TIGHT HAMILTONICITY FROM DENSE LINKS OF TRIPLES

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Abstract. We show that for all $k \geqslant 4$, $\varepsilon > 0$, and n sufficiently large, every k-uniform hypergraph on n vertices in which each set of $k-3$ vertices is contained in at least $(5/8+\varepsilon)\binom{n}{3}$ edges contains a tight Hamilton cycle. This is asymptotically best possible.

§1 Introduction

Our starting point is Dirac's theorem, which states that any graph G on $n \geqslant 3$ vertices and minimum degree $\delta(G) \geqslant n/2$ contains a Hamilton cycle. Moreover, the constant $1/2$ is best possible as exhibited by simple constructions.

Over the past twenty-five years, this result has been extended to the hypergraph setting. Formally, a *k-uniform hypergraph* (*k-graph* for short) G has a set of *vertices* $V(G)$ and a set of *edges* $E(G)$, where each edge consists of k vertices, and we denote the number of edges $|E(G)|$ by $e(G)$. For $1 \leqslant d \leqslant k-1$, the *minimum d-degree of* G, denoted $\delta_d(G)$, is the maximum m such that every set of d vertices is contained in at least m edges. A *tight cycle* $C \subseteq G$ is a subgraph whose vertices are cyclically ordered such that every k consecutive vertices form an edge. Moreover, C is *Hamilton* if it spans all the vertices of G. We define the *Dirac constant* $\hbar_d^{(k)}$ as the (asymptotic) minimum d-degree threshold for tight Hamiltonicity. More precisely, $\hbar_d^{(k)}$ is the infimum $\varsigma \in [0,1]$ such that for every $\varepsilon > 0$ and n sufficiently large, every n-vertex k-graph G with $\delta_d(G) \geqslant (\varsigma + \varepsilon)\binom{n-d}{k-d}$ contains a tight Hamilton cycle. So for example, Dirac's theorem implies that $\hbar_1^{(2)} = 1/2$.

Minimum d-degree thresholds for tight Hamilton cycles were first investigated by Katona and Kierstead [14], who observed that $\hbar_{k-1}^{(k)} \geqslant 1/2$ for all $k \geqslant 2$ and conjectured this to be tight (see Figure 1). This conjecture was resolved by Rödl, Ruciński, and Szemerédi [22,23] by introducing the *absorption method* in this setting. Since then the focus has shifted to degree types d below $k-1$. After advances for nearly spanning cycles by Cooley and Mycroft [3], it was shown by Reiher, Rödl, Ruciński, Schacht, and Szemerédi [21] that $\hbar_1^{(3)} = 5/9$, which resolves the case of $d = k-2$ when $k = 3$. Subsequently, this was generalised to $k = 4$ [19] and finally, Polcyn, Reiher, Rödl, and Schülke [20] and, independently, Lang and Sanhueza-Matamala [15] established $\hbar_{k-2}^{(k)} = 5/9$ for all $k \geqslant 3$.

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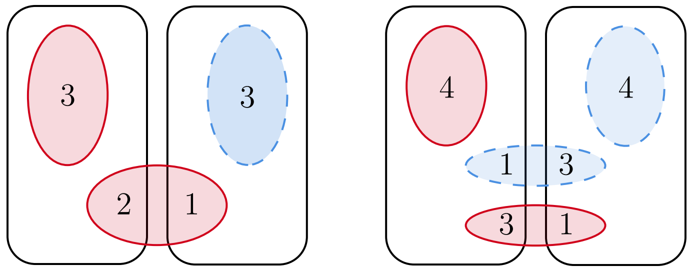

FIGURE 1. The picture shows a 3-graph on the left and a 4-graph on the right. Both hypergraphs have their vertex sets partitioned into two parts of equal size. The drawn edges indicate that the graphs contain all edges of this type. The colours highlight the respective tight components. Since any tight cycle is either red or blue, neither of the hypergraphs admits a tight Hamilton cycle. This gives a lower bound for the corresponding minimum degree thresholds. Indeed, the 3-graph has a relative minimum 2-degree close to $1/2$, while the 4-graph has relative minimum 1-degree close to $5/8$.

We focus on the case $d = k - 3$. Han and Zhao [12] provided a construction that shows $\hbar^{(k)}_{k-3} \geqslant 5/8$ (see Figure 1), which is believed to be optimal. Our main result confirms this conjecture.

Theorem 1.1. *For every $k \geqslant 4$ and $\varepsilon > 0$, there is n_0 such that every k-graph G on $n \geqslant n_0$ vertices with $\delta_{k-3}(G) \geqslant (5/8 + \varepsilon)\binom{n}{3}$ contains a tight Hamilton cycle.*

In the following section, we give an outline of the argument and reduce Theorem 1.1 to two lemmata, whose proofs are given in Sections 3 and 4. We conclude with a discussion and a few open problems in Section 5.

§2 FROM DIRAC TO ERDŐS–GALLAI

Let us recall two classic problems from extremal combinatorics. The first one is Erdős' Matching Conjecture [4], which predicts the size of a largest matching we are guaranteed to find in a k-graph with a given number of vertices and edges. A matching is a subgraph with pairwise disjoint edges.

Conjecture 2.1. *Let G be a k-graph on n vertices that does not contain a matching of more than s edges. Then*

$$e(G) \leqslant \max\left\{\binom{(s+1)k-1}{k}, \binom{n}{k} - \binom{n-s}{k}\right\}.$$

The conjecture was resolved by Erdős and Gallai [5] for $k = 2$ and for $k = 3$ by Łuczak and Mieczkowska [18] (when n is large) and Frankl [7] (for all n). The second problem concerns the Erdős–Gallai Theorem [5], which tells us the size of a longest cycle in a graph of given density. This problem has a natural extension to tight cycles in hypergraphs, and it was first studied by Győri, Katona, and Lemons [11] and Allen, Böttcher, Cooley, and Mycroft [1].

How are these questions related to Dirac-type problems? The idea is to study the internal structure of the neighbourhoods. Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov [2] showed that the problem of determining the minimum d-degree threshold for k-uniform perfect matchings can be reduced to a special case of Erdős' Matching Conjecture for $(k-d)$-graphs, applied to the $(k-d)$-uniform link hypergraphs. Analogously, it was shown by Lang and Sanhueza-Matamala [15] that the problem of determining the minimum d-degree threshold for k-uniform tight Hamilton cycles can be reduced to an Erdős–Gallai-type question for $(k-d)$-uniform link hypergraphs.

Using a standard machinery [1], we can decompose the problem further. For a k-graph G, let G^* be the *dual graph* on the vertex set $E(G)$ with an edge ef whenever $|e \cap f| = k-1$. A subgraph $H \subseteq G$ without isolated vertices is said to be *tightly connected* if $E(H)$ induces a connected subgraph in G^*. Moreover, we refer to edge maximal tightly connected subgraphs as *tight components*.

Theorem 2.2 (Lang and Sanhueza-Matamala [15, Theorem 11.5])**.** *Suppose for every $\varepsilon > 0$, there are $\gamma > 0$ and n_0 such that every 3-graph G on $n \geqslant n_0$ vertices with $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$ contains a subgraph $C \subseteq G$ such that*

(*i*) *C is tightly connected,*
(*ii*) *$e(C) \geqslant (1/2+\gamma)\binom{n}{3}$, and*
(*iii*) *C has a matching of size at least $(1/4+\gamma)n$.*

Then $\hbar^{(k)}_{k-3} = 5/8$ for every $k \geqslant 4$. □

Against this backdrop, our argument proceeds as follows. Let G be an n-vertex 3-graph with $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$. In a first step, we establish the existence of a tight component $C \subseteq G$ that satisfies conditions (*i*) and (*ii*) of Theorem 2.2.

Lemma 2.3 (Connection)**.** *For all $\varepsilon > 0$, there exists n_0 such that every 3-graph G on $n \geqslant n_0$ vertices with $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$ contains a tight component $C \subseteq G$ with $e(C) \geqslant (1/2+\varepsilon)\binom{n}{3}$.*

We then show that any such tight component C provided by Lemma 2.3 contains a large matching satisfying condition (*iii*) of Theorem 2.2.

Lemma 2.4 (Matching)**.** *For all $\varepsilon > 0$, there exist $\gamma > 0$ and n_0 such that every 3-graph G on $n \geqslant n_0$ vertices with $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$ and a tight component $C \subseteq G$ with $e(C) \geqslant (1/2+\varepsilon)\binom{n}{3}$ contains a matching $M \subseteq C$ of size at least $(1/4+\gamma)n$.*

The proof of Lemma 2.3 can be found in the next section. It is an easy consequence of a Kruskal–Katona-type result obtained, independently, by Frankl, Kato, Katona, and Tokushige [8] and by Huang, Linial, Naves, Peled, and Sudakov [13]. For the proof of Lemma 2.4, presented in Section 4, we adapt a strategy of Łuczak and Mieczkowska [18] from their proof of Erdős' Matching Conjecture for 3-graphs.

§3 The connection lemma

We start by stating the aforementioned Kruskal–Katona-type result of Frankl, Kato, Katona, and Tokushige [8] and Huang, Linial, Naves, Peled, and Sudakov [13]. We denote the complete graph on n vertices by K_n.

Theorem 3.1. *For every $\varepsilon > 0$, there exists n_0 such that for every $n \geqslant n_0$ the following holds. Suppose that the edges of a subgraph of K_n are coloured red and blue such that there are at least $\binom{n}{3}/8$ monochromatic triangles in each colour. Then there are fewer than $(5/8+\varepsilon)\binom{n}{3}$ monochromatic triangles all together.* □

Below we derive Lemma 2.3 from Theorem 3.1.

Proof of Lemma 2.3. For given $\varepsilon > 0$, we set n_0 according to Theorem 3.1. Let G be a 3-graph on $n \geqslant n_0$ vertices with $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$. Without loss of generality, we may assume $\varepsilon < 1/16$ and $e(G) \leqslant (5/8+2\varepsilon)\binom{n}{3}$.

Let $C_1, \dots, C_\ell$ be the tight components of G with $e(C_1) \geqslant \dots \geqslant e(C_\ell)$. Aiming for a contradiction, we impose that $e(C_1) < (1/2+\varepsilon)\binom{n}{3}$. We shall group the tight components of G into a partition $R \mathbin{\dot\cup} B = E(G)$ such that

$$\max\left\{|B|, |R|\right\} < \left(\frac{1}{2}+\varepsilon\right)\binom{n}{3} \tag{3.1}$$

and no edge of R intersects with any edge of B in more than one vertex. Indeed, considering the smallest integer i such that

$$e(C_1) + \dots + e(C_i) \geqslant \left(\frac{1}{2}+\varepsilon\right)\binom{n}{3}$$

and recalling the upper bound on $e(G)$ tells us

$$e(C_{i+1}) + \dots + e(C_\ell) \leqslant \left(\frac{1}{8}+\varepsilon\right)\binom{n}{3}.$$

Note that $e(C_i) < (3/8)\binom{n}{3}$, since otherwise the monotonicity of $e(C_j)$ combined with the upper bound $e(G) \leqslant (5/8+2\varepsilon)\binom{n}{3}$ implies $i = 1$, which contradicts the choice of i. Consequently, $R = E(C_1) \cup \dots \cup E(C_{i-1})$ and $B = E(C_i) \cup \dots \cup E(C_\ell)$ form the desired partition $R \mathbin{\dot\cup} B = E(G)$.

Now observe that inequality (3.1) and the given lower bound $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$ yields

$$\frac{1}{8}\binom{n}{3} \leqslant \min\left\{|B|, |R|\right\}.$$

From this, we derive an edge-colouring of K_n by giving an edge colour red if it is contained in a triple of R and colour blue if it is contained in a triple of B. (The remaining edges are coloured arbitrarily.) By Theorem 3.1 it follows that there are fewer than $(5/8+\varepsilon)\binom{n}{3}$ monochromatic triangles. But this contradicts the assumption that $e(G) \geqslant (5/8+\varepsilon)\binom{n}{3}$. □

§4 The matching lemma

In this section, we establish Lemma 2.4. We proceed by studying the extremal function for matchings hosted by a largest tight component in a 3-graph G. If G has edge density above 5/8, then we are guaranteed by Lemma 2.3 a tight component C of density at least 1/2. Applying Erdős' Matching Conjecture (Conjecture 2.1) to C gives a matching of size at least $(1-2^{-1/3})n \approx 0.206n$, which does not suffice for the aspirations set in Lemma 2.4. However, this naïve approach turns out to be suboptimal, because there might be edges in G that lie outside of C, which contribute indirectly to the size of a largest matching by obstructing the space. The extremal constructions arising from these restrictions are no longer captured by Erdős' Matching Conjecture, and we thus require a more nuanced analysis to deduce Lemma 2.4.

In Section 4.1 we formulate the corresponding extremal problem, and we reduce Lemma 2.4 to it (see Lemma 4.2). The rest of Section 4 is devoted to the proof of Lemma 4.2. The proof is based on the approach of Łuczak and Mieczkowska [18] to show the Erdős' Matching Conjecture for 3-graphs. In particular, we also employ the *shifting technique*, which is described in Section 4.2.

4.1. **Extremal function for matchings in tight components.** We write $m(R)$ for the size of a largest matching in a 3-graph R. We call 3-graphs R and B *distinguishable* if the edges of R and B only intersect in single vertices. So in particular, distinct tight components are distinguishable.

Definition 4.1. We define $\mathcal{G}(n,s,t)$ as the family of all pairs (R,B) such that R and B are distinguishable 3-graphs on the vertex set $\{1,\dots,n\}$ with $m(R) \leqslant s$ and $e(R) > t$.

Moreover, we define the extremal number

$$\mu(n,s,t) = \max\left\{e(R \cup B)\colon (R,B) \in \mathcal{G}(n,s,t)\right\},$$

and we denote the family of extremal pairs by

$$\mathcal{M}(n,s,t) = \left\{(R,B) \in \mathcal{G}(n,s,t)\colon e(R \cup B) = \mu(n,s,t)\right\}.$$

In view of Lemma 2.4, we are interested in $\mu(n,s,t)$ for $s \approx n/4$ and $t \approx \binom{n}{3}/2$, which is rendered by the main lemma of this section.

Lemma 4.2. *For each $\gamma > 0$, we have $\mu\big(n, n/4, \binom{n}{3}/2\big) \leqslant (5/8+\gamma)\binom{n}{3}$ for sufficiently large n.*

Below we deduce Lemma 2.4 as a simple consequence of Lemma 4.2 and, consequently, for the proof of Theorem 1.1 it then only remains to establish Lemma 4.2.

Proof of Lemma 2.4. For a given $\varepsilon > 0$, we set $\gamma = \varepsilon/11$ and let n_0 be sufficiently large. Given G and a tight component C satisfying the assumptions of Lemma 2.4, we first fix an arbitrary matching $M' \subseteq C$ of size γn.

Set $R = C - V(M')$, and let B be obtained from $G - V(M')$ by deleting all edges of R. After adding isolated vertices, we can assume that R and B are 3-graphs on the vertex set $\{1, \dots, n\}$ with $e(R) > e(B)$ and

$$e(R \cup B) \geqslant e(G) - 3\gamma n \cdot \binom{n}{2} \geqslant \left(\frac{5}{8} + \varepsilon - 10\gamma\right)\binom{n}{3} = \left(\frac{5}{8} + \gamma\right)\binom{n}{3}$$

for sufficiently large n. Moreover, R and B are distinguishable, since C is a tight component. It follows that R contains a matching M'' of size $n/4$ by Lemma 4.2 and $M = M' \cup M''$ is the desired matching in C. □

4.2. **Shifting.** For a k-graph G with $i,\ j \in V(G)$, the (i,j)-*shift* of G, denoted by $\mathrm{Sh}_{i\to j}(G)$, is obtained from G by replacing each edge $e \in E(G)$ with

$$f = (e \smallsetminus \{i\}) \cup \{j\}$$

provided that

$$i \in e\,, \quad j \notin e \quad \text{and} \quad f \notin E(G)\,.$$

We mainly consider 3-graphs here. However, we will also study the *shadow* ∂G of a 3-graph G, defined as the 2-graph on $V(G)$ containing an edge $e \in E(\partial G)$ whenever there is a triple $f \in E(G)$ with $e \subseteq f$. In our analysis for the proof of Lemma 4.2, we shall use the relation of shifted 3-graphs and its shadow, and for that we defined the (i,j)-shift for k-graphs in general. Below we collect the basic facts about shifted 3-graphs for our proof. For a more comprehensive review we refer to the survey of Frankl [6] and the monograph of Frankl and Tokushige [9].

Lemma 4.3. *For every 3-graph G with $i,\ j \in V(G)$ and distinguishable subgraphs $R,\ B \subseteq G$, the following holds:*

- *(a) $e(\mathrm{Sh}_{i\to j}(G)) = e(G)$,*
- *(b) $m(\mathrm{Sh}_{i\to j}(G)) \leqslant m(G)$, and*
- *(c) $\mathrm{Sh}_{j\to i}(R)$ and $\mathrm{Sh}_{i\to j}(B)$ are distinguishable.*

Proof. Assertion *(a)* follows from the definition of an (i,j)-shift and property *(b)* is a well known fact [18, Lemma 3].

For the proof of *(c)*, we assume by contradiction that there are edges $e \in E(\mathrm{Sh}_{j\to i}(R))$ and $f \in E(\mathrm{Sh}_{i\to j}(B))$ with $|e \cap f| \geqslant 2$. Since R and B are distinguishable, we may assume by symmetry that $e \notin E(R)$, writing $e = uvi$ and $e' = uvj \in E(R)$.

If $uv \subseteq f$, then there would be an edge $uvw \in E(B)$ for some $w \in V(G)$, which contradicts the distinguishness of R and B. Consequently, without loss of generality we have $f = u'vi$ for some $u' \neq u$. If $u' = j$, then f is also an edge of B, and f and e' contradict the distinguishability of R and B. In the remaining case $u' \neq j$, we arrive at $u'vj \in E(B)$, which again contradicts the assumed distinguishedness. □

As usual we shall study 3-graphs (and their shadow), which are fully shifted in one direction. More precisely, we say a k-graph G on the vertex set $\{1, \dots, n\}$ is *left-shifted* if $\mathrm{Sh}_{j\to i}(G) = G$ for all $i < j$ and, similarly, it is *right-shifted* if $\mathrm{Sh}_{i\to j}(G) = G$ for all $i < j$. It is easy to see that we can obtain a left-shifted k-graph from any given G after a finite sequence of (j, i)-shifts with $i < j$. Combining this fact with Lemma 4.3 tells us that there are shifted extremal examples in $\mathcal{M}(n, s, t)$.

Corollary 4.4. *For all integers n, s, and t there is a pair $(R, B) \in \mathcal{M}(n, s, t)$ such that R and ∂R are left-shifted, while B and ∂B are right-shifted.*

Proof. Consider an arbitrary pair $(R', G') \in \mathcal{M}(n, s)$ and let $1 \leqslant i < j \leqslant n$. It follows from Lemma 4.3 *(a)*–*(c)* that $\big(\mathrm{Sh}_{j\to i}(R'), \mathrm{Sh}_{i\to j}(B')\big)$ is also in $\mathcal{M}(n, s, t)$. Since the degree of the vertex j in $\mathrm{Sh}_{j\to i}(R')$ is smaller than in R' if $R' \neq \mathrm{Sh}_{j\to i}(R')$ and, similarly, the degree increases in $\mathrm{Sh}_{i\to j}(B')$ if $B' \neq \mathrm{Sh}_{i\to j}(B')$, it follows that after a finite sequence of such simultaneous (j, i)- and (i, j)-shifts in the respective subgraphs, we arrive at a pair $(R, B) \in \mathcal{M}(n, s, t)$ with R being left-shifted and B being right-shifted.

Finally, let us argue that ∂R is left-shifted. To this end, consider $i < j$ and $vj \in E(\partial R)$. Consequently, there is some vertex u such that $uvj \in E(R)$. Since R is left-shifted, we have either have $u = i$ or $uvi \in E(R)$, but in both cases we arrive at $vi \in E(\partial R)$, and this shows that ∂R is left-shifted indeed. The argument for ∂B follows analogously. □

4.3. **Integrality.** In the proof of Lemma 4.2, we consider weighted graphs and the case analysis will be simplified by moving from arbitrary weights to integer weights. For that we shall appeal to Kőnig's theorem for bipartite graphs in the form stated below. A *fractional independent set* in a graph F is a function $\iota\colon V(F) \longrightarrow [0, 1]$ such that $\iota(u) + \iota(v) \leqslant 1$ for all edges uv of F. The *size* of ι is $\sum_{v\in V(F)} \iota(v)$. We denote by $\alpha^*(F)$ the maximum size of a fractional independent set in F. Since every independent set in F can be interpreted as a fractional independent set taking integral values, it follows that $\alpha^*(F)$ is at least the independence number $\alpha(F)$. For bipartite graphs, this turns out to be tight.

Theorem 4.5 (Kőnig's theorem)**.** *Every bipartite graph F has $\alpha^*(F) = \alpha(F)$.* □

4.4. **Optimisation.** In the proof of Lemma 4.2 we repeatedly evaluate a certain function. To this end, we formulate the following remark that can be easily verified.

Fact 4.6. *For $\sigma > 0$, let $\beta = 1/\sigma - 3$ and $f_{s,p,t}(\sigma) = \sigma^3 (\beta^3 + s\beta^2 + p\beta + t)$. We have $f_{s,p,t}(\sigma) \leqslant 5/8$ for $1 - 2^{-1/3} \leqslant \sigma \leqslant 1/4$ and any triple of coefficients (s, p, t) among $(6, 10, 23)$, $(6, 11, 22)$, $(6, 12, 21)$, $(7, 8, 21)$, $(9, 3, 27)$, $(9, 7, 21)$, and $(9, 9, 17)$.*

4.5. **Proof of Lemma 4.2.** For a (not necessarily uniform) hypergraph H and subsets S, $W \subseteq V(G)$, we denote by $\deg_H(S; W)$ the number of edges $S \cup Y$ in H with $Y \subseteq W$. To emphasise (or specify) the uniformity of an edge in a hypergraph, we sometimes speak of an (unordered) triple, pair or singleton.

Proof of Lemma 4.2. Given $\gamma > 0$, let n be sufficiently large and consider a distinguishable pair $(R, B) \in \mathcal{M}(n, n/4, \binom{n}{3}/2)$. We have to show that $e(R \cup B) \leqslant (5/8 + \gamma)\binom{n}{3}$.

By Corollary 4.4 we can assume that R, ∂R are left-shifted, while B, ∂B are right-shifted. We refer to the edges of R, ∂R and B, ∂B as *red* and *blue*, respectively. Let $M = \{(i_\ell, j_\ell, k_\ell) \colon 1 \leqslant \ell \leqslant s\}$ be a largest matching in R with $i_\ell < j_\ell < k_\ell$ for each ℓ. By the solution of Erdős' Matching Conjecture (Conjecture 2.1) for 3-graphs [7, 18], we can assume that $s = \sigma n$ for some σ satisfying

$$\left(1 - 2^{-1/3}\right) \leqslant \sigma \leqslant \frac{1}{4}\,.$$

We partition the vertex set of M into three parts

$$V(M) = I \cup J \cup K$$

such that for every edge $(i, j, k) \in M$, we have $i \in I$, $j \in J$, and $k \in K$. A set of vertices is called *crossing* if it contains at most one vertex of every matching edge of M. (This includes singletons, naturally.)

The remainder of the argument proceeds by analysing the local configurations of the matching M. More precisely, we give an upper bound to the number of edges each triple of matching edges may intersect with. By double counting, this allows us to bound the number of edges in $R \cup B$.

We define an auxiliary hypergraph H as follows. Denote by W the set of vertices that are not covered by M. Obviously, none of the edges of R are contained in W. Let H be the (non-uniform) hypergraph on $V(M)$ with edge set $M \cup E_1 \cup E_2 \cup E_3$, where

$$\begin{aligned}
E_1 &= \{u \colon \deg_R(u; W) \geqslant 20n\}\,,\\
E_2 &= \{uv \colon uv \text{ is crossing and } \deg_R(uv; W) \geqslant 20\}\,,\\
E_3 &= \{uvw \colon uvw \text{ is crossing}\}\,.
\end{aligned}$$

Note that since R is left-shifted, the hypergraphs with edges E_1, E_2, and E_3 are each left-shifted as well, due to the monotonicity of the degrees. So for instance, $i_1 j_2 \in E_2$ implies that $i_1 i_2 \in E_2$. We call an edge e of R *supported* if $e \cap V(M) \in E(H)$. Observe that the number of unsupported edges of R is at most quadratic in n. Hence, this number can be bounded by $\frac{\gamma}{3}\binom{n}{3}$ for sufficiently large n, which allows us to 'ignore' the unsupported edges. We shall track the following relative degrees

$$\overline{\deg}_R(u) = \frac{\deg_R(u; W)}{\binom{|W|}{2}}\,, \qquad \overline{\deg}_R(uv) = \frac{\deg_R(uv; W)}{|W|}\,,$$

$$\overline{\deg}_B(u) = \frac{\deg_B(u; W)}{\binom{|W|}{2}}\,, \qquad \overline{\deg}_B(uv) = \frac{\deg_B(uv; W)}{|W|}\,.$$

Now consider a triple T of matching edges from M. We denote the crossing subsets of $V(T)$ by $\mathrm{cr}(T)$. Observe that every singleton of $V(M)$ appears in exactly $\binom{|M|-1}{2}$ such triples, while every crossing pair of $V(M)$ appears in exactly $|M|-2$ such triples. Moreover, every crossing triple appears of course only in a single triple. This leads to the following definitions

$$e_1(T) = \frac{\binom{|W|}{2}}{\binom{|M|-1}{2}} \sum_{u\in \mathrm{cr}(T)} \overline{\deg}_R(u) + \overline{\deg}_B(u)\,,$$

$$e_2(T) = \frac{|W|}{|M|-2} \sum_{uv\in \mathrm{cr}(T)} \overline{\deg}_R(uv) + \overline{\deg}_B(uv)\,,$$

$$e_3(T) = |\mathrm{cr}(T) \cap E(R\cup B)|\,.$$

Given this setup, we double count the edges of $R\cup B$ along the triples T of matching edges from M to obtain

$$\begin{aligned} e(R\cup B) &\leqslant \sum_{T\in\binom{M}{3}} \big(e_1(T)+e_2(T)+e_3(T)\big) + \binom{|W|}{3} + \frac{\gamma}{3}\binom{n}{3} \\ &= \sum_{T\in\binom{M}{3}} \left(e_1(T)+e_2(T)+e_3(T) + \frac{\binom{|W|}{3}}{\binom{|M|}{3}}\right) + \frac{\gamma}{3}\binom{n}{3} \\ &\leqslant \sum_{T\in\binom{M}{3}} \left(e_1(T)+e_2(T)+e_3(T) + \left(\frac{1}{\sigma}-3\right)^3\right) + \frac{\gamma}{2}\binom{n}{3}\,, \end{aligned}$$

where we used $|M| = \sigma n$ and $|W| = (1-3\sigma)n$ in the last inequality.

For the remainder, it suffices to show that $e_1(T)+e_2(T)+e_3(T)+(1/\sigma-3)^3 \leqslant (5/8+\gamma/3)/\sigma^3$ for every triple T of matching edges in M. Indeed, for sufficiently large n, this leads to our desired bound

$$e(R\cup B) \leqslant \left(\frac{5}{8}+\frac{\gamma}{3}\right)\frac{1}{\sigma^3}\binom{|M|}{3} + \frac{\gamma}{2}\binom{n}{3} \leqslant \left(\frac{5}{8}+\frac{5}{6}\gamma\right)\binom{n}{3}\,. \tag{4.1}$$

Unfortunately, not every triple of matching edges abides, since some of them may exhibit a rather extrovert degree structure. We capture this by calling a triple T *expanding*, if $V(T)$ contains three pairwise disjoint edges of H whose union intersects I in at most 2 vertices, and *steady* otherwise. Fortunately, it turns out that there cannot be too many expanding triples due to the maximality of the matching, which was already observed by Łuczak and Mieczkowska [18, Claim 4]. We note that they call expanding triples *bad* and steady triples *good*. For the sake of completeness, let us spell out their argument.

Claim 4.7. *No three disjoint triples are expanding.*

Proof. Suppose that there exist 9 disjoint edges $\{(i_\ell, j_\ell, k_\ell) \colon 1\leqslant \ell\leqslant 9\}\subseteq M$ such among their vertices one can find a set of 9 pairwise disjoint edges $H'\subseteq H$, which do not cover the vertices i_3, i_6, and i_9.

Without loss of generality we may assume that i_3 is the minimum among i_3, i_6, and i_9. So in particular, we have $i_6 < i_9 < j_9 < k_9$. Since $e = i_9j_9k_9$ is an edge of the left-shifted hypergraph R, it follows that $e' = i_6i_9j_9$ is also an edge of R by considering an (k_9, i_6)-shift for e. Similarly, the fact that $i_3 < i_6 < i_9 < j_9$ and considering an (j_9, i_3)-shift of e' tells us that $e'' = i_3i_6i_9$ is in R. Therefore, we find 10 pairwise disjoint edges $H'' = H' \cup \{e''\} \subseteq H$.

Furthermore, since edges from $E_1 \cup E_2$ have large degrees, all edges from H'' which belong to $E_1 \cup E_2$ can be simultaneously extended to disjoint edges of R by adding to them vertices from W. But this would lead to a matching M' of size $|M| + 1$ in R contradicting the assumption $(R, B) \in \mathcal{M}_3(n, n/4, \binom{n}{2}/2)$. □

As a consequence of Claim 4.7, there exist six or fewer edges in the matching M such that each expanding triple contains one of these edges. Given the slack in our estimate (4.1), we can hence focus on steady triples. For the remainder of the argument, we may therefore fix a steady triple

$$T = \{M_1, M_2, M_3\}$$

of matching edges in M that maximises the sum $e_1(T) + e_2(T) + e_3(T)$ and reaffirm our goal to show

$$\sigma^3\big(e_1(T) + e_2(T) + e_3(T)\big) + (1 - 3\sigma)^3 \leqslant \frac{5}{8} + \frac{\gamma}{3}\,. \tag{4.2}$$

We begin with two structural observations. The first is a simple consequence of shiftedness.

Claim 4.8. *Every crossing pair of $I \cup J$ is an edge of ∂R. Moreover, for any two matching edges (i_ℓ, j_ℓ, k_ℓ) and (i_p, j_p, k_p) of M with $k_\ell < k_p$, the edges of ∂B between the two form a star centred in k_p.*

Proof. Since $k_\ell < k_p$, the edge j_pk_ℓ is in ∂R. (Otherwise, we could perform a (k_p, k_ℓ)-shift on ∂R replacing j_pk_p with j_pk_ℓ.) Since ∂R is left-shifted, it follows that $j_\ell j_p$, $i_\ell j_p$, i_pj_ℓ, and $i_\ell i_p$ are also in ∂R. Note that this implies in particular that the only possible edges of ∂B are incident to k_p. □

The next observation, proved by Łuczak and Mieczkowska [18, Claim 5], is a consequence of steadiness.

Claim 4.9. *The singletons of H are in I. Moreover, H has at most 5 pairs between any two matching edges of T with equality if and only if all 5 pairs intersect I.*

Proof. For the first part, let $j_1 < j_2 < j_3$ and assume that $E_1 \cap V(T)$ is not a subset of I. Then, since the singletons of H are shifted to the left, i_1, $j_1 \in E_1$, and T is an expanding triple because of the edges i_1, j_1, and $i_2j_2k_2$, a contradiction.

Let us assume by contradiction that 6 pairs of H are contained in $\{i_1, j_1, k_1, i_2, j_2, k_2\}$. Then $j_1j_2 \in E_2$ and at least one of the edges i_1k_2 or i_2k_1 is in E_2, say it is i_1k_2. Then, T is expanding because of the edges j_1j_2, i_1k_2, and $i_3j_3k_3$. □

Next, we argue that the relative degrees appearing in the expressions $e_1(T)$ and $e_2(T)$ can be assumed to take integral values (see Claim 4.11 below). To this end, we record the following constraints, which come from the fact that R and B are distinguishable.

Claim 4.10. *For $u, v, w \in V(T)$, we have*

$$\overline{\deg}_R(u) + \overline{\deg}_B(v) \leqslant 1\,, \qquad \overline{\deg}_R(uv) + \overline{\deg}_B(u) \leqslant 1\,,$$
$$\overline{\deg}_R(u) + \overline{\deg}_B(uw) \leqslant 1\,, \qquad \overline{\deg}_R(uv) + \overline{\deg}_B(uw) \leqslant 1\,.$$

Proof. We focus on the first case, the others follow similarly. Note that since R and B are distinguishable, there are no two vertices x, y such that uxy is in R and vxy is in B. So in particular, $\deg_R(u; W) + \deg_B(v; W) \leqslant \binom{|W|}{2}$, which gives $\overline{\deg}_R(u) + \overline{\deg}_B(v) \leqslant 1$. □

To simplify the analysis, we assume that the functions $\overline{\deg}_R(\cdot)$ and $\overline{\deg}_B(\cdot)$ only have to satisfy the constraints of Claim 4.10. Moreover, in the case that $\overline{\deg}_R(uv)|W| < 20$ or $\overline{\deg}_R(u)\binom{|W|}{2} < 20n$, we set these two values to be zero. To account for the error arising from this perturbance, we mildly amplify our ambitions to bounding the right side of (4.2) with $5/8 + \gamma/4$.

Claim 4.11. *We can assume that $\overline{\deg}_R(\cdot)$ and $\overline{\deg}_B(\cdot)$ take values 0 and 1.*

Proof. Let us introduce a bipartite graph F that captures the constraints of Claim 4.10. The vertices of F consist of red and blue copies of each crossing pair and vertex of T. We put an edge between a red and a blue vertex in F, if the corresponding objects in T are related via one of the constraints of Claim 4.10. Define a function $\iota\colon V(F) \longrightarrow [0, 1]$ by setting $\iota(x) = \overline{\deg}_R(x)$ for red vertices and $\iota(x) = \overline{\deg}_B(x)$ for blue vertices. Note that ι is a fractional independent set in F. We conclude the argument by Kőnig's theorem (Theorem 4.5). □

For the remainder of the proof, we assume that $\overline{\deg}_R(\cdot)$ and $\overline{\deg}_B(\cdot)$ take integral values. In preparation for the conclusive structural analysis, we define a red and blue edge-coloured (non-uniform) hypergraph Q on vertex set $V(T)$ whose edges are contained in the crossing singletons, pairs, and triples of T. Specifically, a singleton u is red if $\overline{\deg}_R(u) = 1$ and blue if $\overline{\deg}_B(u) = 1$. Similarly, a pair uv is red if $\overline{\deg}_R(uv) = 1$ and blue if $\overline{\deg}_B(uv) = 1$. Lastly, a triple uvw is red if $uvw \in E(R)$ and blue if $uvw \in E(B)$.

For $1 \leqslant r \leqslant 3$, we denote the number of edges of uniformity r in Q, regardless of their colour, by $e_r(Q)$. So in particular, $e_3(Q) = e_3(T)$. Note that the red edges of Q are shifted to the left, and the blue edges of Q are shifted to the right. Let us emphasise that red and blue triples in Q intersect in at most one vertex. Moreover, Claim 4.10 translates to the following observation.

Remark 4.12. *Red and blue pairs do not intersect at all, red singletons are not contained in blue pairs (same with reversed colours) and Q cannot have both a red and a blue singleton.*

Recall the definitions of $e_1(T)$, $e_2(T)$, and $e_3(T)$. We set

$$\beta = \frac{1 - 3\sigma}{\sigma} \geqslant 1\,,$$

where the lower bound follows from $\sigma \leqslant 1/4$.

Since $|M| = \sigma n$, $|W| = (1-3\sigma)n$, and n is large, we may approximate

$$\frac{\binom{|W|}{2}}{\binom{|M|-1}{2}} \leqslant \beta^2 + \frac{\gamma}{8\cdot 9} \quad \text{and} \quad \frac{|W|}{|M|-2} \leqslant \beta + \frac{\gamma}{8\cdot 27}\,.$$

We therefore have $e_1(T) + e_2(T) + e_3(T) \leqslant \beta^2 e_1(Q) + \beta e_2(Q) + e_3(Q) + \gamma/4$, and this allows us to recontextualise our mildly amplified goal (4.2) to

$$\sigma^3 \left(\beta^3 + \beta^2 e_1(Q) + \beta e_2(Q) + e_3(Q)\right) \leqslant \frac{5}{8}\,. \tag{4.3}$$

We count the edges of Q by focusing on the structure between two matching edges at a time. For a pair (M_i, M_j) of distinct matching edges in T, let $Q(M_i, M_j)$ be the subgraph of Q on vertex set $M_i \cup M_j$, which only contains those singletons of Q that are in M_i and all pairs of Q between M_i and M_j. Then

$$e(Q) = e(Q(M_1, M_2)) + e(Q(M_2, M_3)) + e(Q(M_3, M_1)) + e_3(Q)\,.$$

To study these terms, let us denote $M_\ell = (i_\ell, j_\ell, k_\ell)$ for $1 \leqslant \ell \leqslant 3$.

Claim 4.13. *We have $e(Q(M_i, M_j)) \leqslant 6$ for each $i \neq j$.*

Proof. We focus on the pair (M_1, M_2) and abbreviate $Q' = Q(M_1, M_2)$. By Claim 4.9, there are at most 5 red pairs between in Q', while the only possible red singleton of Q' is i_1 (due to the one-sided definition). Moreover, by Claim 4.8 the blue pairs form a star centered in k_1 or k_2. So in particular k_1k_2 is a blue pair if Q' has at least one blue pair, due to right-shiftedness. Moreover, there are at most 3 blue pairs and (trivially) at most 3 blue singletons in Q'. We also note that due to Remark 4.12, there is no red pair that intersects with a blue pair and all singletons of Q' have the same colour.

Suppose first that all pairs of Q' are monochromati Should they be blue, then we have $e(Q') \leqslant 3+3$ by the above. If they are red, then we are done as long as Q' has only 1 singleton as there are at most 5 red pairs. On the other hand, if Q' has 2 or 3 singletons (which then must be blue), it follows that Q' has at most 3 or 0 red edges, respectively.

So we can assume that there is a red pair and a blue pair and, in particular, 3 blue singletons are impossible. Due to shiftedness, it follows that i_1i_2 is red and k_1k_2 is blue. So there are at most 4 red pairs, all of which are contained in $I \cup J$. If Q' has at most 1 singleton, we are done. On the other hand, if Q' has 2 singletons (which then must be blue), it follows that Q' has at most 2 red edges and at most 2 blue edges and this completes the proof. □

Without loss of generality, we assume for the rest of the proof that $k_1 < k_2 < k_3$.

Claim 4.14. *Each of the blue triples of Q contains both k_2 and k_3. In particular, Q has at most 3 blue triples.*

Proof. By Claim 4.8 all edges of ∂B between pairs of M_1, M_2, and M_3 are incident with k_2 or k_3. So a blue triple that does contain only one of k_2 and k_3 would yield a blue shadow edge that contradicts this. □

Claim 4.15. *We have the following:*

(*1*) *If ∂B has at least 2 pairs in T, then $e_3(Q) \leqslant 22$.*
(*2*) *If ∂B has at least 3 pairs in T, then $e_3(Q) \leqslant 21$.*
(*3*) *If ∂B has at least 8 pairs in T, then $e_3(Q) \leqslant 17$.*

Proof. The claim follows by simply counting how many red triples are forbidden by a blue shadow pair. Obviously, one such pair excludes 3 red triples and two such pairs exclude 5 or 6 red triples depending on whether the blue pairs are contained in a crossing triple of vertices or not, and part (*1*) follows.

The same reasoning reveals that three blue shadow pairs exclude at least 7 red triples. Consequently, in case there is at most one blue triple, then we have at most $27 - 7 + 1 = 21$ crossing triples in Q in total. In the other case, when at least two blue triples are present, then Claim 4.14 and shiftedness tells us that k_3, k_2, k_1 together with some vertex from J span two blue triples. It is easy to check that the corresponding 5 underlying pairs of ∂B prohibit 11 red crossing triples. In view of Claim 4.14, this leads to $e_3(Q) \leqslant 27 - 11 + 3 = 19$ and part (*2*) follows.

For the proof of part (*3*), we appeal to Claim 4.8 and infer that there is at least one blue star of size 3 centred at k_3. This star excludes 9 red triples. Moreover, all three pairs of the triangle k_1k_2, k_1k_3, and k_2k_3 must be present in the blue shadow (with one of them contained in the star already). It is easy to check that the union of the star and the triangle forbids 13 red triples and combined with Claim 4.14, assertion (*3*) follows. □

For the remainder of the argument, we assume by contradiction that inequality (4.3) fails. We start by excluding the situation when very many red triples are present.

Claim 4.16. *There are at most* 23 *red triples in Q.*

Proof. Suppose there are at least 24 red triples in Q. Consequently, the vertices $J \cup K$ host at least 5 triples. It follows that there are two disjoint red triples e and f that together cover $J \cup K$.

Now suppose that Q has a red singleton. Then the red singleton is in I, since the singletons of Q are shifted to the left. Consequently, T is expanding as witnessed by that singleton together with e and f, which is absurd.

Similarly, if Q spans a red pair, then one such pair is contained in I and together with e and f we arrive at the same contradiction that T is expanding. Moreover, since Q contains at most 9 (blue) singletons and at most 1 (blue) pair (see assertion (*1*) of Claim 4.15), we arrive contrary to our assumption at the bound (4.3) by evaluating Fact 4.6 for $(s, p, t) = (9, 3, 27)$. This concludes the proof of the claim, if there is no blue triple and in the other case it follows trivially from assertion (*2*) of Claim 4.15. □

The remainder of the analysis is separated into two cases depending on whether Q is governed by many or few blue pairs.

First Case. *Each of* (M_1, M_2), (M_2, M_3), *and* (M_3, M_1) *contains a crossing pair of* ∂B.

In particular k_1k_2, k_2k_3, and k_3k_1 are in ∂B due to right-shiftedness. Here we proceed in two steps.

Claim 4.17. *There are at most* 7 *pairs of* ∂B *in* T.

Proof. Otherwise, part *(3)* of Claim 4.15 yields $e_3(Q) \leqslant 17$. Meanwhile, Claim 4.13 tells us $e_1(Q) + e_2(Q) \leqslant 18$. Since $\beta \geqslant 1$ the term

$$\sigma^3 \left(\beta^3 + \beta^2 e_1(Q) + \beta e_2(Q) + e_3(Q)\right)$$

is maximised for $e_1(Q) = e_2(Q) = 9$. Therefore, the claim follows from Fact 4.6 applied for $(s, p, t) = (9, 9, 17)$. □

By Claim 4.15, the assumption of the case yields $e_3(Q) \leqslant 21$. This allows us to restrict the number of singletons.

Claim 4.18. *There are at most* 6 *singletons in* Q.

Proof. Suppose that there are at least 7 singletons, which must be blue by Claim 4.9 and the fact that all singletons have the same colour. In view of Remark 4.12, Q contains at most one red pair using the two possible non-blue vertices. In case there is no red pair, then we can appeal to Claim 4.17 and the claim follows from Fact 4.6 applied for $(s, p, t) = (9, 7, 21)$.

Hence, we assume that Q contains exactly one red pair and exactly 7 blue singletons. Again the claim follows from Claim 4.17 combined with Fact 4.6 applied for $(s, p, t) = (7, 8, 21)$. □

Finally, we recall that Claim 4.13 tells us $e_1(Q) + e_2(Q) \leqslant 18$ and in light of Claim 4.18 and $\beta \geqslant 1$, Fact 4.6 applied for $(s, p, t) = (6, 12, 21)$ concludes the discussion of the First Case.

Second Case. *The pair* (M_1, M_2) *contains no crossing pair of* ∂B.

Note that in this case there are no blue triples in Q. Again we make a few observations regarding the number of singletons and pairs. We start with the following déjà vu.

Claim 4.19. *There are at most* 6 *singletons in* Q.

Proof. Let us assume that there are at least 7 singletons, which must be blue by Claim 4.9. In view of Remark 4.12, this implies that there is at most one red pair in Q. By Claim 4.13 and the assumption of the case that (M_1, M_2) does not contain a blue pair of $\partial(B)$, it follows that there are at most 6 blue pairs in Q. So $e_2(Q) \leqslant 7$. If there are at most 21 red triples in Q, then the claim follows from Fact 4.6 for $(s, p, t) = (9, 7, 21)$.

So together with Claim 4.16, we can assume that there are 22 or 23 red triples in Q. But then Claim 4.15 tells us that Q has at most two blue pairs. So in this situation the claim easily follows from Fact 4.6 for $(s, p, t) = (9, 3, 27)$. □

We recall again that Claim 4.13 implies $e_1(Q) + e_2(Q) \leqslant 18$ and that $\beta \geqslant 1$. If there are at most 21 red triples, then we are done by Claim 4.19 and Fact 4.6 for $(s, p, t) = (6, 12, 21)$.

Consequently, we may assume that there are at least 22 red triples. Since at most 19 triples contain a vertex from I and the red triples are left-shifted, this implies that the red triple $j_1j_2j_3$ must be present in Q.

Claim 4.20. *Each $Q(M_i, M_{i+1})$ contains at most 4 red pairs.*

Proof. If there are at least 5 red pairs between, say, M_1 and M_2, then the pairs i_1k_2 and i_2k_1 must be red in $Q(M_1, M_2)$ by Claim 4.9. Together with the triple $j_1j_2j_3$ this yields the contradiction that T is expanding. □

Claim 4.21. *We have $e(Q(M_1, M_2)) \leqslant 5$ and $e_1(Q) + e_2(Q) \leqslant 17$.*

Proof. In view of Claim 4.13, it suffices to show that $e(Q(M_1, M_2)) \leqslant 5$. This is immediate from Claim 4.20, if $Q(M_1, M_2)$ contains at most one singleton. If it contains two singletons, these must be blue by Claim 4.9, and hence there are at most 3 red pairs in $Q(M_1, M_2)$ by Remark 4.12. □

If there are exactly 22 red triples in Q, then the fact that $\beta \geqslant 1$, Claim 4.19 and Claim 4.21 conclude the case by Fact 4.6 for $(s, p, t) = (6, 11, 22)$.

In view of Claim 4.16, it remains to address the situation, when there are exactly 23 red triples in Q. In this case, our aim is to sharpen Claim 4.21 and show

$$e_1(Q) + e_2(Q) \leqslant 16\,. \tag{4.4}$$

Note that establishing inequality (4.4) completes the proof of the Second Case by another reference to Fact 4.6 — this time with $(s, p, t) = (6, 10, 23)$.

For the proof of inequality (4.4), we show that in addition to Claim 4.21, we also have $e(Q(M_2, M_3)) \leqslant 5$. Since we have 23 red triples, part *(1)* of Claim 4.15 tells us that there is at most one blue pair in Q, and without loss of generality we may assume that this lies in $Q(M_1, M_3)$. In particular, $Q(M_2, M_3)$ contains no blue pair and as argued in the proof of Claim 4.21, we obtain $e(Q(M_2, M_3)) \leqslant 5$. This bound combined with $e(Q(M_1, M_2)) \leqslant 5$ from Claim 4.21 yields inequality (4.4) and this concludes the proof of Lemma 4.2. □

§5 Conclusion

We determined the minimum d-degree threshold for k-uniform Hamilton cycles when $d = k - 3$ by studying an extension of the Erdős–Gallai Theorem for 3-graphs. We believe that a similar approach could be used to tackle the thresholds $\hbar_d^{(k)}$ for $k - d \geqslant 4$, as suggested by Lang and Sanhueza-Matamala [15, Conjecture 11.6]. This echoes a conjecture of Polcyn, Reiher, Rödl, and Schülke [20] — namely $\hbar_d^{(k)}$ being determined by $k - d$. In the remainder, we discuss two further avenues of research that appear to be worth exploring.

Connectivity. An important part of our proof concerns connectivity in dense hypergraphs. Originally an auxiliary concept, the structure and interplay of tight components has become an object of study on its own over the recent years [10, 16, 17]. We therefore suggest to further investigate the extremal behaviour of the function $c_k(\lambda)$, which we define as the limes supremum of the edge density a k-graph on n vertices may have without containing a tight component on more than $\lambda\binom{n}{k}$ edges. By Lemma 2.3 and the construction in Figure 1, we have $c_3(1/2) = 5/8$. We believe for odd k that $c_k(1/2)$ is attained by the k-graphs defined as follows: the vertex set consists of disjoint sets X and Y with $|X| \geqslant |Y|$ and its edges are all k-sets but those with $\lfloor k/2 \rfloor$ vertices in X and $\lceil k/2 \rceil$ vertices in Y. In light of the proof of Lemma 2.3, a plausible approach to this problem would be to study hypergraph versions of Theorem 3.1, which is a natural question in itself.

Cycles. What is the maximal number of edges a k-graph on n vertices may have that does not contain a tight cycle of length at least ℓ? For $k = 2$, this was answered by Erdős and Gallai [5], and the extremal construction turns out to be a union of cliques of order at most $\ell - 1$. In the hypergraph setting, a similar result was shown by Allen, Böttcher, Cooley, and Mycroft [1] for $\ell = o(n)$. However, when ℓ becomes large enough new extremal constructions appear (see Figure 1). The study of this phenomenon strikes us as interesting.

Let us define the function $\mathrm{EG}_3(\lambda)$ as the limes supremum of the edge density a 3-graph on n vertices may have without containing a cycle of length λn. Using the approach of Allen, Böttcher, Cooley, and Mycroft [1], we can distill the fact that $\mathrm{EG}_3(3/4) = 5/8$ from Lemmata 2.3 and 2.4 combined with the construction of Figure 1. We are convinced that a more careful analysis of our proof should disclose that this type of construction is sharp for all $\lambda \geqslant 3/5$. In particular, in this regime we expect at most two tight components in the extremal constructions. For smaller λ however, a more complex picture emerges, since more tight components may arise. Consider for instance the complement of the canonical extremal construction for Turán's conjecture for the tetrahedron.

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