Loose Hamiltonian cycles forced by $(k-2)$-degree – approximate version

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Abstract

We prove that for all $k \geqslant 3$ and $1 \leqslant \ell < k/2$, every k-uniform hypergraph $\mathcal{H}$ on n vertices with $\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} + o(1)\right)\binom{n}{2}$ contains a Hamiltonian ℓ-cycle if $k-\ell$ divides n. This degree condition is asymptotically best possible.

Keywords: hypergraphs, Hamiltonian cycles, degree conditions

1 Introduction

A k-uniform hypergraph $\mathcal{H}$ is a pair (V, E) with vertex set V and edge set E such that each edge is a subset of k vertices. Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ and $S \in \binom{V}{s}$, we denote by $\deg(S)$ the number of edges of $\mathcal{H}$ containing S and we denote by $N(S)$ the $(k-s)$-element sets $T \in \binom{V}{k-s}$ such

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that $T \cup S$ is an edge of E, i.e. $\deg(S) = |N(S)|$. We define the *minimum s-degree* of $\mathcal{H}$, denoted by $\delta_s(\mathcal{H})$, as the minimum of $\deg(S)$ over all s-vertex sets $S \in \binom{V}{s}$.

We say that a k-uniform hypergraph $\mathcal{C}$ is an ℓ-cycle if there exists a cyclic ordering of its vertices such that every edge of $\mathcal{C}$ is composed of k consecutive vertices, two consecutive edges share exactly ℓ vertices, and every vertex is contained in an edge.

We are interested in the problem of finding minimum degree conditions that ensure the existence of Hamiltonian cycles. This problem was first studied by Katona and Kierstead in [7]. They posed a conjecture, which was confirmed by the following result of Rödl, Ruciński, and Szemerédi [11, 12]: For every $k \geqslant 3$, if $\mathcal{H}$ is a k-uniform n-vertex hypergraph with $\delta_{k-1}(\mathcal{H}) \geqslant (1/2 + o(1))n$, then $\mathcal{H}$ contains a Hamiltonian $(k-1)$-cycle. Their proof introduces the so-called *Absorbing Method*, which we will use in our proof as well. In [10] Kühn and Osthus investigated a similar question for 1-cycles, proving that 3-uniform hypergraphs $\mathcal{H}$ with $\delta_2(\mathcal{H}) \geqslant (1/4 + o(1))n$ contain a Hamiltonian 1-cycle. This result was generalized to arbitrary k and ℓ-cycles with $1 \leqslant \ell < k/2$ by Hàn and Schacht [4] (see also [8]).

Theorem 1.1 *For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and*

$$\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma \right) n$$

contains a Hamiltonian ℓ-cycle. □

To see the optimality of the minimum degree condition, we consider the following well-known example. Let $\mathcal{H}_{k,\ell} = (V, E)$ be a k-uniform hypergraph on n vertices such that E is the set of all edges with at least one vertex from $A \subset V$, where $|A| = \left\lceil \frac{n}{2(k-\ell)} - 1 \right\rceil$. Note that an ℓ-cycle on n vertices contains $n/(k-\ell)$ edges and for $k < \ell/2$ every vertex is contained in at most two edges of any ℓ-cycle. So the hypergraph $\mathcal{H}_{k,\ell}$ does not contain a Hamiltonian ℓ-cycle and has $\delta_{k-1}(\mathcal{H}_{k,\ell}) = \left\lceil \frac{n}{2(k-\ell)} - 1 \right\rceil$. In [5] Han and Zhao proved a version of Theorem 1.1 with this sharp degree condition.

Kühn, Mycroft, and Osthus [9] generalized Theorem 1.1 to $1 \leqslant \ell < k-1$, solving the problem of finding minimum $(k-1)$-degree conditions that ensure the existence of Hamiltonian ℓ-cycles in k-uniform hypergraphs. A natural question is to ask for minimum d-degree conditions forcing the existence of

Hamiltonian ℓ-cycles for $d < k-1$. In this direction Buß, Hàn, and Schacht proved the following asymptotically optimal result in [3].

Theorem 1.2 *For all $\gamma > 0$ there exists an n_0 such that every 3-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in 2\mathbb{N}$ and*

$$\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right) n$$

contains a Hamiltonian 1-cycle. □

Note that the optimality again follows from the hypergraph $\mathcal{H}_{k,\ell}$ considered above for $k = 3$ and $\ell = 1$. The sharp bound for $\delta_1(\mathcal{H})$ was proved by Han and Zhao in [6]. We generalize Theorem 1.2 to k-uniform hypergraphs and give an asymptotically optimal bound on the minimum $(k-2)$-degree for the existence of Hamiltonian ℓ-cycles for all $1 \leqslant \ell < k/2$.

Theorem 1.3 (Main result) *For all integers $k \geqslant 4$ and $1 \leqslant \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} + \gamma\right)\binom{n}{2}$$

contains a Hamiltonian ℓ-cycle.

The hypergraph $\mathcal{H}_{k,\ell}$ motivates the following notion of extremality. Let $k \geqslant 3$ and $\ell \geqslant 1$ be integers and let $0 < \xi < 1$. A k-uniform hypergraph $\mathcal{H} = (V, E)$ is called (ℓ, ξ)-*extremal* if there exists a set $B \subset V$ such that $|B| = \left\lfloor \frac{2(k-\ell)-1}{2(k-\ell)} n \right\rfloor$ and $e(B) \leqslant \xi \binom{n}{k}$, where $e(B)$ stands for the number of edges in the subhypergraph of $\mathcal{H}$ induced by B. Our main result follows directly from the following theorem.

Theorem 1.4 *For any $0 < \xi < 1$ and all integers $k \geqslant 4$ and $1 \leqslant \ell < k/2$, there exists $\gamma > 0$ such that the following holds for sufficiently large n. Suppose $\mathcal{H}$ is a k-uniform hypergraph on n vertices with $n \in (k-\ell)\mathbb{N}$ such that $\mathcal{H}$ is not (ℓ, ξ)-extremal and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma\right)\binom{n}{2}.$$

Then $\mathcal{H}$ contains a Hamiltonian ℓ-cycle.

We remark that for $k = 3$ and $\ell = 1$, the corresponding version of Theorem 1.4 appeared in the so-called non-extremal case of the sharp version of Theorem 1.2 in [5]. As a result, it will be sufficient to address the extremal case for a sharp version of Theorem 1.3 and we shall return to this in the near future [2]. For details about this approach see [5, 6]. It is easy to check that if $\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} + \gamma\right)\binom{n}{2}$, then there exists $\xi = \xi(k, \ell, \gamma) > 0$ such that $\mathcal{H}$ is not (ℓ, ξ)-extremal. Consequently, Theorem 1.3 follows from Theorem 1.4.

2 Outline of the proof of Theorem 1.4

The proof follows the *Absorbing Method* introduced by Rödl, Ruciński, and Szemerédi in [11]. It consists of three main parts: an absorbing part, a connecting part, and an almost spanning path-tiling part.

2.1 Absorption

We call an ℓ-path $\mathcal{A} \subseteq \mathcal{H}$ a *β-absorbing path* for a k-uniform n-vertex hypergraph $\mathcal{H}$ if for every subset $U \subset V(\mathcal{H}) \smallsetminus V(\mathcal{A})$ of size at most βn there exists an ℓ-path $\mathcal{Q}$ such that $V(\mathcal{Q}) = V(\mathcal{A}) \cup U$ and $\mathcal{Q}$ has the same ends as $\mathcal{A}$, for some $\beta > 0$. We can ensure the existence of a β-absorbing path $\mathcal{A}$, which reduces the problem of finding a Hamiltonian ℓ-cycle to that of finding an almost spanning ℓ-cycle that contains $\mathcal{A}$. To show this, we prove that for every set of $(k - \ell)$ vertices there exist many short "absorbing paths" and by taking a small random selection of these short paths and connecting them we obtain the required β-absorbing path. For this step a minimum $(k-2)$-degree of cn^2 is sufficient, for any constant $c > 0$.

2.2 Connecting

To obtain an almost spanning ℓ-cycle, we first find a bounded number of ℓ-paths covering almost all vertices of $V(\mathcal{H}) \smallsetminus \mathcal{A}$ and then connect these paths and $\mathcal{A}$ using only vertices from a small set, a so-called *reservoir set* that we fix beforehand. To obtain this reservoir set, we show that any bounded number of ℓ-sets can be connected with short ℓ-paths using only vertices from a set R, provided that all $(k-2)$-tuples of vertices of the hypergraph extend to enough edges in R. By taking a suitably sized random set we can ensure this condition and obtain the reservoir set R with the property that any bounded number of disjoint ℓ-paths can be connected to an ℓ-cycle, only using vertices from R. Again a minimum $(k-2)$-degree of cn^2 is sufficient, for any $c > 0$.

2.3 Path-Tiling

We can choose the sizes of $\mathcal{A}$ and R linear in n but small enough, so that the remaining hypergraph satisfies almost the same degree condition as $\mathcal{H}$. To complete the proof it is only left to show the existence of a collection of ℓ-paths covering almost all vertices of $V(\mathcal{H}) \smallsetminus (\mathcal{A} \cup R)$. This is the only point in the proof where we use the exact value of the degree condition and the non-extremality of $\mathcal{H}$. In fact, the path-tiling step for a direct proof of Theorem 1.3, which allows us to utilize a slightly larger degree condition, is a bit simpler.

For this step we use the weak hypergraph regularity lemma, the straightforward generalisation of Szemerédi's regularity lemma for graphs [13]. This reduces the problem to that of finding a fractional packing of the "cherry" graph $\mathcal{C}_\ell$, the 2-edge k-uniform hypergraph spanning $2(k-\ell)$ vertices. On each of the regular tuples given by this fractional packing it is easy to greedily obtain a bounded number of ℓ-paths spanning almost all vertices.

To obtain the fractional packing, we show that any packing that does not already have almost full weight can be improved by a small, but constant, fraction. For this we consider the so-called link graphs of $(k-2)$-tuples of vertices that are not fully covered and find many small local improvements that we can aggregate.

The ℓ-paths found in the path-tiling step and $\mathcal{A}$ can be connected by using vertices from R to an almost spanning ℓ-cycle containing $\mathcal{A}$. Since this ℓ-cycle contains almost all vertices of $\mathcal{H}$, the absorbing property of $\mathcal{A}$ allows us to absorb the leftover vertices, i.e. vertices that are not contained in any of the ℓ-paths and vertices that were not used to connect the ℓ-paths. The resulting ℓ-cycle is the desired Hamiltonian ℓ-cycle. The full details of the proof of Theorem 1.4 can be found in [1].

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