planar graphs G are $(\beta, \Delta(G))$-graphs for any $\beta > 0$ [3]. Our first theorem is an analogue of the result in [9] for (β, Δ)-graphs.

Theorem 1.2. *For every $\gamma > 0$ and natural number Δ, there exist a constant $\beta > 0$ and natural number n_0 such that for every (β, Δ)-graph H on $n \geq n_0$ vertices with a legal 2-coloring $\chi : V(H) \to [2]$ where $t_1 = |\chi^{-1}(1)|$ and $t_2 = |\chi^{-1}(2)|$, $t_1 \leq t_2$, we have $R(H, H) \leq (1 + \gamma) \max\{2t_1 + t_2, 2t_2\}$.*

For more recent results on the Ramsey number of graphs of higher chromatic number and sublinear bandwidth, we refer the reader to the recent paper by Allen, Brightwell and Skokan [1].
For $r \geq 3$ less is known about Ramsey numbers. Proving a conjecture of Faudree and Schelp [5], it was shown in [8] that for sufficiently large n $R(P_n, P_n, P_n) = 2n - 1$, for odd n and $R(P_n, P_n, P_n) = 2n - 2$, for even n. Asymptotically this was also proved independently by Figaj and Łuczak [6]. Benevides and Skokan proved [2] that $R(C_n, C_n, C_n) = 2n$ for sufficiently large even n. In our second theorem we extend these results (asymptotically) to balanced (β, Δ)-graphs.

Theorem 1.3. *For every $\gamma > 0$ and natural number Δ, there exist a constant $\beta > 0$ and natural number n_0 such that for every balanced (β, Δ)-graph H on $n \geq n_0$ vertices we have $R(H, H, H) \leq (2 + \gamma)n$.*

In particular, Theorems 1.2 and 1.3 determine asymptotically the two and three color Ramsey numbers for grid graphs.
We conclude this section with a few words about the proof method for our main theorems. The proof of Theorem 1.2 combines ideas from [9] and [4], while the proof of Theorem 1.3 follows a similar approach as in [6], again, together with the result in [4]. Since the strategies for both theorems are close to each other, we focus on the proof of Theorem 1.3, for which we present an outline in the next section. Details can be found in [12].

2 Sketch of the proof of Theorem 1.3

Here we will sketch the main ideas of our proof. The proof relies on the regularity method for graphs and we refer the reader to the survey [10] for related notation and definitions.

The first part of the proof follows the same pattern as the proof by Figaj and Łuczak [6] for the case where H is a path. Namely, we apply a multicolored variant of Szemerédi's Regularity Lemma [14] to the 3-colored complete graph K_N with $N = (2+\gamma)n$ and get a partition with a very dense reduced graph. The edges of the reduced graph inherit the majority color of the respective pair. Applying Lemma 8 from [6] gives us a monochromatic tree T in the reduced graph that contains a matching M covering almost half of the vertices.
Switching back from the reduced graph to the colored complete graph, we denote by G_T the subgraph of K_N whose vertices are contained in the clusters represented by the vertices of T and whose edges run inside the pairs represented by the edges of T and have the same color as the edges of T. Thus G_T is a monochromatic subgraph of K_N whose regular pairs are arranged in a structure mirroring that of T and all have density at least $1/3$. Finally, we localize almost spanning super-regular subgraphs in the pairs in G_T represented by edges in M and denote the subgraph formed by the union of these pairs by $G_M \subset G_T$.

To understand the motivation for the second part, recall that our overall goal is to embed H into G_T. Notice that G_M has in fact enough vertices to accomodate all of H. Indeed, most of the vertices of H will be mapped to G_M, and we will only have to use parts from $G_T \setminus G_M$ because we may need to connect the various parts of H embedded into G_M. Let us explain this more precisely by assuming for the moment that H is just a path. Let m be an integer which is just a bit smaller than the size of the clusters in G_T (that we assume to be all of the same size). Applying the Blow-up Lemma by Komlós, Sárközy and Szemerédi [11], we then embed the first m vertices from each color class of H into the super-regular pair represented by the first matching edge in M.
To be able to 'reach' the next super-regular pair in G_M, where we can embed the next $m + m$ vertices of H, we need to make use of the fact

that the vertices representing these two pairs are connected by a path in T. This path translates into a sequence of regular pairs in G_T, into each of which we embed an intermediate edge of H, thereby 'walking' towards the next super-regular pair in G_M. In this way, we only use few edges of the regular pairs in $G_T \setminus G_M$, thus keeping them regular all the way through, and leaving a bit of space in the super-regular pairs in G_M, in case we need to walk through them later again.

The task for the second part of our proof is to restructure our balanced (β, Δ)-graph H in such a way that it behaves like the path in the embedding approach described before. Here two major problems occur:

- Suppose for example that H is a graph consisting of a path whose vertices are labelled by $1, \dots, n$, with some additional edges between vertices whose label differ by at most βn and have different parity (because H is bipartite). For such a graph 'making the connections' as above is now more difficult. Suppose, for instance, that we have a chain of regular pairs (V_i, V_{i+1}) in G_T for $i = 1, \dots, 4$ and want to use it to 'walk' with H from V_1 to V_5. We cannot simply assign vertex 1 to V_1, then 2 to V_2 and so on up to 5 to V_5, because maybe $\{2, 5\}$ forms an edge in H but (V_2, V_5) is not a regular pair in G_T.

  The solution to this problem is to walk more slowly: assign vertex 1 to V_1, then with the vertices $2, 3, \dots, \beta n + 1$ alternate between V_2 and V_3, the next βn vertices continue the zig-zag pattern between V_3 and V_4, and finally we send the last vertex to V_5. What does this buy us? Consider, e.g., the final vertex y that got mapped to V_5. Due to the bandwidth condition, all its potential neighbours were embedded in V_3 or V_4, and due to the parity condition, they must all lie in V_4. This is good, because we have a regular pair (V_4, V_5).

- The second problem that we have to face is as follows. By definition, H has a 2-coloring of its vertices that uses both colors similarly often *in total*, but this does not have to be true *locally* – among the first $m+m$ vertices of H, there could be far more vertices of color 1 than of color 2, which means that our approach to embed them into a super-regular pair with two classes of the same size would fail.

  The solution to this problem is to re-balance H. We use an ordering of H with bandwidth at most βn and cut H into small blocks of size ξn, where $\beta n \ll \xi n \ll m$. Then it is not hard to see that we can obtain a new ordering of the vertices of H by changing the order in which the blocks appear, so that in every interval of blocks summing to roughly m consecutive vertices of H the two colors are balanced up to $2\xi n$ vertices. We can now assign the blocks forming these intervals to super-regular pairs in G_M in such a way that they there represent a balanced 2-coloring and can therefore be embedded via the Blow-up Lemma into the super-regular pair.

Both these problems can appear at the same time, but one can combine these two solutions. Hence H can indeed be embedded into G_T similarly to the example of the path example given above, which finishes the proof.

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