

UNAVOIDABLE SUBGRAPHS IN RAMSEY GRAPHS

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ABSTRACT. We study subgraphs that appear in large Ramsey graphs for a given graph F . The recent girth Ramsey theorem of the first two authors asserts that there are Ramsey graphs such that all small subgraphs are ‘forests of copies of F ’ amalgamated on vertices and edges. We derive a few further consequences from this structural result and investigate to which extent such forests of copies must be present in Ramsey graphs.

§1 INTRODUCTION

Some of the most fundamental and general types of questions in combinatorics aim for understanding the interplay of *local* and *global* properties in discrete structures. As an insightful example for our discussion serve those local properties that are captured by the appearance of subgraphs of fixed isomorphism type in large graphs with high chromatic number. We consider having a high chromatic number to be a global property of graphs, since in general it cannot be bounded by studying subgraphs of given size. In fact, this is a consequence of the work of Erdős [4], which asserts the existence of graphs with arbitrarily high chromatic number all of whose ‘small’ subgraphs are forests. This result is qualitatively complemented by the simple graph theoretic fact, that any graph of sufficiently high chromatic number must contain all forests of fixed size.

Natural generalisations of these results, in the context of Ramsey theory for *vertex colourings*, were obtained by Nešetřil and Rödl [8] and by Diskin et al. [3]. Here we investigate to which extent these results generalise to *edge colourings*. A central tool for our investigation is the recent girth Ramsey theorem of the first two authors and we start with its formulation.

1.1. Girth Ramsey theorem. Given two graphs F and G and a number of colours $r \geq 2$ we write $G \longrightarrow (F)_r$ for the statement that every r -colouring of the edge set of G yields a monochromatic copy of F which is induced in G . In the situation, when we do not require the monochromatic copy to be necessarily induced, we write $G \xrightarrow{\text{n.n.i.}} (F)_r$. We may also say G is a *Ramsey graph* for F and r colours in these situations. If a graph G fails to

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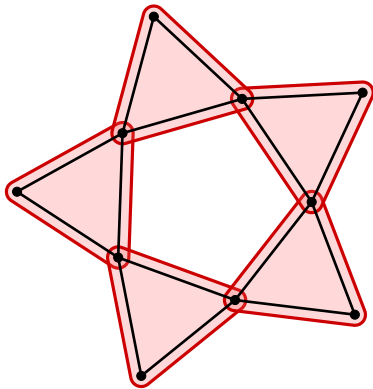
be such a Ramsey graph, then we write $G \twoheadrightarrow (F)_r$ and $G \xrightarrow{\text{n.n.i.}} (F)_r$, respectively. It is easy to see that Ramsey graphs for a given graph F must contain copies of F sharing vertices and edges. Roughly speaking, the girth Ramsey theorem (see Theorem 1.2 below) asserts, that there are Ramsey graphs with the property that locally the copies of F have a forest-like structure. In order to make this precise we define forests in this context.

Definition 1.1 (Forests of copies). A set of graphs \mathcal{F} is called a *forest of copies* if there exists an enumeration $\mathcal{F} = \{F_1, \dots, F_{|\mathcal{F}|}\}$ such that for every $j \in [2, |\mathcal{F}|]$ the set

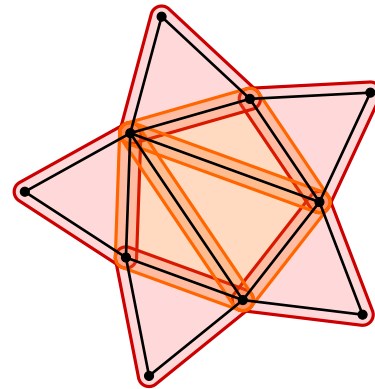
$$V(F_j) \cap \bigcup_{i < j} V(F_i)$$

is either empty or a single vertex, or an edge belonging both to $E(F_j)$ and to $\bigcup_{i < j} E(F_i)$.

We remark that for a forest of copies \mathcal{F} of a graph F the graph $\mathfrak{F} = \bigcup \mathcal{F}$ may contain additional copies of F , which are not present in \mathcal{F} itself. It is also important to note that contrary to forests in graphs, subsets of forests of copies might not be such forests themselves, as the example in Figure 1.1 illustrates. On the left hand side we see a ‘cycle’ of triangles, i.e., a collection of five triangles not forming a forest of copies. By adding two further edges, however, we can hide this configuration inside a forest of eight triangles (see Figure 1.1b).



(a) A ‘cycle’ of triangles ...



(b) ... contained in a forest of triangles.

FIGURE 1.1. Some subforests fail to be forests.

We denote by $\binom{G}{F}$ the set of all induced copies of F in G , i.e., the set of all induced subgraphs of G isomorphic to F . For a subsystem $\mathcal{G} \subseteq \binom{G}{F}$ and a number of colours r the partition relation $\mathcal{G} \twoheadrightarrow (F)_r$ indicates that for every r -colouring of $E(G)$ one of the copies in \mathcal{G} is monochromatic. The rôle of \mathcal{X} in the following theorem is due to the fact that, for the above reason, \mathcal{F} can only be demanded to be a subset of a forest copies rather than

an actual forest of copies. With this notation at hand we can state the following version of the girth Ramsey theorem of the first two authors [10].

Theorem 1.2 (Girth Ramsey theorem). *Given a graph F and $r, \ell \in \mathbb{N}$ there exists a graph G together with a system of copies $\mathcal{G} \subseteq \binom{G}{F}$ satisfying not only $\mathcal{G} \rightarrow (F)_r$, but also the following statement: For every $\mathcal{F} \subseteq \mathcal{G}$ with $|\mathcal{F}| \in [2, \ell]$ there exists a set $\mathcal{X} \subseteq \mathcal{G}$ such that $|\mathcal{X}| \leq |\mathcal{F}| - 2$ and $\mathcal{F} \cup \mathcal{X}$ is a forest of copies.* \square

Similarly, as Erdős' theorem concerns graphs G yielding monochromatic copies of K_2 for vertex colourings, Theorem 1.2 concerns graphs G yielding monochromatic copies of F for edge colourings. Both theorems then assert that there are such graphs G with the additional property of containing locally only forests of the target graph F .

1.2. New results. Our first two results address the question to which extent the converse of Theorem 1.2 holds. More precisely, for which graphs F every forest of copies necessarily occurs in every Ramsey graph for sufficiently many colours. In the positive direction we show that this is indeed true for cycles and cliques.

Theorem 1.3. *Let F be a cycle or a balanced, complete, multipartite graph with at least two edges. For every forest \mathcal{F} of copies of F , there exists some $r \geq 2$ such that every Ramsey graph $G \xrightarrow{n.n.i.} (F)_r$ contains a subgraph isomorphic to $\bigcup \mathcal{F}$.*

The proof of Theorem 1.3 will be presented in §2. It seems plausible that Theorem 1.3 extends to a larger family of graphs. However, our next result shows that there are graphs for which such a statement fails.

Theorem 1.4. *There exist infinitely many graphs F such that for every integer $r \geq 2$ there exists a Ramsey graph $G \rightarrow (F)_r$ with the following properties.*

- (a) *All copies of F in G are induced.*
- (b) *Any two induced copies of F in G , sharing an edge xz , also share a vertex y that is neither adjacent to x nor adjacent to z in G .*

At first sight, property (b) seems quite orthogonal to anything the girth Ramsey theorem might be capable of doing for us. Nevertheless, the proof of Theorem 1.4 presented in §4 relies on a version of the girth Ramsey theorem for ordered hypergraphs, which we discuss in §3. It might also be interesting to point out that the graphs F constructed to this end are dense and highly connected, so the phenomenon described here is not just some pathologic degenerate case.

The two remaining results concern other local structures in Ramsey graphs. The next theorem establishes the existence of Ramsey graphs for many colours, which locally fail to

have the Ramsey property for two colours. In that sense it parallels the simple consequence of Erdős' theorem that there are high chromatic graphs, which are locally bipartite.

Theorem 1.5. *For every graph F containing a cycle and all integers $r \geq 2$ and $n \in \mathbb{N}$ there exists a Ramsey graph $G \rightarrow (F)_r$, such that for every subgraph $H \subseteq G$ on n vertices we have $H \xrightarrow{n.n.i.} (F)_2$.*

We remark that Theorem 1.5 does not hold for all graphs F . For example, it obviously fails for matchings and stars. We deduce Theorem 1.5 from the girth Ramsey theorem and we present the details of that proof in §5.

Our last result shows that for bipartite graphs F there exist Ramsey graphs containing F -free subgraphs with positive density.

Theorem 1.6. *Let F be a bipartite graph that cannot be disconnected by deleting either one vertex or two vertices forming an edge, and let $\ell(F)$ be the number of edges of a longest path in F .*

For every integer $r \geq 2$ there exists a Ramsey graph $G \rightarrow (F)_r$ with the property that every subgraph $H \subseteq G$ contains an F -free subgraph $H' \subseteq H$ with $|E(H')| \geq \frac{\ell(F)-1}{2\ell(F)}|E(H)|$.

Theorem 1.6 is remotely related to Pisier's problem and we refer to the survey [5], where the result for $F = C_4$ was stated. We remark that, due to limitations of our approach, the promised subgraph H' has less than half of the edges of H and, hence, the corresponding statement trivially holds for graphs with chromatic number at least three. In view of that we put forward the following question.

Question 1.7. Does there exist an $\alpha > 1/2$ such that for every integer $r \geq 2$ there is a Ramsey graph $G \rightarrow (K_3)_r$ with the property that every $H \subseteq G$ contains a triangle-free subgraph $H' \subseteq H$ with $|E(H')| \geq \alpha|E(H)|$.

The proof of Theorem 1.6 presented in §6 also relies on the girth Ramsey theorem for ordered graphs stated in §3.

1.3. Ramsey infinite graphs. It turns out that the proof of Theorem 1.5 also shows, that graphs containing a cycle have infinitely many minimal Ramsey graphs. We recall that a graph F is *Ramsey infinite*, if there are infinitely many minimal graphs G satisfying the partition relation $G \xrightarrow{n.n.i.} (F)_2$ and otherwise we say F is *Ramsey finite*.

Burr, Erdős, and Lovász [2] expressed the belief that with a few exceptions most graphs are Ramsey infinite and this was confirmed through the work of Murty (see, e.g., [2]), of Burr, Erdős, and Lovász [2], of Nešetřil and Rödl [9], of Faudree [6], and of Rödl and Ruciński (see, e.g., Łuczak [7]). As it turned out the only Ramsey finite graphs are certain

types of star forests, among which are matchings of any size and stars with an odd number of edges (see the work of Faudree [6] for a full classification). The complementing case, that all graphs containing a cycle are Ramsey infinite, is a consequence of the work of Rödl and Ruciński [11, 12] on thresholds for Ramsey properties in random graphs.

Theorem 1.8. *Every graph containing a cycle is Ramsey infinite.* \square

In §5 we see that this follows easily from Theorem 1.5.

§2 FORCING FORESTS OF CLIQUES AND CYCLES

In this section we derive Theorem 1.3. The proof for cycles and the proof for balanced, complete, multipartite graphs are different and we start the discussion with the latter.

2.1. Forests of copies of balanced, complete, multipartite graphs. The proof relies on a theorem of de Bruijn and Erdős on set mappings. A *set mapping* over a set X is defined to be a map $f: X \rightarrow \wp(X)$ with $x \notin f(x)$ for every $x \in X$. Subsets $Y \subseteq X$ such that $y \notin f(y')$ holds for all $y, y' \in Y$ are called *free*. Let us quote [1, Theorem 3].

Theorem 2.1 (de Bruijn and Erdős). *Let f be a set mapping over a (finite or infinite) set X , and let k be a positive integer. If $|f(x)| \leq k$ for every $x \in X$, then X can be partitioned into $2k + 1$ free sets.* \square

We call a graph F *strongly edge transitive* if for all ordered pairs $(x, y), (x', y') \in V(F)^2$ with $xy, x'y' \in E(F)$ there exists an automorphism of F sending x to x' and y to y' . For instance, all balanced, complete multipartite graphs, Kneser graphs, and cycles are strongly edge-transitive. Given a strongly edge-transitive graph F with at least one edge and a positive integer m we denote the graph obtained by gluing m copies of F along an edge by (m, F) . For clarity we point out that this graph has $m \cdot (v(F) - 2) + 2$ vertices and $m \cdot (e(F) - 1) + 1$ edges. The edge where the gluing occurs is called the *kernel* of (m, F) . In the special case $m = 1$ every edge can be viewed as a kernel of $F \cong (1, F)$.

Lemma 2.2. *Let F be a balanced, complete, multipartite graph with at least two edges. If for some positive integer m a graph H has no subgraphs isomorphic to (m, F) , then*

$$H \xrightarrow{\text{n.n.i.}} (F)_{4mv(F)}.$$

Proof. We define a set mapping f over $E(H)$ as follows. Given $e \in E(H)$ we set $f(e) = \emptyset$ if e does not belong to any n.n.i. copy of F in H . Otherwise we let m_e be maximal such that H contains a n.n.i. copy M_e of (m_e, F) with kernel e . Since H is (m, F) -free, we know $m_e < m$. For every vertex z of M_e not belonging to e let $E(z, e)$ be the set

of the edges containing z and one vertex of e . We note $1 \leq |E(z, e)| \leq 2$, as F is a complete, multipartite graph. Let $f(e) = \bigcup_z E(z, e) \subseteq E(H)$ be the set of up to at most $2(v(M_e) - 2) = 2m_e(v(F) - 2)$ edges arising in this way. Owing to $|f(e)| < 2mv(F)$ for every $e \in E(H)$, Theorem 2.1 yields a partition of $E(H)$ into $4mv(F)$ sets which are free with respect to f . In other words, there is a colouring $h: E(H) \rightarrow [4mv(F)]$ whose colour classes are free sets.

It suffices to show that no n.n.i. copy of F is monochromatic with respect to h . Assume for the sake of contradiction that F_\star is a counterexample and let $e = xy \in E(F_\star)$ be arbitrary. By the maximal choice of M_e there is a vertex $z \in V(M_e) \cap V(F_\star)$ distinct from x and y . Consequently,

$$E(F_\star) \cap f(e) \supseteq E(F_\star) \cap E(z, xy) \neq \emptyset$$

and, as the colour classes of h are free, we arrive at a contradiction to F_\star being monochromatic. \square

Proof of Theorem 1.3 for balanced, complete, multipartite graphs. Let us call a forest \mathcal{F} of copies of F *edgy* if it has an enumeration $\mathcal{F} = \{F_1, \dots, F_m\}$ such that for every $j \in [2, m]$ the set

$$q_j = V(F_j) \cap \bigcup_{i < j} V(F_i)$$

is an edge in $E(F_j) \cap \bigcup_{i < j} E(F_i)$. It can easily be confirmed that for every forest \mathcal{F} of copies of F there is an edgy forest \mathcal{F}' of copies of F such that $\bigcup \mathcal{F}'$ has a subgraph isomorphic to $\bigcup \mathcal{F}$. In fact, this holds for some \mathcal{F}' with $|\mathcal{F}'| = O(|\mathcal{F}|)$, but this bound is of no importance to what follows. In any case, this argument shows that it suffices to prove our theorem for edgy forests \mathcal{F} .

We handle this case by induction on $|\mathcal{F}|$. In the base case, $|\mathcal{F}| = 1$, it suffices to take $r = 1$. Now consider an edgy forest $\mathcal{F} = \{F_1, \dots, F_m\}$ with $m \geq 2$ enumerated as in the previous paragraph. Set $\mathcal{F}^- = \mathcal{F} \setminus \{F_m\}$, let r^- be a natural number obtained by applying the induction hypothesis to \mathcal{F}^- , and set $r = r^- + 4mv(F)^2$. Given any graph H with $H \xrightarrow{\text{n.n.i.}} (F)_r$ we define H' to be the spanning subgraph of H whose edges are the kernels of n.n.i. copies of $(mv(F), F)$ in H . By Lemma 2.2 the spanning subgraph H'' of H with edge set $E(H) \setminus E(H')$ satisfies $H'' \xrightarrow{\text{n.n.i.}} (F)_{4mv(F)^2}$ and thus we have $H' \xrightarrow{\text{n.n.i.}} (F)_{r^-}$. Our choice of r^- yields a n.n.i. copy of $\bigcup \mathcal{F}^-$ in H' . Denote the edge of H' playing the rôle of q_m in that copy by q . Since q is the kernel of a n.n.i. copy of $(mv(F), F)$ in H and

$$v(\mathcal{F}^-) = (m - 1)(v(F) - 2) + 2 < mv(F)$$

there is a n.n.i. copy of F intersecting our copy of $\bigcup \mathcal{F}^-$ only in q . This proves that H contains indeed a n.n.i. copy of $\bigcup \mathcal{F}$. \square

2.2. Forests of copies of cycles. The proof of Theorem 1.3 for cycles follows the one for balanced, complete, multipartite graphs with the key lemma, Lemma 2.2, replaced by the following statement.

Lemma 2.3. *For all integers $\ell \geq 3$, $m \geq 1$ there exists a constant r such that every graph H without subgraphs isomorphic to (m, C_ℓ) satisfies $H \xrightarrow{n.n.i.} (C_\ell)_r$.*

Proof. Let us call a number *bounded* if it is $O_{\ell,m}(1)$, i.e., if it has an upper bound in terms of ℓ and m , which is independent of H . Similarly, a set is *bounded* if its cardinality is.

We define recursively for every edge xy of H an increasing chain $(K_i(xy))_{i \in [\ell]}$ of bounded vertex sets. In the first step, we choose m_{xy} maximal such that H contains a n.n.i. copy of (m_{xy}, C_ℓ) with kernel xy and let $K_1(xy)$ be the vertex set of such a copy. In the degenerate case $m_{xy} = 0$, i.e., if no n.n.i. copy of C_ℓ in H passes through xy , we simply set $K_1(xy) = \{x, y\}$. Notice that our assumption on H yields $m_{xy} < m$, whence $|K_1(xy)| \leq 2 + (\ell - 2)m$ is indeed bounded.

Now suppose that for some positive $\lambda < \ell$ the bounded set $K_\lambda(xy) \subseteq V(H)$ has already been defined. To every pair $p = (uv, k) \in K_\lambda(xy)^{(2)} \times [2, \ell)$ we assign a bounded set B_p as follows. If H contains more than $2\ell m$ internally vertex-disjoint paths of length k from u to v we put $B_p = \emptyset$. Otherwise we consider a maximal collection of such paths and let B_p denote the set of their inner vertices, so that $|B_p| \leq 2k\ell m < 2\ell^2 m$ is bounded. The number of pairs p considered here is at most $|K_\lambda(xy)|^2 \ell$ and, therefore, bounded. Thus the union $K_{\lambda+1}(xy)$ of $K_\lambda(xy)$ with all these vertex sets B_p is bounded as well.

Having thereby completed our recursive definition we employ Theorem 2.1 in order to get a colouring $h: E(H) \rightarrow [r]$ with boundedly many colours such that for every $xy \in E(H)$ the only edge $e \subseteq K_\ell(xy)$ with the same colour as xy is xy itself.

Claim 2.4. If $x_1 \dots x_\ell$ denotes a n.n.i. copy of C_ℓ in H and

$$K_1(x_1x_2) \cap \{x_1, \dots, x_\ell\} \subseteq \{x_1, \dots, x_i\} \tag{2.1}$$

for some $i \in [\ell]$, then the path $x_1 \dots x_i$ fails to be monochromatic with respect to h .

Proof. Arguing indirectly we consider a counterexample such that i is as small as possible. Due to $x_1, x_2 \in K_1(x_1x_2)$ we have $i \geq 2$. In fact, our maximal choice of $K_1(x_1x_2)$ guarantees that the left side of (2.1) is larger than $\{x_1, x_2\}$, whence $i \geq 3$. Now $i = 3$ entailed $x_3 \in K_1(x_1x_2)$ and our choice of h gave rise to the contradiction $h(x_1x_2) \neq h(x_2x_3)$.

For all these reasons we have $i \geq 4$. The minimality of i applied to the cycle

$$x_{i-1} \dots x_1 x_\ell \dots x_i$$

yields $K_1(x_{i-1}x_{i-2}) \cap \{x_1, \dots, x_\ell\} \not\subseteq \{x_1, \dots, x_{i-1}\}$ and, consequently, there is some vertex in $K_1(x_{i-1}x_{i-2}) \cap \{x_i, \dots, x_\ell\}$.

In particular, there is a pair $(j, \lambda) \in [i, \ell] \times [\ell]$ such that $x_j \in K_\lambda(x_{i-1}x_{i-2})$ and $j + \lambda \leq \ell + 1$. Indeed, we have just shown that even with $\lambda = 1$ such a pair exists. For the remainder of the argument we fix $\lambda_\star \leq \ell$ as large as possible such that there is an index $j_\star \in [i, \ell]$ with

$$x_{j_\star} \in K_{\lambda_\star}(x_{i-1}x_{i-2}) \quad \text{and} \quad j_\star + \lambda_\star \leq \ell + 1.$$

Due to $j_\star \geq i > 1$ we have $\lambda_\star < \ell$.

If we had $j_\star = i$, then $x_i \in K_{\lambda_\star}(x_{i-1}x_{i-2}) \subseteq K_\ell(x_{i-1}x_{i-2})$ would imply

$$h(x_{i-1}x_{i-2}) \neq h(x_{i-1}x_i),$$

contrary to the fact that the path $x_1 \dots x_i$ is monochromatic with respect to h .

Consequently, we have $j_\star > i$ and for the definition of the set $K_{\lambda_\star+1}(x_{i-1}x_{i-2})$ the pair

$$p = (x_{i-1}x_{j_\star}, j_\star - i + 1)$$

was considered.

Suppose first that $B_p \neq \emptyset$, so that B_p needs to intersect $\{x_i, \dots, x_{j_\star-1}\}$ in some vertex $x_{j'}$, say. But having $x_{j'} \in K_{\lambda_\star+1}(x_{i-1}x_{i-2})$ and $j' + (\lambda_\star + 1) \leq j_\star + \lambda_\star \leq \ell + 1$ is contrary to the maximality of λ_\star . This contradiction proves $B_p = \emptyset$, which means that there are more than $2\ell m$ internally vertex disjoint path of length $j_\star - i + 1$ from x_{i-1} to x_{j_\star} . At most $|K_1(x_1x_2)| \leq \ell m$ of these paths have an inner vertex in $K_1(x_1x_2)$, and at most ℓ have an inner vertex in $\{x_1, \dots, x_\ell\}$. Consequently, there is a path $x_{i-1}y_i \dots y_{j_\star-1}x_{j_\star}$ such that

$$x_1 \dots x_{i-1}y_i \dots y_{j_\star-1}x_{j_\star} \dots x_\ell$$

is a cycle in H and $\{y_i, \dots, y_{j_\star-1}\} \cap K_1(x_1, x_2) = \emptyset$. Clearly this cycle contradicts the minimal choice of i at the beginning of the proof of our claim. \square

It remains to observe that the case $i = \ell$ of Claim 2.4 shows, in particular, that no ℓ -cycle is monochromatic with respect to h . \square

Imitating the proof of Theorem 1.3 for balanced, complete, multipartite graphs with Lemma 2.2 replaced by Lemma 2.3 yields Theorem 1.3 for cycles and we omit the details.

§3 ORDERED HYPERGRAPHS

We shall need the following version of Theorem 1.2 for ordered, linear hypergraphs. Recall that a hypergraph is *linear* if any pair of distinct edges shares at most one vertex. An *ordered hypergraph* $H = (V, E, <)$ is a hypergraph (V, E) equipped with a total order $<$ of its vertex set; all subgraphs of H inherit the ordering induced by H . For two ordered

hypergraphs H and F we require for copies of F in H to display the same ordering as F itself. Moreover, in this context $\binom{H}{F}_<$ stands for the set of all ordered, induced copies of F in H . We shall also consider unordered, induced copies of F in the unordered hypergraph H and for the set $\binom{(V(H), E(H))}{(V(F), E(F))}$ of those copies we shall simply write $\binom{H}{F}$. In the context of linear hypergraphs, forests of copies can be defined in the same way as for graphs, i.e., one just replaces the word ‘graphs’ in the first line of Definition 1.1 by ‘linear hypergraphs’. The variant of the girth Ramsey theorem for ordered linear hypergraphs can be stated as follows.

Theorem 3.1 (Girth Ramsey theorem for ordered hypergraphs). *Given an ordered linear k -uniform hypergraph F and $r, n \in \mathbb{N}$ there exists an ordered linear k -uniform hypergraph H together with a system of ordered copies $\mathcal{H} \subseteq \binom{H}{F}_<$ satisfying not only $\mathcal{H} \rightarrow (F)_r$, but also the following statement: For every set of ordered copies $\mathcal{N} \subseteq \mathcal{H}$ with $|\mathcal{N}| \in [2, n]$ there exists a set $\mathcal{X} \subseteq \mathcal{H}$ such that $|\mathcal{X}| \leq |\mathcal{N}| - 2$ and $\mathcal{N} \cup \mathcal{X}$ is a forest of copies. \square*

We remark that the system \mathcal{H} in Theorem 3.1 is necessary for linear hypergraphs F that are not sufficiently connected, since in this case the hypergraph H may contain some ‘unwanted’ copies spread over several copies from \mathcal{H} . Roughly speaking, this situation can be excluded if F cannot be disconnected by removing the vertex set of an edge or fewer vertices. Let us recall here that a hypergraph F is said to be *connected* if for every partition $V(F) = X \cup Y$ with $X, Y \neq \emptyset$ there is an edge $e \in E(F)$ such that $e \cap X, e \cap Y \neq \emptyset$. The stronger notion we require is rendered by the following definition.

Definition 3.2. We say a linear k -uniform hypergraph F is (e, v) -*inseparable*, if it satisfies the following properties

- (a) F is connected and contains at least two edges,
- (b) for every $z \subseteq V(F)$ with $|z| < k$ and no two vertices of which are contained in one edge the hypergraph $F - z$ remains connected,
- (c) and for every edge $e \in E(F)$ the hypergraph $F - e$, obtained from F by removing all vertices of e , remains connected.

This property is precisely what we need for proving the following result.

Lemma 3.3. *Let \mathcal{F} be a forest of linear k -uniform hypergraphs and let J be an induced subgraph of $\bigcup \mathcal{F}$. If J is (e, v) -inseparable, then there is some $F \in \mathcal{F}$ such that J is an induced subhypergraph of F .*

Proof. A straightforward argument by induction on $|\mathcal{F}|$ reduces the claim to its special case $|\mathcal{F}| = 2$. So we may assume $\mathcal{F} = \{F_1, F_2\}$ for two linear k -uniform hypergraphs F_1

and F_2 intersecting either at most in a vertex or in a common edge. Set

$$A = V(F_1) \cap V(F_2) \cap V(J) \quad \text{and} \quad B_i = (V(F_i) \cap V(J)) \setminus A$$

for $i = 1, 2$, so that $V(J) = A \cup B_1 \cup B_2$. Clearly if $B_1 = \emptyset$, then $V(J) \subseteq F_2$ and J is an induced subhypergraph of F_2 . Similarly if $B_2 = \emptyset$, then J is induced in F_1 and we are done again.

Thus it suffices to derive a contradiction from the assumption that $B_1, B_2 \neq \emptyset$. Now the partition $B_1 \cup B_2$ witnesses that $J - A$ is disconnected.

If $|A| < k$, then the (e, v) -inseparability of J yields an edge $e \in E(J)$ such that $|A \cap e| \geq 2$. Owing to $|A| \geq 2$ the copies F_1, F_2 intersect in a common edge e_\star . Now $A \subseteq e_\star$ yields $|e \cap e_\star| \geq 2$, linearity gives $e = e_\star$, and $e_\star \in E(J)$ entails $A = e_\star$, which contradicts $|A| < k$.

It remains to consider the case $|A| \geq k$. Due to $A \subseteq V(F_1) \cap V(F_2)$ this is only possible if F_1 and F_2 intersect in a common edge, namely A . But the fact that J is induced in $F_1 \cup F_2$ reveals $A \in E(J)$ and we reach a contradiction to the (e, v) -inseparability of J . \square

The following corollary of Theorem 3.1 will be employed in the proofs of Theorem 1.4 and Theorem 1.6.

Corollary 3.4. *Given an ordered, linear, (e, v) -inseparable, k -uniform hypergraph F and integers $r, n \in \mathbb{N}$ there exists an ordered linear k -uniform hypergraph H satisfying not only $H \rightarrow (F)_r$, but also the following statements:*

- (i) *For every set $\mathcal{N} \subseteq \binom{H}{F}_<$ with $|\mathcal{N}| \in [2, n]$ there exists a set $\mathcal{X} \subseteq \binom{H}{F}_<$ such that $|\mathcal{X}| \leq |\mathcal{N}| - 2$ and $\mathcal{N} \cup \mathcal{X}$ is a forest of copies.*
- (ii) *We have $|\binom{H}{F}| = |\binom{H}{F}_<|$, i.e., H contains no induced copies of F that are not order isomorphic to F .*
- (iii) *Every edge of H is contained in some ordered copy from $\binom{H}{F}_<$.*

Compared to Theorem 3.1 the main new claim is part (ii), and this is where the (e, v) -inseparability is used. We would like to point out that already the case $n = 1$ of Corollary 3.4 seems nontrivial to us, even though its clause (i) is void.

Proof of Corollary 3.4. Given $F = (V_F, E_F, <_F)$ and integers r and n we may assume that $n \geq |E_F|$. We apply Theorem 3.1 and obtain $H = (V_H, E_H, <_H)$ and a system of ordered copies $\mathcal{H} \subseteq \binom{H}{F}_<$. Clearly, $H \rightarrow (F)_r$ and without loss of generality we assume that $H = \bigcup \mathcal{H}$. This is because we can simply delete edges of H that belong to no copy of F from \mathcal{H} without affecting the Ramsey property. Consequently, the hypergraph H enjoys property (iii).

For properties (i) and (ii) we prove that every copy from $\binom{H}{F}_<$ equipped with the vertex order induced by the ordering of H belongs to \mathcal{H} (and is, therefore, order isomorphic

to $(V_F, E_F, <_F)$). For that we consider any copy $F_\star \in \binom{H}{F}$ and for every edge e of F_\star we fix an ordered copy of F from \mathcal{H} containing e . Owing to Theorem 3.1 there exists a forest $\mathcal{F} = \{F_1, \dots, F_t\} \subseteq \mathcal{H}$ of copies of F for some t with $1 < t \leq 2|E_F| - 2$ such that the ordered hypergraph $\bigcup \mathcal{F}$ contains all edges of F_\star . Since F_\star is an (e, v) -inseparable, induced subhypergraph of $\bigcup \mathcal{F}$, Lemma 3.3 yields $F_\star \in \mathcal{F} \subseteq \mathcal{H}$. \square

§4 RAMSEY GRAPHS CONTAINING NO FOREST OF COPIES

In this section we establish Theorem 1.4, which concerns the existence of Ramsey graphs $G \rightarrow (F)_r$ that only contain trivial forests of copies of F . The proof naturally splits into two parts. In §4.1 we construct suitable graphs F and in §4.2 we apply the girth Ramsey theorem to obtain the corresponding Ramsey graphs G .

4.1. Graphs derived from ordered linear hypergraphs. The graphs F will be derived from ordered, linear 3-uniform hypergraphs $S = (V, E, <)$. For that, for an ordered hypergraph S we define the (unordered) graph $F_S = (V, E')$ on the same vertex set by

$$xz \in E' \iff \text{there is } xyz \in E \text{ with } x < y < z. \quad (4.1)$$

In other words, the edge set of the graph F_S is obtained by removing the ‘intermediate vertices’ from all edges of S . For the proof of Theorem 1.4 we require that S is (e, v) -inseparable and F_S is 4-connected.

Proposition 4.1. *There are infinitely many ordered, (e, v) -inseparable, linear 3-uniform hypergraphs S such that the graph F_S is 4-connected.*

Proof. For any sufficiently large integer n divisible by 16, consider the ordered, linear 3-uniform hypergraph $S = (V, E, <)$ defined on $V = [n]$ with its natural ordering and

$$E = \{xyz \in [n]^{(3)} : x + y + z = n \text{ or } x + y + z = 2n\}.$$

Clearly, S is linear. Since for edges of S the vertices x, y , and z must be distinct, one may observe that every fixed vertex x forms an edge in E with all other vertices except one of $n - 2x$ or $2n - 2x$ and with vertices of the form $(n - x)/2$ or $(2n - x)/2$, if these are integers. Consequently, for distinct vertices x and x' there are at least $n - 8$ choices for $b \in [n] \setminus \{x, x'\}$ for which there are vertices a, a' such that

$$xab \in E \quad \text{and} \quad ba'x' \in E.$$

Since any vertex can play each of the rôles a, b, a' at most once, the removal of any vertex different from x and x' can destroy at most three such x - x' paths. Consequently, x and x' cannot be separated in S by removing less than $(n-8)/3$ vertices and the (e, v) -inseparability of S follows.

We prepare the proof of the 4-connectivity of F_S by analysing its minimum degree. This will later exclude the existence of ‘small’ components.

Claim 4.2. We have $\delta(F_S) \geq n/9$.

Proof. We show that for every vertex $x \in [n]$ there are at least $n/8 - 2 \geq n/9$ pairs $\{a, b\} \in [n]^{(2)}$ such that $xab \in E$ and either $x < \min\{a, b\}$ or $x > \max\{a, b\}$. In fact, one can check that depending on the choice of x those pairs $\{a, b\}$ can be found with at least one of the vertices a and b from the interval $I(x)$ defined by

$$I(x) = \begin{cases} [n/4, n/2], & \text{if } x \in [1, n/4 - 1] \\ [3n/4, n], & \text{if } x \in [n/4, n/2 - 1] \\ [1, n/4], & \text{if } x \in [n/2, 3n/4 - 1] \\ [n/2 + 1, 3n/4 - 1], & \text{if } x \in [3n/4, n]. \end{cases}$$

This proves the claim. \square

The proof of the fact that F_S is sufficiently connected is rendered by the following claim.

Claim 4.3. The derived graph F_S is $n/100$ -connected.

Proof. Let $U \subseteq [n]$ be an arbitrary set of at most $n/100$ vertices and set

$$x_\star = \min([n] \setminus U) \quad \text{and} \quad x^\star = \max([n] \setminus U).$$

We plan to show that these vertices are in the same component of $F_S - U$ and that this component consists of ‘almost all’ vertices. Together with the minimum degree property of F_S this will show that $F_S - U$ is indeed connected.

Clearly, we have $x_\star \leq |U| + 1 \leq n/16$ and $x^\star \geq n - |U| \geq 15n/16$. Thus for every positive integer $k < n/12$ the edges

$\{x_\star, n/8 + k - x_\star, 7n/8 - k\}$, $\{n/2 - k, 5n/8 + 2k, 7n/8 - k\}$, $\{n/2 - k, 3n/2 + k - x^\star, x^\star\}$ of S yield a path $x_\star - (7n/8 - k) - (n/2 - k) - x^\star$ in F_S . As these paths are internally vertex disjoint, at least one of them is present in $F_S - U$ and, consequently, there is a component C of $F_S - U$ containing both x_\star and x^\star .

Due to $x_\star < n/4$ for $k \in [n/2 - 2x_\star]$ the edges $\{x_\star, x_\star + k, n - 2x_\star - k\}$ of S certify $N_{F_S}(x_\star) \supseteq [n/2, n - 2x_\star - 1]$ and we arrive at whence

$$|[n/2, n] \setminus V(C)| \leq 2x_\star + 1 + |U| \leq 3|U| + 3.$$

Similarly, $x^\star > 3n/4$ leads to $N_{F_S}(x^\star) \supseteq [2n - 2x^\star + 1, n/2 - 1]$, for which reason

$$|[1, n/2 - 1] \setminus V(C)| \leq 2n - 2x^\star + |U| \leq 3|U|.$$

Altogether, we obtain $|[n] \setminus V(C)| \leq 6|U| + 3$. Combining the minimum degree $\delta(F_S) \geq n/9$ provided by Claim 4.2 and $|U| \leq n/100$, we infer that $F_S - U$ is indeed connected. \square

This concludes the proof of Proposition 4.1. \square

4.2. Ramsey graphs for graphs derived from linear hypergraphs. The proof of Theorem 1.4 relies on Corollary 3.4 and Proposition 4.1.

Proof of Theorem 1.4. Let $S = (V_S, E_S, <_S)$ be an ordered (e, v) -inseparable linear hypergraph provided by Proposition 4.1 and let $F = F_S$ be the derived 4-connected graph. For $r \geq 2$ let $H = (V_H, E_H, <_H)$ be the ordered, linear hypergraph provided by Corollary 3.4 applied to S with $n = |E(F)|$. We derive the graph $G = F_H$ from H in the same way as F was derived from S , i.e.,

$$xz \in E(G) \quad \iff \quad \text{there is } xyz \in E(H) \text{ with } x < y < z.$$

Owing to the linearity of H this construction yields a one-to-one correspondence between $E(H)$ and $E(G)$.

Claim 4.4. Every copy $S_\star \in \binom{H}{S}$ gives rise to an induced copy $F_{S_\star} \in \binom{G}{F}$.

Proof. By Corollary 3.4(ii) we know that S_\star appears with the correct ordering in H and, therefore, $F_\star = F_{S_\star}$ is a subgraph of H isomorphic to F . We need to show that this subgraph is induced.

Suppose to this end that $x, z \in V(F_\star)$ are connected by an edge $xz \in E(G)$. This means that for some vertex $y \in V(H)$ with $x < y < z$ there is an edge $xyz \in E(H)$. Since H is linear, it has no other edges containing x and z . We need to show that $xyz \in E(S_\star)$.

By Corollary 3.4(iii) there is a copy $S_{\star\star} \in \binom{H}{S}$ containing the edge xyz . If $S_\star = S_{\star\star}$ we are done, so suppose that this is not the case. By Corollary 3.4(i) the set $\{S_\star, S_{\star\star}\}$ is a forest of copies of S . As the set $q = V(S_\star) \cap V(S_{\star\star})$ contains x and z , this is only possible if $q = xyz$ is an edge of S_\star . \square

This immediately implies that G enjoys the Ramsey property $G \rightarrow (F)_r$. Indeed, any r -colouring of $E(G)$ defines an r -colouring of $E(H)$. Some copy $S_\star \in \binom{H}{S}$ needs to be monochromatic with respect to this induced colouring and by Claim 4.4 we get a monochromatic copy $F_{S_\star} \in \binom{G}{F}$.

Claim 4.5. Every n.n.i. copy F_\star of F in G is of the form $F_\star = F_{S_\star}$ for some $S_\star \in \binom{H}{S}_{<}$.

Proof. For every edge $e = xz$ of F_\star there exists a unique triple $T_e = xyz$ in $E(H)$ and by property (iii) of Corollary 3.4 the triple T_e is contained in some copy of S in H . Appealing to property (i) of Corollary 3.4 we obtain a forest of copies $\mathcal{S} = \{S_1, \dots, S_t\}$ of S for some

$t \leq 2|E(F)| - 2$ with $\bigcup \mathcal{S}$ containing all triples T_e for $e \in E(F_\star)$. We may assume that t is minimal with that property. If $t = 1$ we are done, so suppose for the sake of contradiction that $t \geq 2$. We set $q = V(S_t) \cap \bigcup_{j < t} V(S_j)$. Due to the minimal choice of t , the copy S_t contains at least one of those triples T_e and $T_e \setminus q \neq \emptyset$.

If $|q| \leq 1$, then $T_e \setminus q$ contains at least one of the two vertices of e and, since F_\star is 4-connected, we arrive at $V(F_\star) = V(S_t)$ and by Claim 4.4 we are done.

In the remaining case $|q| = 3$ and $q \in E(H)$. Consequently, the linearity of H implies again $|T_e \cap q| \leq 1$ and the same argument as in the case $|q| \leq 1$ yields the claim. \square

Both claims together establish part (a) of Theorem 1.4. For part (b) we consider two copies $F_\star, F_{\star\star}$ in G sharing an edge $xz \in E(G)$ with $x < z$. The definition of G yields a vertex y with $x < y < z$ and $xyz \in E(H)$. Moreover, Claim 4.5 implies $xyz \in E(S_\star) \cap E(S_{\star\star})$ and, hence, y is a common vertex of $F_\star, F_{\star\star}$. Finally, $xy, yz \notin E(G)$ follows from the linearity of H . \square

§5 GLOBALITY OF RAMSEY PROPERTIES AND RAMSEY INFINITE GRAPHS

In this section we shall prove Theorem 1.5 and Theorem 1.8 (see §5.2). In those proofs the 2-density of a graph will play an important rôle and we discuss this parameter first.

5.1. The 2-density of a graph. For every graph F with at least one edge we define its *2-density*

$$d_2(F) = \begin{cases} \frac{|E(F)|-1}{|V(F)|-2}, & \text{if } |V(F)| \geq 3, \\ 1, & \text{if } F = K_2, \end{cases}$$

and denote by $m_2(F)$ the *maximum 2-density*

$$m_2(F) = \max \{d_2(F') : F' \subseteq F \text{ and } e(F') \geq 1\}.$$

We say a graph F with at least two edges is *strictly 2-balanced* if $d_2(F) > d_2(F')$ for every proper subgraph $F' \subsetneq F$ with at least one edge. In particular, for any strictly 2-balanced graph F we have $d_2(F) > 1$ and, consequently, $|E(F)| \geq |V(F)|$.

The 2-density is a well-known parameter in random graph theory, as $p = n^{-1/d_2(F)}$ marks the edge probability in $G(n, p)$ where the expected number of copies of F is as large as the expected number of edges.

We first observe that building forests of copies does not increase the maximum 2-density.

Proposition 5.1. *For every nonempty forest of copies \mathcal{F} of a graph F with at least one edge we have $m_2(\mathfrak{F}) = m_2(F)$ for the graph $\mathfrak{F} = \bigcup \mathcal{F}$.*

Proof. The statement is obvious if \mathcal{F} contains only one copy of F and the general case follows from a simple inductive argument on the number of copies in \mathcal{F} .

For some $\ell > 1$ let $\mathcal{F} = \{F_1, \dots, F_\ell\}$ be a forest of copies of F and set

$$q = V(F_\ell) \cap \bigcup_{i < \ell} V(F_i).$$

Recall that the set q contains at most two vertices and if $|q| = 2$, then it spans an edge from the set $E(F_\ell) \cap \bigcup_{i < \ell} E(F_i)$.

We shall show that $d_2(J) \leq m_2(F)$ for every subgraph $J \subseteq \bigcup \mathcal{F}$. Owing to the induction hypothesis, it suffices to consider such a graph J with both subgraphs $J_1 = J \cap \bigcup_{i < \ell} F_i$ and $J_2 = J \cap F_\ell$ containing at least one edge. We set $q_J = q \cap V(J)$. Note that without loss of generality we may assume that J is an induced subgraph and, hence, if $|q_J| = |q| = 2$, then the edge spanned by q_J is also contained in J . Let $\zeta \in \{0, 1\}$ indicate if q_J spans an edge. We have

$$1 - \zeta \leq (2 - |q_J|) \cdot m_2(F)$$

because if $\zeta = 0$, then $|q_J| \leq 1 \leq m_2(F)$. For $j \in \{1, 2\}$ the estimate $d_2(J_j) \leq m_2(F)$ yields

$$|E(J_j)| - 1 \leq (|V(J_j)| - 2) \cdot m_2(F).$$

By adding all three inequalities we infer

$$|E(J_1)| + |E(J_2)| - 1 - \zeta \leq (|V(J_1)| + |V(J_2)| - 2 - |q_J|) \cdot m_2(F),$$

i.e., $|E(J)| - 1 \leq (|V(J)| - 2) \cdot m_2(F)$, whence $d_2(J) \leq m_2(F)$. \square

For strictly 2-balanced graphs similar calculations yield the following relation.

Proposition 5.2. *For any union H of two distinct copies of a strictly 2-balanced graph F , which intersect in at least two edges, we have $d_2(H) > d_2(F)$.*

Proof. Consider a graph $H = F_1 \cup F_2$ being the union of two copies F_1 and F_2 of F . We denote by $I = (V(F_1) \cap V(F_2), E(F_1) \cap E(F_2))$ the intersection of both copies and our assumption yields $|E(I)| \geq 2$ and $|V(I)| \geq 3$. Since F is strictly 2-balanced, we have $d_2(I) < d_2(F)$, whence

$$|E(I)| - 1 < (|V(I)| - 2) \cdot d_2(F).$$

By subtracting this from $2(|E(F)| - 1) = 2(|V(F)| - 2) \cdot d_2(F)$ we obtain

$$\begin{aligned} |E(H)| - 1 &= 2|E(F)| - |E(I)| - 1 > (2|V(F)| - |V(I)| - 2) \cdot d_2(F) \\ &= (|V(H)| - 2) \cdot d_2(F), \end{aligned}$$

which implies $d_2(H) > d_2(F)$. \square

We complement Proposition 5.1 with the observation that certain ‘cycles of copies’ have larger 2-density.

Proposition 5.3. *Let F be a graph with $d_2(F) > 1$. Suppose that for some $\ell \geq 3$ we have a cyclic sequence F_1, \dots, F_ℓ of copies of F such that no edge belongs to three of these copies and for all distinct $i, j \in \mathbb{Z}/\ell\mathbb{Z}$ we have*

$$|E(F_i) \cap E(F_j)| = \begin{cases} 1 & \text{if } j = i \pm 1 \\ 0 & \text{else.} \end{cases}$$

Then the union $\mathfrak{C} = \bigcup_{i \in \mathbb{Z}/\ell\mathbb{Z}} F_i$ satisfies $d_2(\mathfrak{C}) > d_2(F)$.

Proof. Notice that $|V(\mathfrak{C})| \leq \ell(|V(F)| - 2) + 1$ and $|E(\mathfrak{C})| = \ell(|E(F)| - 1)$. So by subtracting $d_2(F) > 1$ from $\ell(|V(F)| - 2) \cdot d_2(F) = \ell(|E(F)| - 1)$ we obtain

$$(|V(\mathfrak{C})| - 2) \cdot d_2(F) < |E(\mathfrak{C})| - 1,$$

whence $d_2(F) < d_2(\mathfrak{C})$. □

The following observation appeared in slightly stronger form in the work of Rödl and Ruciński [11] and it establishes a link between the 2-density of F and the 2-density of its Ramsey graphs.

Proposition 5.4. *For every graph F containing a cycle and every graph G satisfying the Ramsey property $G \xrightarrow{n.n.i.} (F)_2$ we have $m_2(G) > m_2(F)$.*

Proof. Since any given graph F containing a cycle satisfies $m_2(F) > 1$, there exists a strictly 2-balanced subgraph $F_\star \subseteq F$ with $d_2(F_\star) = m_2(F)$ and we fix such an F_\star .

Assume for the sake of contradiction that $m_2(G) \leq m_2(F) = d_2(F_\star)$. This implies that all copies of F_\star in G are induced. We consider the auxiliary $|E(F_\star)|$ -uniform hypergraph A with vertex set $V(A) = E(G)$ whose hyperedges correspond to the copies of F_\star in G . In view of Propositions 5.2 and 5.3, the hypergraph A is linear and contains no cycles. Consequently, A admits a 2-colouring of its vertices without monochromatic hyperedges. However, this colouring signifies $G \xrightarrow{n.n.i.} (F_\star)_2$, which is absurd. □

Together with Proposition 5.1 this has the following m_2 -free consequence.

Corollary 5.5. *For every graph F containing a cycle every forest \mathcal{F} of copies of F satisfies $\bigcup \mathcal{F} \xrightarrow{n.n.i.} (F)_2$.* □

5.2. Proofs of the theorems. In this section we deduce Theorem 1.5 and Theorem 1.8. Both results are consequences of the girth Ramsey theorem combined with Corollary 5.5.

Proof of Theorem 1.5. Given a graph F that is not a forest and integers $r \geq 2$ and $n \geq 1$ let the graph G and a system of copies \mathcal{G} be provided by Theorem 1.2 applied with F , r , and $\ell = \binom{n}{2}$. Without loss of generality we may assume $G = \bigcup \mathcal{G}$ and obviously G enjoys the Ramsey property $G \rightarrow (F)_r$.

Let $H \subseteq G$ be a subgraph on n vertices and for every edge e of H fix a copy F_e of F from the system \mathcal{G} containing e . Theorem 1.2 tells us that the set $\{F_e : e \in E(H)\}$ is contained in a forest \mathcal{F} of copies of F and by definition we have $H \subseteq \bigcup \mathcal{F}$. On the other hand, Corollary 5.5 implies $\bigcup \mathcal{F} \xrightarrow{n, \eta, 1} (F)_2$. \square

Proof of Theorem 1.8. Suppose there are only finitely many minimal Ramsey graphs for a given graph F containing a cycle and that $n \in \mathbb{N}$ exceeds the number of vertices of each such graph. By Theorem 1.5 applied to F , $r = 2$, and n there is a Ramsey graph $G \rightarrow (F)_2$ containing none of the minimal Ramsey graphs for F , which is absurd. \square

§6 RAMSEY GRAPHS WITH LARGE FREE SUBGRAPHS

We deduce Theorem 1.6 from the girth Ramsey theorem in form of Corollary 3.4.

Proof of Theorem 1.6. Let $P = x_0 \dots x_\ell$ be a longest path in F and fix some total order $<_F$ of the vertex set of F such that the path is monotone $x_0 <_F \dots <_F x_\ell$. We shall show that for every integer $r \geq 2$ the ordered Ramsey graph G provided by Corollary 3.4 applied to F , r , and $n = 1$ has the desired property.

Let $H \subseteq G$ be an arbitrary subgraph of G . By a standard averaging argument we obtain an ℓ -partite subgraph H_0 with vertex partition $V_1 \cup \dots \cup V_\ell$ such that

$$|E(H_0)| \geq \left(1 - \frac{1}{\ell}\right) |E(H)|.$$

We partition the edge set of H_0 into two classes depending on whether the vertex ordering $<_G$ induced by G complies with the labelling of the vertex classes of H_0 or not, i.e., we consider

$$\bar{E} = \{xy \in E(H_0) : x <_G y \text{ and } x \in V_i, y \in V_j \text{ for some } i < j\} \quad \text{and} \quad \bar{\bar{E}} = E(H_0) \setminus \bar{E}.$$

Note that neither the edges of \bar{E} nor the edges of $\bar{\bar{E}}$ can form a monotone path with $\ell + 1$ vertices. However, owing to property (ii) of Corollary 3.4, every copy of F in G must contain such a monotone path on $\ell + 1$ vertices. Consequently, both subgraphs $(V(H), \bar{E})$ and $(V(H), \bar{\bar{E}})$ are F -free and one of them contains at least

$$\frac{1}{2} |E(H_0)| \geq \frac{\ell - 1}{2\ell} |E(H)|$$

edges and may serve as the promised subgraph $H' \subseteq H$. \square

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