Regular partitions of hypergraphs and property testing

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Zusammenfassung

Die *Regularitätsmethode* für Graphen wurde vor über 30 Jahren von Szemerédi, für den Beweis seines Dichteresultates über Teilmengen der natürlichen Zahlen, welche keine arithmetischen Progressionen enthalten, entwickelt. Grob gesprochen besagt das Regularitätslemma, dass die Knotenmenge eines beliebigen Graphen in konstant viele Klassen so zerlegt werden kann, dass fast alle induzierten bipartiten Graphen quasi-zufällig sind, d.h. sie verhalten sich wie zufällige bipartite Graphen mit derselben Dichte.

Das Regularitätslemma hatte viele weitere Anwendungen, vor allem in der extremalen Graphentheorie, aber auch in der theoretischen Informatik und der kombinatorischen Zahlentheorie, und gilt mittlerweile als eines der zentralen Hilfsmittel in der modernen Graphentheorie. Vor wenigen Jahren wurden Regularitätslemmata für andere diskrete Strukturen entwickelt. Insbesondere wurde die Regularitätsmethode für uniforme Hypergraphen und dünne Graphen verallgemeinert.

Ziel der vorliegenden Arbeit ist die Weiterentwicklung der Regularitätsmethode und deren Anwendung auf Probleme der theoretischen Informatik. Im Besonderen wird gezeigt, dass vererbbare (entscheidbare) Hypergrapheneigenschaften, das sind Familien von Hypergraphen, welche unter Isomorphie und induzierten Untergraphen abgeschlossen sind, testbar sind. D.h. es existiert ein randomisierter Algorithmus, der in konstanter Laufzeit mit hoher Wahrscheinlichkeit zwischen Hypergraphen, welche solche Eigenschaften haben und solchen die „weit“ davon entfernt sind, unterscheidet.

Abstract

About 30 years ago Szemerédi developed the *regularity method* for graphs, which was a key ingredient in the proof of his famous density result concerning the upper density of subsets of the integers which contain no arithmetic progression of fixed length. Roughly speaking, the regularity lemma asserts, that the vertex set of every graph can be partitioned into a constant number of classes such that almost all of the induced bipartite graphs are quasi-random, i.e., they mimic the behavior of random bipartite graphs of the same density.

The regularity lemma had have many applications mainly in extremal graph theory, but also in theoretical computer science and additive number theory, and it is considered one of the central tools in modern graph theory. A few years ago the regularity method was extended to other discrete structures. In particular extensions for uniform hypergraphs and sparse graphs were obtained.

The main goal of this thesis is the further development of the regularity method and its application to problems in theoretical computer science. In particular, we will show that hereditary, decidable properties of hypergraphs, that are properties closed under isomorphism and vertex removal, are testable. I.e., there exists a randomised algorithm with constant running time, which distinguishes between Hypergraphs displaying the property and those which are “far” from it.

Contents

1 Introduction

1.1 Background

The main focus of this thesis concerns the development of the so-called regularity method and its applications. Szemerédi's regularity lemma for graphs is one of the most important tools in extremal graph theory. It has many applications not only in graph theory, but also in combinatorial number theory, discrete geometry, and theoretical computer science.

The first form of this lemma was invented by Szemerédi [Sze75] as a tool for the resolution of a famous conjecture of Erdős and Turán [ET36], stating that any sequence of integers with a positive upper density must contain arithmetic progressions of any finite length.

Theorem 1.1 (Szemerédi's theorem)**.** *For every integer $k \geq 3$ and every $\delta > 0$ there exists an integer n_0 such that for every $n \geq n_0$ every subset $A \subseteq [n] = \{1, \ldots, n\}$ with*

$$|A| \geq \delta n$$

contains an arithmetic progression of length k, i.e., there exist elements $a_1, \ldots, a_k \in A$ such that $a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0$. □

Szemerédi's theorem led to a lot of research in several branches of mathematics and by now several different proofs of Theorem 1.1 are known.

In 1977 Furstenberg [Fur77] found a proof based on ergodic theory. Generalizations and extensions of this approach, due to Furstenberg and Katznelson [FK78, FK85, FK91], yielded several other density results including a multidimensional version of Theorem 1.1 and a density version of the Hales-Jewett theorem [HJ63]. Another proof of Theorem 1.1 based on harmonic analysis and additive number theory was found by Gowers [Gow01]. This approach, which can be viewed as an extension of the proof of Roth [Rot53] for the case $k = 3$, also gives the best quantitative bounds on n_0 in Theorem 1.1. In [Gow01] a bound for n_0 was derived, which grows doubly exponential in poly$(1/\delta)$ for fixed k and better bounds for $k = 3$ and 4 were established by Bourgain [Bou08] and Green and Tao [GT09]. A few years ago a new proof of Theorem 1.1 and its multidimensional version based on the regularity method for hypergraphs was found independently by Nagle, Rödl, Schacht, and Skokan [NRS06a, RS04, RS06] and Gowers [Gow07] (see also [Tao06b] for a more concise proof given subsequently by Tao). Many of the key ideas and techniques which appeared in these different proofs of Theorem 1.1 were very fruitful and could be applied to other problems in the respective areas. Moreover, Theorem 1.1 itself had several interesting applications. For example it was one of the key ingredients for the proof of the Green-Tao theorem [GT08] which states the the set of primes contains arbitrarily long arithmetic progressions.

1.1.1 Regularity lemma for graphs

One of the central lemmas in the original proof of Theorem 1.1 of Szemerédi was the regularity lemma for graphs. Since its invention it became an important and widely used tool in modern graph theory. This lemma roughly states that every graph may be approximated by a union of induced random-like (quasi-random) bipartite subgraphs. The quasi-randomness brings important additional information and allows one to import probabilistic intuition to deterministic problems and in many applications the original problems did not suggest a probabilistic approach.

More precisely, for a graph $G = (V, E)$ and two disjoint subsets X and $Y \subseteq V$ we denote by

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

the *density* of the bipartite subgraph $G[X, Y]$ induced on X and Y. We say a the pair (X, Y) is ε-regular for some $\varepsilon > 0$, if

$$|d(X', Y') - d(X, Y)| \leq \varepsilon \tag{1.1}$$

for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$. The modern form of the regularity lemma, which first appeared in [Sze78], states that every graph admits a vertex partition into a bounded number of classes such that most induced bipartite graphs are ε-regular. We call a partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ of V *t-equitable* (or simply *equitable*) if

$$|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1\,.$$

Moreover, we say the graph $G = (V, E)$ is *ε-regular w.r.t. $\mathscr{P}$* if all but at most εt^2 pairs (V_i, V_j) are *ε-regular.*

Theorem 1.2 (Szemerédi's regularity lemma)**.** *For any positive real ε and any integer t_0, there exist positive integers $t_{Sz} = t_{Sz}(\varepsilon, t_0)$ and $n_{Sz} = n_{Sz}(\varepsilon, t_0)$ such that the following holds.*

For every graph $G = (V, E)$ with $|V| = n \geq n_{Sz}$ vertices there exists a partition $\mathscr{P}$ of V such that

(i) $\mathscr{P} = \{V_i \colon i \in [t]\}$ is t-equitable, where $t_0 \leq t \leq t_{Sz}$ and

(ii) G is ε-regular w.r.t. $\mathscr{P}$. □

We refer to the surveys of Komlós and Simonovits [KS96] and Komlós, Shokoufandeh, Simonovits, and Szemerédi [KSSS02] for a detailed overview on the applications of Theorem 1.2.

In one of the first applications Ruzsa and Szemerédi answered a question of Brown, Erdős, and T. Sós, [BES73, SEB73] established the so-called triangle removal lemma for graphs. They proved that every graph which does not contain many triangles can be made triangle free by removing few edges.

Theorem 1.3 (Triangle removal lemma)**.** *For every $\eta > 0$ there exists $c > 0$ and an integer n_0 such that the following holds.*

If a graph G on $n \geq n_0$ vertices contains at most cn^3 triangles, then G can be made triangle free by removing at most $\eta\binom{n}{2}$ edges. □

More general statements of that type concerning graphs were successively proved by several authors in [AFKS00, AS08a, AS08b, EFR86, Für95]. In particular, the result of Alon and Shapira in [AS08a] is a generalization, which extends all the previous results of this type, where the triangle is replaced by a possibly infinite family of graphs and containment is induced. One of the main results of this thesis, Theorem 1.19, is an extension of the result of Alon and Shapira from graphs to hypergraphs.

1.1.2 Removal lemma

A *k-uniform hypergraph* $H^{(k)}$ on the vertex set V is some family of k-element subsets of V, i.e.,

$$H^{(k)} \subseteq \binom{V}{k} = \{K \subseteq V\colon\ |K| = k\}\,.$$

Note that we identify hypergraphs with their set of edges. For a given k-uniform hypergraph $H^{(k)}$, we denote by $V(H^{(k)})$ and $E(H^{(k)})$ its vertex and edge set, respectively. We only consider uniform hypergraphs, where the uniformity is some fixed number independent of the size of the hypergraph. We usually indicate the uniformity by a superscript.

It was shown by Ruzsa and Szemerédi [RS78] that Theorem 1.3 can be used to deduce Theorem 1.1 for progressions of length 3, which was earlier (and with better quantitative bounds) proved by Roth [Rot53]. This connection was generalized by Frankl and Rödl [FR02, Röd91], who showed that a removal lemma (see Theorem 1.4 below) for the complete k-uniform hypergraph with $k+1$ vertices implies Szemerédi's theorem for arithmetic progressions of length $k+1$. Moreover, Frankl and Rödl [FR02] verified such a removal lemma for $k = 3$ (see also [NR03] for the general removal lemma for 3-uniform hypergraphs) and Rödl and Skokan [RS05] for $k = 4$. The general result for k-uniform hypergraphs Theorem 1.19, based on generalizations of the regularity lemma and the local counting lemma for hypergraphs, was obtained independently by Gowers [Gow07] and by Nagle, Rödl, Schacht, and Skokan [NRS06a, RS04, RS06].

Furthermore, Solymosi [Sol04] and Tengan, Tokushige, Rödl, and Schacht [RSTT06] showed that this result also implies multidimensional versions of Szemerédi's theorem first obtained by Furstenberg and Katznelson [FK78, FK85] (see also [Gow07, RNS+05, RS06, Sol05, Tao06b] for more details).

Theorem 1.4 (Removal lemma)**.** *For all k-uniform hypergraphs $F^{(k)}$ on ℓ vertices and every $\eta > 0$ there exists a $c > 0$ and an integer n_0 such that the following holds.*

Suppose $H^{(k)}$ is a k-uniform hypergraph on $n \geq n_0$ vertices. If $H^{(k)}$ contains at most cn^ℓ copies of $F^{(k)}$, then one can delete $\eta\binom{n}{k}$ edges from $H^{(k)}$ so that the resulting sub-hypergraph contains no copy of $F^{(k)}$. □

We present a proof of Theorem 1.4 for graphs (i.e., for $k = 2$) in Section 2.10 (see Theorem 2.20).

1.2 Summary of results

The main results in this work concern generalizations and applications of the regularity lemma (Theorem 1.2) and the removal lemma (Theorem 1.4). The results in Chapters 2, 4, and 5 are joint work with Vojtěch Rödl [RS, RS07b, RS07c]. Chapter 3 is based on joint work with Yoshiharu Kohayakawa, Brendan Nagle, and Vojtěch Rödl [KNRS] and the results in Chapter 6 are joint work with Yoshiharu Kohayakawa, Vojtěch Rödl, and Endre Szemerédi [KRSS].

1.2.1 Variants of the regularity lemma for graphs

For some applications variants of the regularity lemma were considered by several researchers. In Chapter 2 we revisit several of those variants of Theorem 1.2 and their relation to each other. We focus mainly on the lemmas proved by Frieze and Kannan [FK99] and by Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00]. We show how these lemmas compare to Szemerédi's original lemma and how they relate to some other variants. Another thorough discussion of the connections of those regularity lemmas, from an analytical and geometrical perspective was given recently by Lovász and B. Szegedy in [LS07]. In Section 2.7 we discuss the so-called *counting lemmas*. We close this chapter with a brief discussion of the *limit approach* of Lovász and B. Szegedy and its relation to the regularity lemmas.

1.2.2 The weak regularity lemma for hypergraphs

In Chapter 3 we focus on an application of the so-called weak regularity lemma for hypergraphs, Theorem 3.1. This regularity lemma can be viewed as the straight forward extension of Theorem 1.2. Although the quasi-randomness provided by this lemma does not suffice to embed hypergraphs in general, it turns out that this lemma is well suited for embedding problems concerning linear hypergraphs (see, e.g., [CHPS, LPRS09, PS09] for more applications). In Chapter 3 we focus on an application related to the notion of quasi-random hypergraphs.

A graph $G = (V, E)$ is said to be (ϱ, d)-*quasi-random* if any subset $U \subseteq V$ of size $|U| \geq \varrho|V|$ induces $(d \pm \varrho)\binom{|U|}{2}$ edges, i.e,

$$(d-\varrho)\binom{|U|}{2} \leq e(U) \leq (d+\varrho)\binom{|U|}{2}.$$

Such graphs, first systematically studied by Thomason [Tho87a, Tho87b] and Chung, Graham, and Wilson [CGW89], share several properties with genuine random graphs of the same edge density. For example, it was shown that if $\varrho = \varrho(d, \ell)$ is sufficiently small, then any (ϱ, d)-quasi-random graph G is ℓ-universal, meaning that G contains approximately the same number of copies of any ℓ-vertex graph F as the random graph of the same density.

Theorem 1.5. *For every graph F, every $d > 0$ and every $\gamma > 0$, there exists a $\varrho > 0$ and an integer n_0 such that the following holds.*

If G is a (ϱ, d)-quasi-random on $n \geq n_0$ vertices, then G contains $(1 \pm \gamma) d^{e_F} n^{v_F}$ labeled copies of F. □

As usual, in the result above we write e_F for the number of edges in F and we write v_F for the number of vertices in F. In Chapter 3, we address the extent to which Theorem 1.5 can be generalized to hypergraphs.

Definition 1.6. *A k-uniform hypergraph $H^{(k)}$ is (ϱ, d)-quasi-random if for any subset $U \subseteq V(H^{(k)})$ of size $|U| \geq \varrho|V|$, we have $e(U) = (d \pm \varrho)\binom{|U|}{k}$.*

It is known that Theorem 1.5 does not generally extend to k-uniform hypergraphs, for $k \geq 3$. Indeed, let $F_0^{(3)}$ be the 3-uniform hypergraph consisting of two triples intersecting in two vertices, and consider the following two (ϱ, d)-quasi-random n-vertex hypergraphs $H_1^{(3)}$ and $H_2^{(3)}$. Let $H_1^{(3)} = G^{(3)}(n, 1/8)$ be the random 3-uniform hypergraph on n vertices whose triples appear independently with probability $1/8$. Let $H_2^{(3)} = K_3(G(n, 1/2))$ be the 3-uniform hypergraph whose triples correspond to triangles of the random graph $G(n, 1/2)$ on n vertices, where the edges of $G(n, 1/2)$ appear independently with probability $1/2$. It is easy to check that, w.h.p., both $H_1^{(3)}$ and $H_2^{(3)}$ are $(\varrho, 1/8)$-quasi-random for any $\varrho > 0$. However, w.h.p., $H_1^{(3)}$ contains $(1 \pm o(1))n^4/64$ copies of $F_0^{(3)}$, while $H_2^{(3)}$ contains $(1 \pm o(1))n^4/32$ such copies, approximately twice as many.

The hypergraph $F_0^{(3)}$, while very elementary, has one property which causes the extension of Theorem 1.5 to fail: it contains two vertices belonging to more than one edge. We will show that removing this "obstacle" allows an extension of Theorem 1.5.

Definition 1.7. *We say a k-uniform hypergraph $F^{(k)}$ is linear if $|e \cap f| \leq 1$ for all distinct edges e and f of $F^{(k)}$. We denote by $\mathscr{L}^{(k)}$ the family of all k-uniform, linear hypergraphs and set $\mathscr{L}_\ell^{(k)} = \{F^{(k)} \in \mathscr{L}^{(k)} \colon |V(F^{(k)})| = \ell\}$.*

Theorem 1.8. *For every integer $k \geq 2$, $d > 0$ and $\gamma > 0$, and every $F^{(k)} \in \mathscr{L}_\ell^{(k)}$, there exist $\varrho > 0$ and n_0 so that any (ϱ, d)-quasi-random k-uniform hypergraph $H^{(k)}$ on $n \geq n_0$ vertices contains $(1 \pm \gamma) d^{e(F^{(k)})} n^\ell$ labeled copies of $F^{(k)}$.*

We will also consider some other related results that extend known graph results to hypergraphs in a similar way to how Theorem 1.8 extends Theorem 1.5.

Definition 1.9. *A k-uniform hypergraph $H^{(k)}$ is (ϱ, d)-dense if for any subset $U \subseteq V(H^{(k)})$ of size $|U| \geq \varrho|V|$, we have $e(U) \geq d\binom{|U|}{k}$.*

For graphs, a simple induction on $\ell \geq 2$ shows that every (ϱ, d)-dense graph on sufficiently many vertices contains a copy of K_ℓ, as long as $\varrho \leq d^{\ell-2}$. However, the analogous statement for $k \geq 3$ fails. Indeed, the following simple construction was considered by several researchers and can be traced back to Erdős and Hajnal [EH72]. Let T_n be a tournament on n vertices chosen uniformly at random, and let $H^{(3)} = H^{(3)}(T_n)$ be the 3-uniform hypergraph whose triples correspond to cyclically oriented triangles of T_n. Then, w.h.p., $H^{(3)}$ is (ϱ, d)-dense for any $\varrho > 0$ and $0 < d < 1/4$. (In fact, $H^{(3)}$ is $(\varrho, 1/4)$-quasi-random.) However, since every tournament on four vertices contains at most two cyclically oriented triangles, $H^{(3)}$ is $K_4^{(3)}$-free. (In fact, $H^{(3)}$ does not even contain three triples on any four vertices.) We prove that, on the other hand, a (ϱ, d)-dense hypergraph $H^{(3)}$ will contain (many) copies of linear hypergraphs of fixed size.

Definition 1.10. *For integers $\ell \geq k$ and $\xi > 0$, we say a k-uniform hypergraph $H^{(k)}$ with n vertices is $(\xi, \mathscr{L}_\ell^{(k)})$-universal if the number of copies of any $F^{(k)} \in \mathscr{L}_\ell^{(k)}$ is at least $\xi n^{\ell|}$.*

Theorem 1.11. *For all integers $\ell \geq k \geq 2$ and every $d > 0$, there exist $\varrho = \varrho(\ell, k, d)$, $\xi = \xi(\ell, k, d) > 0$, and $n_0 = n_0(\ell, k, d)$ so that every (ϱ, d)-dense k-uniform hypergraph $H^{(k)}$ on $n \geq n_0$ vertices is $(\xi, \mathscr{L}_\ell^{(k)})$-universal.*

We shall also prove an easy corollary of Theorem 1.11 (upcoming Corollary 1.12), which roughly asserts the following. Suppose $H^{(k)}$ is a 'non-universal' hypergraph of density d. We prove that V may be partitioned into nearly equal-sized classes $V_1, \dots, V_t$ so that the number of edges of $H^{(k)}$ crossing at least two such classes is slightly larger than it would be expected if $V = V_1 \cup \dots \cup V_t$ were a random partition. More precisely, for $t \in \mathbb{N}$, let $\tau_t(H)$ be the *maximal t-cut-density* of H, defined by

$$\tau_t(H^{(k)}) = \max\Big\{\hat{d}(U_1, \dots, U_t) \colon U_1 \cup \dots \cup U_t = V(H^{(k)}) \\ \text{and } |U_1| \leq \dots \leq |U_t| \leq |U_1| + 1\Big\},$$

where

$$\hat{d}(U_1, \dots, U_t) = \frac{|E(H^{(k)}) \setminus \bigcup_{i=1}^t \binom{U_i}{k}|}{\binom{|V(H^{(k)})|}{k} - \sum_{i=1}^t \binom{|U_i|}{k}}.$$

Corollary 1.12. *For all integers $\ell \geq k \geq 2$ and every $d > 0$, there exist $t \in \mathbb{N}$, $\beta = \beta(\ell, k, d), \xi = \xi(\ell, k, d) > 0$ and $n_0 = n_0(\ell, k, d)$ so that every k-uniform hypergraph $H^{(k)}$ on $n \geq n_0$ vertices and $e(H^{(k)}) \geq d\binom{n}{k}$ edges satisfies the following. If $H^{(k)}$ is not $(\xi, \mathscr{L}_\ell^{(k)})$-universal, then $\tau_t(H^{(k)}) \geq d + \beta$.*

Corollary 1.12 is related to a result from [Röd86] and its strengthening due to Nikiforov [Nik06]. The proofs of Theorems 1.8 and 1.11 and Corollary 1.12 are presented in Chapter 3.

Related problems

Subgraphs of locally dense graphs. The following question seems interesting already for graphs. Recall from Theorem 1.5 that a (ϱ, d)-quasi-random n-vertex graph H contains $(1 \pm o(1))d^{e_F}n^{v_F}$ labeled copies of any fixed graph F. It is conceivable that replacing (ϱ, d)-quasi-randomness by (ϱ, d)-denseness would not decrease this number. We believe the following question has an affirmative answer.

Question 1.13. Is it true that for any $\gamma, d > 0$ and any graph F, there exist $\varrho > 0$ and n_0 so that any (ϱ, d)-dense graph H on $n \geq n_0$ vertices contains at least $(1-\gamma)d^{e_F}n^{v_F}$ labeled copies of F?

One may check that the answer to Question 1.13 is positive when F is a clique or more generally, a complete ℓ-partite graph for some fixed ℓ. If F is the line graph of a Boolean cube, then a result in [CHPS] shows that the same follows.

Sidorenko [Sid91, Sid93] made a related conjecture stating that any graph G with at least $d\binom{n}{2}$ edges contains at least $(1-o(1))d^{e_F}n^{v_F}$ labeled copies of any given bipartite graph F. Sidorenko's conjecture is known to be true for even cycles, complete bipartite graphs and was recently proved for a certain family of graphs including Boolean cubes [Hat]. Since our assumption in Question 1.13 is stronger than that made in Sidorenko's conjecture, the positive answer to Sidorenko's conjecture would also validate Question 1.13 for all bipartite graphs. To our knowledge, the smallest non-bipartite graph for which Question 1.13 is open is the 5-cycle.

Regularity and partial Steiner systems. In this chapter, we established that a fairly weak concept of regularity provides a counting lemma for linear hypergraphs. In order to extend this result to partial Steiner (r, k)-systems (k-uniform hypergraphs in which every r-set is covered at most once), a stronger concept of regularity will be needed. For example, when $r = 3 \leq k$, one will need a concept of regularity for k-uniform hypergraphs $H^{(k)}$ which relates the edges of $H^{(k)}$ to certain subgraphs of $K^{(2)}_{|V(H^{(k)})|}$ (rather than to subsets of $V(H^{(k)})$). Such concepts of regularity for $k = 3$ were considered in [FR02, Gow06]. For arbitrary $r \leq k$, one will need that $H^{(k)}$ should be regular with respect to certain sub-hypergraphs $G^{(r)}$ of $K^{(r)}_{|V(H^{(k)})|}$, where $G^{(r)}$ has to be regular with respect to certain sub-hypergraphs $G^{(r-1)}$ of $K^{(r-1)}_{|V(H^{(k)})|}$, and so on. This stronger concept of regularity is related to the hypergraph regularity lemmas from [Gow07, RS04, Tao06b] (see also Chapters 4 and 5).

Remark on Theorem 1.8. Note that the parameter ϱ in the concept of (ϱ, d)-quasi-randomness plays two roles. On the one hand, it "governs the locality", i.e., the size of the subsets to which the condition of uniform edge distribution applies. On the other hand, it "governs the precision" of that condition. The following result shows that, in fact, one can (formally) relax the condition on the locality, if the precision remains high enough (for graphs, a result similar in nature was proved in [Röd86, Theorem 2]).

Theorem 1.14. *For all integers $k \geq 2$, $\gamma, d > 0$, $1/k > \varepsilon > 0$ and every $F \in \mathscr{L}^{(k)}$, there exist $\delta > 0$ and n_0 so that any k-uniform hypergraph $H = (V, E)$ on $n \geq n_0$ vertices* *with the property that* $e_H(U) = (d \pm \delta)\binom{|U|}{k}$ *for every* $U \subseteq V$ *with* $|U| \geq \varepsilon|V|$ *contains* $(1 \pm \gamma)d^{e_F} n^{v_F}$ *labeled copies of* F. □

Theorem 1.14 can be proved in a similar way to Theorem 1.8, and so we omit the details. The main idea, however, is to show first that a hypergraph satisfying the assumptions of Theorem 1.14 is, in fact, (ϱ, d)-quasi-random for some $\varrho = \varrho(\delta)$ with $\varrho(\delta) \to 0$ as $\delta \to 0$.

Non-universality and large cuts. For graphs, Corollary 1.12 has the consequence that if one selects, uniformly at random, a set $I \in \binom{[t]}{t/2}$ (say, w.l.o.g., that t is even), then the set $U = \bigcup_{i \in I} V_i$ induces a cut larger than $(d+\beta)(n/2)^2 = (d+\beta-o(1))(1/2)\binom{n}{2}$, for some small $\beta > 0$ independent of n (see [KR03b, Nik06] for related results). For $k \geq 3$, Corollary 1.12 does not seem to yield immediately a similar result, and the following question remains open.

Question 1.15. Is it true that for all integers $\ell \geq k \geq 3$ and $d, \xi > 0$, there exist $\beta > 0$ and n_0 so that if $H^{(k)}$ is a k-uniform hypergraph on $n \geq n_0$ vertices and $d\binom{n}{k}$ edges which is not $(\xi, \mathscr{L}_\ell^{(k)})$-universal, then there exists a set $U \subseteq V$ of size $\lfloor n/2 \rfloor$ such that

$$\left|\left\{e \in E(H^{(k)})\colon\ 1 \leq |e \cap U| \leq k-1\right\}\right| \geq (d+\beta)\left(1 - \frac{1}{2^{k-1}}\right)\binom{n}{k}\ ?$$

1.2.3 Strong regular partitions of hypergraphs

Chapter 4 contains the main part of this thesis. In this chapter we continue the line of research from [FR02, NRS06a, RS04] and obtain a stronger and easier to use regularity lemma for hypergraphs – Theorem 4.15. We also give a proof of the corresponding counting lemma – Theorem 4.18. A standard application of those theorems, following the lines of [EFR86, FR02, Gow07, RS06] (see also proof of Theorem 2.20), yields a proof of Theorem 1.4.

As a byproduct we obtain a result for hypergraphs, Theorem 4.12 (see also Theorem 2.11 in Chapter 2), which might be of independent interest. Roughly speaking, in the context of graphs Theorem 4.12 says that for every fixed $\nu > 0$ any graph on n vertices can be approximated, by adding and deleting at most νn^2 edges, by an ε-regular graph on a vertex partition into t parts, where $\varepsilon = \varepsilon(t)$ is an arbitrary function of t, and thus we may have $\varepsilon(t) \ll \frac{1}{t}$. This may perhaps be somewhat surprising, since it follows from the work of Gowers [Gow97], that there are graphs which if not changed admit only an ε-regular partition with t classes, where $t \gg \frac{1}{\varepsilon}$. In fact Gowers constructed graphs with number of partition classes in any ε-regular partition being bigger than a tower of height polynomial in $1/\varepsilon$. We defer the somewhat technical statement of the main results in that chapter to Section 4.1.

1.2.4 Generalizations of the removal lemma

One possible generalization of Theorem 1.4 is to replace the single hypergraph $F^{(k)}$ by a possibly infinite family $\mathscr{F}$ of k-uniform hypergraphs. Such a result was first proved for graphs by Alon and Shapira [AS05b, AS08b] in the context of property testing (see Section 1.2.5 below). For a family of graphs $\mathscr{F}$ consider the class $\mathrm{Forb}(\mathscr{F})$ of all graphs H containing no member of $\mathscr{F}$ as a subgraph. Clearly $\mathrm{Forb}(\mathscr{F})$ is monotone, i.e., if $H \in \mathrm{Forb}(\mathscr{F})$ and H' is a subgraph of H (obtained from H by successive vertex and edge deletions), then $H' \in \mathrm{Forb}(\mathscr{F})$. Moreover, it is easy to see that for every monotone family of graphs $\mathscr{P}$ (so-called monotone property $\mathscr{P}$) there exists a family $\mathscr{F}$ such that $\mathscr{P} = \mathrm{Forb}(\mathscr{F})$. Alon and Shapira proved the following in [AS08b].

Theorem 1.16. *For every (possibly infinite) family of graphs $\mathscr{F}$ of graphs and every $\eta > 0$ there exist constants $c > 0$, $C > 0$, and n_0 such that the following holds.*

Suppose H is a graph on $n \geq n_0$ vertices. If for every $\ell = 1, \dots, C$ and every $F \in \mathscr{F}$ on ℓ vertices, H contains at most cn^ℓ copies of F, then one can delete $\eta\binom{n}{2}$ edges from H so that the resulting subgraph H' contains no copy of any member of $\mathscr{F}$, i.e., $H' \in \mathrm{Forb}(\mathscr{F})$. □

Clearly, Theorem 1.4 for $k = 2$ is equivalent to Theorem 1.16 in the special case when $\mathscr{F}$ consists of only one graph. While for finite families $\mathscr{F}$ Theorem 1.16 can be proved along the lines of the proof of Theorem 1.4 (alternatively, it can easily be deduced from Theorem 1.4 directly), for infinite families $\mathscr{F}$ the proof of Theorem 1.16 is more sophisticated.

Perhaps one of the earliest results of this nature was obtained by Bollobás, Erdős, Simonovits, and Szemerédi [BESS78], who essentially proved Theorem 1.16 for the special family $\mathscr{F}$ of blow-up's of odd cycles. In [DR85] answering a question of Erdős (see, e.g., [Erd90]) Duke and Rödl generalized the result from [BESS78] and proved Theorem 1.16 for the families of $(r+1)$-chromatic graphs $r \geq 2$.

The proof of Theorem 1.16 for arbitrary families $\mathscr{F}$ relies on a strengthened version of Szemerédi's regularity lemma, which was obtained by Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00] by iterating the regularity lemma for graphs (see Section 2.1 for details).

Theorem 1.16 was extended by Avart, Rödl, and Schacht in [ARS07] from graphs to hypergraphs. The proof in [ARS07] follows the approach of Alon and Shapira and is based on two successive applications of the hypergraph regularity lemma from Chapter 4.

Another very natural variant of Theorem 1.4 would be an *induced* version. For graphs this was first considered by Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00]. Note that in this case in order to obtain an induced F-free graph, we may need to not only remove, but also to add edges.

Theorem 1.17. *For all graphs F on ℓ vertices and and every $\eta > 0$ there exist $c > 0$ and n_0 so that the following holds.*

Suppose H is a graph on $n \geq n_0$ vertices. If H contains at most cn^ℓ induced copies of F, then one can change $\eta\binom{n}{2}$ pairs from $V(H)$ (deleting or adding the edge) so that the resulting graph H' contains no induced copy of F. □

An extension of Theorem 1.17 from graphs to 3-uniform hypergraphs was obtained by Kohayakawa, Nagle, and Rödl in [KNR02].

Recently, in [AS05a, AS08a] Alon and Shapira proved a common generalization of Theorem 1.16 and Theorem 1.17, extending Theorem 1.17 from one forbidden induced graph F to a forbidden family of induced graphs $\mathscr{F}$ (see Theorem 2.21). In Chapter 5 we extend their result to k-uniform hypergraphs and prove Theorem 1.19.

For a family of k-uniform hypergraphs $\mathscr{F}$, let $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ be the family of all hypergraphs $H^{(k)}$ which contain no induced copy of any member of $\mathscr{F}$. Clearly, $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ is a *hereditary* family (or *hereditary property*) of hypergraphs, i.e., if $H^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and $\widetilde{H}^{(k)}$ is an induced sub-hypergraph of $H^{(k)}$, then $\widetilde{H}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

Definition 1.18 (η**-far**)**.** *For a constant $\eta \geq 0$ and a possibly infinite family of k-uniform hypergraphs $\mathscr{P}$ we say a given hypergraph $H^{(k)}$* ***is η-far from*** *$\mathscr{P}$ if every hypergraph $G^{(k)}$ on the same vertex set $V(H^{(k)})$ with*

$$|G^{(k)} \triangle H^{(k)}| \leq \eta \binom{|V(H^{(k)})|}{k}$$

satisfies $G^{(k)} \notin \mathscr{P}$, where $G^{(k)} \triangle H^{(k)}$ denotes the symmetric difference of the edge sets of $G^{(k)}$ and $H^{(k)}$.

The main objective of Chapter 5 is to prove the following.

Theorem 1.19. *For every (possibly infinite) family $\mathscr{F}$ of k-uniform hypergraphs and every $\eta > 0$ there exist constants $c > 0$, $C > 0$, and n_0 such that the following holds.*

Suppose $H^{(k)}$ is a k-uniform hypergraph on $n \geq n_0$ vertices. If for every $\ell = 1, \ldots, C$ and every $F^{(k)} \in \mathscr{F}$ on ℓ vertices, $H^{(k)}$ contains at most cn^ℓ induced copies of $F^{(k)}$, then $H^{(k)}$ is not η-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

In other words one can change (add/delete) up to at most $\eta\binom{n}{k}$ k-tuples in $V(H^{(k)})$ (to/from $H^{(k)}$) so that the resulting hypergraph $G^{(k)}$ contains no induced copy of any member of $\mathscr{F}$, i.e., so that $G^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

Moreover, since $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ is a subset of the family $\overline{\mathscr{F}}$ of all hypergraphs not contained in $\mathscr{F}$, such a hypergraph $H^{(k)}$ is also not η-far from $\overline{\mathscr{F}}$.

For graphs Theorem 1.19 was first obtained by Alon and Shapira [AS08a]. The proof presented in [AS08a] is again based on the strong version of Szemerédi's regularity lemma from [AFKS00]. Another proof of Theorem 1.19 for graphs was found by Lovász and B. Szegedy [LS05] (see also [BCL+06]). Below we discuss a few consequences of Theorem 1.19, which motivated the original work for graphs.

1.2.5 Property Testing

Recall that for every hereditary property $\mathscr{P}$ of k-uniform hypergraphs, there exists a family of k-uniform hypergraphs $\mathscr{F}$ such that $\mathscr{P} = \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$. Consequently, Theorem 1.19 states that if $H^{(k)}$ is η-far from some hereditary property $\mathscr{P} = \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$, then it must contain many (at least $cn^{|V(F^{(k)})|}$) induced copies of some "forbidden" hypergraph $F^{(k)} \in \mathscr{F}$ of size at most C, which "proves" that $H^{(k)}$ is not in $\mathscr{P}$. In other words, if $H^{(k)}$ is η-far from some given hereditary property $\mathscr{P}$, then it is "easy" to detect that $H^{(k)} \notin \mathscr{P}$. This implies Corollary 1.20, which we discuss after the following remark.

Note that if $\mathscr{P}$ is $\overline{\mathscr{F}}$, the complement of some family $\mathscr{F}$, then $\mathscr{P}$ is not necessarily hereditary. If $H^{(k)}$ is η-far from $\mathscr{P}$ in this case, then the "moreover-part" of Theorem 1.19 still implies that $H^{(k)}$ contains many induced copies of some forbidden hypergraph $F^{(k)} \in \mathscr{F}$ of bounded size. In this case, however, containing a forbidden hypergraph does not necessarily imply that $H^{(k)} \notin \mathscr{P}$. Hence, an analogous statement of Corollary 1.20 for arbitrary properties $\mathscr{P}$ (which is known to be false) is not implied.

Let us return to hereditary properties $\mathscr{P}$. For such properties Theorem 1.19 has an interesting consequence in the area of *property testing* (see, e.g., [GGR98] for the definitions). We say a property $\mathscr{P}$ of hypergraphs (i.e., a family of hypergraphs) is *testable with one-sided error* if for every $\eta > 0$ there exists a constant $q = q(\mathscr{P}, \eta)$ and a randomized algorithm $\mathscr{A}$ which does the following:

For a given hypergraph $H^{(k)}$ the algorithm $\mathscr{A}$ can query some oracle whether a k-tuple K of $V(H^{(k)})$ spans and edge in $H^{(k)}$ or not. After at most q queries the algorithm outputs

1. *$H^{(k)} \in \mathscr{P}$ with probability 1 if $H^{(k)} \in \mathscr{P}$ and*

2. *$H^{(k)} \notin \mathscr{P}$ with probability at least 2/3 if $H^{(k)}$ is η-far from $\mathscr{P}$.*

If $H^{(k)} \notin \mathscr{P}$ and $H^{(k)}$ is not η-far from $\mathscr{P}$, then there are no guarantees for the output of $\mathscr{A}$.

Furthermore, we say a property $\mathscr{P}$ is *decidable* if there exists an algorithm which for every hypergraph $H^{(k)}$ distinguishes in finite time if $H^{(k)} \in \mathscr{P}$ or $H^{(k)} \notin \mathscr{P}$. In this context Theorem 1.19 implies the following.

Corollary 1.20. *Every decidable and hereditary property of k-uniform hypergraphs is testable with one-sided error.*

Proof. Let a decidable and monotone property $\mathscr{P} = \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and some $\eta > 0$ be given. By Theorem 1.19, there are some constants $c > 0$, $C > 0$, and $n_0 \in \mathbb{N}$ such that any k-uniform hypergraph on $n \geq n_0$ vertices, which is η-far from exhibiting $\mathscr{P}$ contains at least $cn^{|V(F_0^{(k)})|}$ induced copies of some $F_0^{(k)} \in \mathscr{F}$ with $|V(F_0^{(k)})| \leq C$.

Let $s \in \mathbb{N}$ be such that $(1-c)^{s/C} < 1/3$ and set $m_0 = \max\{s, n_0\}$. We claim that there exists a one-sided tester with query complexity $\binom{m_0}{k}$ for $\mathscr{P}$. For that let $H^{(k)}$ be a k-uniform hypergraph on n vertices. If $n \leq m_0$, then the tester simply queries all edges of $H^{(k)}$ and since $\mathscr{P}$ is decidable, there is an exact algorithm with running time only depending on the fixed m_0, which determines correctly if $H^{(k)} \in \mathscr{P}$ or not.

Consequently, let $n > m_0$. Then we choose uniformly at random a set S of s vertices from H. Consider the hypergraph $H^{(k)}[S] = H^{(k)} \cap \binom{S}{k}$ induced on S. If $H^{(k)}[S]$ has $\mathscr{P}$, then the tester says "yes" and otherwise "no." Since $\mathscr{P}$ is decidable and s is fixed the algorithm decides whether or not $H^{(k)}[S]$ is in $\mathscr{P}$ in constant time (constant only depending on s and $\mathscr{P}$).

Clearly, if $H^{(k)} \in \mathscr{P}$ or $n \leq m_0$, then the output of the tester is correct and hence it is one-sided. On the other hand, if $H^{(k)}$ is η-far from $\mathscr{P}$ and $n > m_0$, then due to

Theorem 1.19 the random set S spans a copy of $F_0^{(k)}$ for some $F_0^{(k)} \in \mathscr{F}$ on $f_0 \le C$ vertices, with probability at least

$$\frac{cn^{f_0}}{\binom{n}{f_0}} \ge c. \tag{1.2}$$

Hence the probability that S does not span any copy of $F_0^{(k)}$ is at most

$$(1-c)^{s/f_0} \le (1-c)^{s/C} < \frac{1}{3}\,.$$

In other words, S spans a copy of $F_0^{(k)}$ with probability at least $2/3$, which shows that the tester works as specified. □

1.2.6 Regularity method for sparse graphs

As we discussed above, the regularity method has proved to be a powerful tool in asymptotic combinatorics. Regular decompositions of graphs and hypergraphs reveal much of the structure of such objects, and have been fundamental in approaching diverse problems in the area. The regularity method for *dense graphs* is the best developed direction in this line of research, with a long history of applications and such surprising tools as the blow-up lemma [KSS97, KSS98] and due to the recent advances [Gow07, NRS06a, RS04], one is now able to apply the regularity method to *hypergraphs*.

The regularity method for *sparse graphs* is, however, still under development: it appears that even the embedding or counting lemma for graphs of constant size has not been proved in its full generality or strength (see, e.g., [GS05, Koh97, KR03b]). In this work we contribute to the development of the regularity method for sparse graphs, providing an embedding strategy for large graphs of bounded degree in the sparse setting. As an application, we prove a numerical result in Ramsey theory: we prove an upper bound for a variant of the Ramsey number for graphs of bounded degree (for numbers in Ramsey theory, see [GR87]).

For graphs G and H, write $G \longrightarrow H$ if G contains a monochromatic copy of H for every 2-coloring of the edges of G. Erdős, Faudree, Rousseau and Schelp [EFRS78] considered the question of how few edges G may have if $G \longrightarrow H$. Following [EFRS78] we denote the *size-Ramsey number*

$$\hat{r}(H) = \min\{e(G)\colon\ G \longrightarrow H\}\,,$$

where $e(G)$ denotes the cardinality of the edge set of G.

For example $\hat{r}(K_{1,n}) = 2n-1$ for the star $K_{1,n}$ on $n+1$ vertices. In [Bec83] Beck disproved a conjecture of Erdős [Erd81] and showed that

$$\hat{r}(P_n) \le 900n\,.$$

More generally, it follows from the result of Friedman and Pippenger [FP87] that the size-Ramsey number of bounded degree trees grows linearly with the size of the tree (for further results in this direction, see [Bec90, HK95]). Moreover, it was proved by Haxell, Kohayakawa, and Łuczak [HKŁ95] that cycles also have linear size-Ramsey numbers. Beck asked in [Bec90] if $\hat{r}(H)$ is always linear in the number of vertices of H for graphs H of bounded degree. This was disproved by Rödl and Szemerédi [RS00], who proved that there are graphs of order n, maximum degree three, and

$$\hat{r}(H) \geq n \log^c n$$

for some constant $c > 0$. These authors also conjectured that, for every $\Delta \geq 3$, there exists $\varepsilon = \varepsilon(\Delta) > 0$ such that

$$n^{1+\varepsilon} \leq \hat{r}_{\Delta,n} := \max\{\hat{r}(H)\colon\ H \in \mathcal{H}_{\Delta,n}\} \leq n^{2-\varepsilon}\,, \tag{1.3}$$

where $H_{\Delta,n}$ is the class of all n-vertex graphs with maximum degree at most Δ, up to isomorphism. In Chapter 6 we prove the upper bound conjectured in (1.3).

In fact, our proof method yields a stronger result. Let us say that a graph is $\mathcal{H}_{\Delta,n}$-*universal* if it contains every member of $\mathcal{H}_{\Delta,n}$ as a subgraph. Furthermore, let us say that a graph is *partition universal for the class of graphs* $\mathcal{H}_{\Delta,n}$ if any 2-coloring of its edges contains a monochromatic $\mathcal{H}_{\Delta,n}$-universal graph. We shall establish for every Δ the existence of a graph G with $O(n^{2-1/\Delta}\log^{1/\Delta} n)$ edges that is partition universal for $\mathcal{H}_{\Delta,n}$.

Theorem 1.21. *For every $\Delta \geq 2$ there exist constants B and C such that for every n and N satisfying $N \geq Bn$ there exists a graph G on N vertices and with at most $CN^{2-1/\Delta}\log^{1/\Delta} N$ edges that is partition universal for $\mathcal{H}_{\Delta,n}$. In particular, we have $G \longrightarrow H$ for every $H \in \mathcal{H}_{\Delta,n}$.*

Remark 1.22. (*i*) As observed in [ACK⁺00], one can show that the number of edges in any $\mathcal{H}_{\Delta,n}$-universal graph is $\Omega(n^{2-2/\Delta})$ and, hence, the exponent $2-1/\Delta$ of N in Theorem 1.21 cannot be reduced to $2-2/\Delta-\varepsilon$ for any given $\varepsilon > 0$. For completeness, let us quickly see how to obtain this lower bound on the number of edges M in an $\mathcal{H}_{\Delta,n}$-universal graph G. Let us suppose first that $n\Delta$ is even. Note that we must have

$$\binom{M}{n\Delta/2} \geq \frac{1}{n!}L_{\Delta,n}, \tag{1.4}$$

where $L_{\Delta,n}$ denotes the number of labeled graphs on n vertices that are Δ-regular. Bender and Canfield [BC78] showed that, for any fixed Δ, as $n \to \infty$ with $n\Delta$ even, we have

$$L_{\Delta,n} = (1+o(1))\sqrt{2}e^{-(\Delta^2-1)/4}\left(\frac{\Delta^{\Delta/2}}{e^{\Delta/2}\Delta!}\right)^n n^{\Delta n/2}.$$

Therefore, for $n\Delta$ even, $L_{\Delta,n} = \Omega(c^n n^{n\Delta/2})$ for a constant $c = c(\Delta)$. Combining this with (1.4), we see that

$$\left(\frac{2\mathrm{e}M}{n\Delta}\right)^{n\Delta/2} \geq \binom{M}{n\Delta/2} \geq \frac{L_{\Delta,n}}{n!} = \Omega\left(\frac{c^n n^{n\Delta/2}}{n^n}\right),$$

whence $M = \Omega(n^{2-2/\Delta})$, as required. If $n\Delta$ is odd, simply observe that an $\mathcal{H}_{\Delta,n}$-universal graph is also $\mathcal{H}_{\Delta-1,n}$-universal.

A recent, remarkable result of Alon and Capalbo [AC08] confirms the existence of $\mathcal{H}_{\Delta,n}$-universal graphs with $O(n^{2-2/\Delta})$ edges (see also [ACK+00, ACK+01, AC07] for more results).

(*ii*) A weaker version of Theorem 1.21, with

$$|E(G)| = N^{2-\frac{1}{2\Delta}+o(1)},$$

was proved earlier by Kohayakawa, Rödl, and Szemerédi (unpublished).

Let $G(N,p)$ be the standard random graph on N vertices, with all the edges present with probability p, independently of one another (see [Bol01, JŁR00] for the theory of random graphs). To prove Theorem 1.21, we shall show that $G(N,p)$ with an appropriate choice of $p = p(N)$ is as required with high probability.

Theorem 1.23. *For every $\Delta \geq 2$ there exist constants B and C for which the following holds. Let $p = p(N) = C(\log N/N)^{1/\Delta}$. Then*

$$\lim_{N\to\infty} \mathbb{P}\left(G(N,p) \text{ is partition universal for } \mathcal{H}_{\Delta,N/B}\right) = 1. \tag{1.5}$$

Remark 1.24. (*i*) In Theorem 1.21, we have restricted ourselves to the 2-color case for simplicity. One may easily prove the same result for more than two colors (the constants B and C would then depend on both Δ and on the number of colors). Similarly, Theorem 1.23 holds as stated for any fixed number of colors, that is, we may generalize the notion of partition universality to any fixed number of colors r and prove the same result (the constant C would then depend on both Δ and r).

(*ii*) Theorem 1.21 follows from Theorem 1.23. In Chapter 6, we focus our attention on the proof of Theorem 1.23.

The main tool in our proof of Theorem 1.23 is the regularity method, adapted to the appropriate sparse and random setting. The key novel ingredient in our approach is an embedding strategy that allows one to embed bounded degree graphs of linear order in suitably pseudorandom graphs (see the proof of Lemma 6.14). Crucial in the proof is a rather surprising phenomenon, namely, the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed. This phenomenon was first spelt out in full in [KR03a], but we use an improved version proved in [GKRS07].

Open Questions

Theorem 1.21 asserts the existence of a partition universal graph G for the class of graphs $\mathcal{H}_{\Delta,n}$ with G having $O(n^{2-1/\Delta}\log^{1/\Delta} n)$ edges. We believe it would be rather interesting to decide whether one can substantially improve on this upper bound. In particular, we believe that bringing this bound down to a bound of the form $O(n^{2-1/\Delta-\varepsilon})$ for some $\varepsilon > 0$ would require a completely new idea. The only lower bound that we know is of the form $\Omega(n^{2-2/\Delta})$ (see Remark 1.22(i)).

Our proof of Theorem 1.21 is heavily based on random graphs, and we do not know how to prove this result or anything numerically similar by constructive means. In particular, for instance, we do not know whether (N, d, λ)-graphs with reasonable parameters are partition universal for $\mathcal{H}_{\Delta,n}$.

Another interesting question is whether one can prove Theorem 1.21 without the regularity method.

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2 Regularity lemmas for graphs

In this Chapter we discuss several regularity lemmas for graphs. We start our discussion with the regularity lemma of Frieze and Kannan [FK99] in the next section. In Section 2.2 we show how Szemerédi's regularity lemma, Theorem 1.2, can be deduced from the weaker lemma of Frieze and Kannan by iterated applications. In Section 2.3 we discuss the (ε, r)-regularity lemma, whose analog for 3-uniform hypergraphs was introduced by Frankl and Rödl [FR02]. We continue in Section 2.4 with the regularity lemma of Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00], which can be viewed as an iterated version of Szemerédi's regularity lemma. In Section 2.5 we introduce the regular approximation lemma whose hypergraph variant will be proved in Chapter 4. Finally, in Section 2.6 we briefly discuss the original regularity lemma of Szemerédi [Sze75] for bipartite graphs and a multipartite version of it from [DLR95].

2.1 The Frieze-Kannan lemma

The following variant of Szemerédi's regularity lemma was introduced by Frieze and Kannan [FK99] for the design of an efficient approximation algorithm for the MAX-CUT problem in dense graphs.

Theorem 2.1. *For every $\varepsilon > 0$ and every $t_0 \in \mathbb{N}$ there exist $T_{FK} = T_{FK}(\varepsilon, t_0)$ and n_0 such that for every graph $G = (V, E)$ with at least $|V| = n \geq n_0$ vertices the following holds. There exists a partition $\mathscr{P}$ of V such that*

(i) $\mathscr{P} = \{V_i\colon\ i \in [t]\}$ is t-equitable, where $t_0 \leq t \leq T_{FK}$, and

(ii) for every $U \subseteq V$

$$\left| e(U) - \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} d(V_i, V_j)|U \cap V_i||U \cap V_j| \right| \leq \varepsilon n^2 \,, \tag{2.1}$$

where $e(U)$ denotes the number of edges of G contained in U.

Definition 2.2. *A partition that satisfies properties (i) and (ii) will be referred to as* $(\varepsilon, t_0, T_{\mathrm{FK}})$-FK-partition. *Sometimes we may omit t_0 and T_{FK} and simply refer to such a partition as* ε-FK-partition.

The essential properties of the partition provided by Theorem 2.1 are property the boundedness of t and (ii). Property (i) bounds the number of partition classes by a constant independent of G and n and, roughly speaking, property (ii) asserts that the number of edges of any large set U can be fairly well approximated by the densities

$d(V_i, V_j)$ given by the partition $V_1 \cup \ldots \cup V_t = V$. More precisely, $e(U) \approx e(U')$ for any choice of U and U' satisfying for example $|U \cap V_i| \approx |U' \cap V_i|$ for all $i \in [t]$. Moreover, we note that conclusion (ii) can be replaced by the following:

(ii') for all (not necessarily disjoint) sets $U, W \subseteq V$

$$\left| e(U,W) - \sum_{i=1}^{t} \sum_{j \in [t]\setminus\{i\}} d(V_i, V_j)|U \cap V_i||W \cap V_j| \right| \leq 6\varepsilon n^2\,, \tag{2.2}$$

where edges contained in $U \cap W$ are counted twice in $e(U,W)$.

Indeed, if (ii) holds, then we infer (ii') from the identity

$$e(U,W) = e(U \cup W) - e(U) - e(W) + 3e(U \cap W)\,.$$

The proof of Theorem 2.1 relies on the *index* of a partition, a concept which was first introduced and used by Szemerédi.

Definition 2.3. *For a partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ of the vertex sets of a graph $G = (V, E)$, i.e., $V_1 \cup \ldots \cup V_t = V$ we define the* index *of $\mathscr{P}$ by*

$$\operatorname{ind}(\mathscr{P}) = \frac{1}{\binom{|V|}{2}} \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} d^2(V_i, V_j)|V_i||V_j|\,.$$

Note that it follows directly from the definition of the index that for any partition $\mathscr{P}$ we have

$$0 \leq \operatorname{ind}(\mathscr{P}) \leq 1\,.$$

For the proof of Theorem 2.1 we will use the following consequence of the Cauchy-Schwarz inequality.

Lemma 2.4. *Let $1 \leq M < N$, let $\sigma_1, \ldots, \sigma_N$ be positive and $d_1, \ldots, d_N$, and d be reals. If $\sum_{i=1}^{N} \sigma_i = 1$ and $d = \sum_{i=1}^{N} d_i \sigma_i$ then*

$$\sum_{i=1}^{N} d_i^2 \sigma_i \geq d^2 + \left(d - \frac{\sum_{i=1}^{M} d_i \sigma_i}{\sum_{i=1}^{M} \sigma_i} \right)^2 \frac{\sum_{i=1}^{M} \sigma_i}{1 - \sum_{i=1}^{M} \sigma_i}\,.$$

For completeness we include the short proof of Lemma 2.4.

Proof. For $M = 1$ and $N = 2$ the statement follows from the identity

$$\hat{d}_1^2 \hat{\sigma}_1 + \hat{d}_2^2 \hat{\sigma}_2 = \hat{d}^2 + (\hat{d} - \hat{d}_1)^2 \frac{\hat{\sigma}_1}{\hat{\sigma}_2}\,. \tag{2.3}$$

which is valid for positive $\hat{\sigma}_1, \hat{\sigma}_2$ with $\hat{\sigma}_1 + \hat{\sigma}_2 = 1$ and $\hat{d} = \hat{d}_1 \hat{\sigma}_1 + \hat{d}_2 \hat{\sigma}_2$.

For general $1 \le M < N$ we infer from the Cauchy-Schwarz inequality applied twice in the form $(\sum d_i\sigma_i)^2 \le \sum d_i^2\sigma_i \sum \sigma_i$

$$\begin{aligned}\sum_{i=1}^{N} d_i^2\sigma_i &= \sum_{i=1}^{M} d_i^2\sigma_i + \sum_{i=M+1}^{N} d_i^2\sigma_i \\ &\ge \frac{\left(\sum_{i=1}^{M} d_i\sigma_i\right)^2}{\sum_{i=1}^{M}\sigma_i} + \frac{\left(\sum_{i=M+1}^{N} d_i\sigma_i\right)^2}{\sum_{i=M+1}^{N}\sigma_i} \\ &= \left(\frac{\sum_{i=1}^{M} d_i\sigma_i}{\sum_{i=1}^{M}\sigma_i}\right)^2 \sum_{i=1}^{M}\sigma_i + \left(\frac{\sum_{i=M+1}^{N} d_i\sigma_i}{\sum_{i=M+1}^{N}\sigma_i}\right)^2 \sum_{i=M+1}^{N}\sigma_i \,.\end{aligned}$$

Setting

$$\hat{\sigma}_1 = \sum_{i=1}^{M}\sigma_i, \qquad \hat{\sigma}_2 = \sum_{i=M+1}^{N}\sigma_i,$$

$$\hat{d}_1 = \frac{\sum_{i=1}^{M} d_i\sigma_i}{\sum_{i=1}^{M}\sigma_i}, \qquad \hat{d}_2 = \frac{\sum_{i=M+1}^{N} d_i\sigma_i}{\sum_{i=M+1}^{N}\sigma_i},$$

and

$$\hat{d} = \hat{d}_1\hat{\sigma}_1 + \hat{d}_2\hat{\sigma}_2$$

we have $\hat{d} = \sum_{i=1}^{N} d_i\sigma_i = d$ and from (2.3) we infer

$$\sum_{i=1}^{N} d_i^2\sigma_i \ge \left(\sum_{i=1}^{N} d_i\sigma_i\right)^2 + \left(\sum_{i=1}^{N} d_i\sigma_i - \frac{\sum_{i=1}^{M} d_i\sigma_i}{\sum_{i=1}^{M}\sigma_i}\right)^2 \frac{\sum_{i=1}^{M}\sigma_i}{\sum_{i=M+1}^{N}\sigma_i}\,,$$

which is what we claimed. □

After those preparations we prove Theorem 2.1.

Proof of Theorem 2.1. The proof is based on an idea which was already present in the original work of Szemerédi. Starting with an arbitrary equitable vertex partition $\mathscr{P}_0$ with t_0 classes, we consider a sequence of partitions $\mathscr{P}_0, \mathscr{P}_1, \dots$ such that $\mathscr{P}_j$ always satisfies property (i). As soon as $\mathscr{P}_j$ also satisfies (ii) we can stop. On the other hand, if $\mathscr{P}_j$ does not satisfy (ii) we will show that there exists a partition $\mathscr{P}_{j+1}$ whose index increased by $\varepsilon^2/2$. Since $\mathrm{ind}(\mathscr{P}) \le 1$ for any partition $\mathscr{P}$, we infer that after at most $2/\varepsilon^2$ steps this procedure must end with a partition satisfying properties (i) and (ii) of the theorem.

So suppose $\mathscr{P}_j = \mathscr{P} = \{V_i \colon i \in [t]\}$ is a partition of V which satisfies (i) and (ii), but there exists a set $U \subseteq V$ such that (2.1) fails. We are going to construct a partition $\mathscr{R} = \mathscr{P}_{j+1}$ satisfying

$$\mathrm{ind}(\mathscr{R}) \ge \mathrm{ind}(\mathscr{P}) + \varepsilon^2/2\,. \tag{2.4}$$

For that set

$$U_i = V_i \cap U \quad \text{and} \quad \bar{U}_i = V_i \setminus U\,.$$

We define a new partition $\mathscr{Q}$ by replacing every vertex class V_i by U_i and $\bar{U}_i$

$$\mathscr{Q} = \{U_1, \bar{U}_1 \dots, U_t, \bar{U}_t\}\,.$$

Next we show that the index of $\mathscr{Q}$ increased by ε^2 compared to $\mathrm{ind}(\mathscr{P})$. For every $1 \le i < j \le t$ we set

$$\varepsilon_{ij} = d(U_i, U_j) - d(V_i, V_j)\,.$$

Since we may assume $t \ge t_0 \ge 1/\varepsilon$, which yields $\sum_{i=1}^{t} e(V_i) \le \varepsilon n^2/2$, we infer from the assumption that (2.1) fails, that

$$\Big|\sum_{i<j} \varepsilon_{ij}|U_i||U_j|\Big| \ge \varepsilon n^2 - \sum_{i=1}^{t} e(U_i) \ge \varepsilon n^2 - \sum_{i=1}^{t} e(V_i) \ge \frac{\varepsilon}{2} n^2\,, \tag{2.5}$$

Since $V_i = U_i \cup \bar{U}_i$ for every $i \in [t]$ we obtain

$$\begin{aligned} d(V_i, V_j)|V_i||V_j| = d(U_i, U_j)|U_i||U_j| &+ d(\bar{U}_i, U_j)|\bar{U}_i||U_j| \\ &+ d(U_i, \bar{U}_j)|U_i||\bar{U}_j| + d(\bar{U}_i, \bar{U}_j)|\bar{U}_i||\bar{U}_j| \end{aligned}$$

and

$$|V_i||V_j| = |U_i||U_j| + |\bar{U}_i||U_j| + |U_i||\bar{U}_j| + |\bar{U}_i||\bar{U}_j|\,.$$

Combining those identities with Lemma 2.4, we obtain

$$\begin{aligned} d^2(U_i, U_j)|U_i||U_j| + d^2(\bar{U}_i, U_j)|\bar{U}_i||U_j| + d^2(U_i, \bar{U}_j)|U_i||\bar{U}_j| + d^2(\bar{U}_i, \bar{U}_j)|\bar{U}_i||\bar{U}_j| \\ \ge d^2(V_i, V_j)|V_i||V_j| + \varepsilon_{ij}^2 \left(\frac{|U_i||U_j|}{1 - \frac{|U_i||U_j|}{|V_i||V_j|}}\right) \\ \ge d^2(V_i, V_j)|V_i||V_j| + \varepsilon_{ij}^2 |U_i||U_j|\,. \end{aligned}$$

Summing over all $1 \le i < j \le t$ we obtain

$$\begin{aligned} \mathrm{ind}(\mathscr{Q}) &\ge \mathrm{ind}(\mathscr{P}) + \frac{1}{\binom{n}{2}} \sum_{i<j} \varepsilon_{ij}^2 |U_i||U_j| \\ &\ge \mathrm{ind}(\mathscr{P}) + \frac{\left(\sum_{i<j} \varepsilon_{ij}|U_i||U_j|\right)^2}{\binom{n}{2} \sum_{i<j} |U_i||U_j|} \\ &\overset{(2.5)}{\ge} \mathrm{ind}(\mathscr{P}) + \frac{(\varepsilon n^2/2)^2}{\binom{n}{2}\binom{n}{2}} \\ &\ge \mathrm{ind}(\mathscr{P}) + \varepsilon^2. \end{aligned} \tag{2.6}$$

We now find an equitable partition $\mathscr{R}$ which is a refinement of $\mathscr{P}$ (and almost a refinement of $\mathscr{Q}$) for which (2.4) holds. For that subdivide each vertex class V_i of $\mathscr{P}$ into sets $W_{i,a}$ of size $\lfloor \varepsilon^2 n/(5t) \rfloor$ or $\lfloor \varepsilon^2 n/(5t) \rfloor + 1$ in such a way that for all but at most one of these sets either $W_{i,a} \subseteq U_i$ or $W_{i,a} \subseteq \bar{U}_i$ holds. For every $i \in [t]$ let $W_{i,0}$ denote the exceptional set if it exists and let $W_{i,0}$ be arbitrary otherwise. Let $\mathscr{R}$ be the resulting partition. Moreover, we consider the partition $\mathscr{R}^*$ which is a refinement of $\mathscr{R}$ obtained by replacing $W_{i,0}$ by possibly two classes $U_i \cap W_{i,0}$ and $\bar{U}_i \cap W_{i,0}$. Since the contribution of the index of $\mathscr{R}$ and $\mathscr{R}^*$ may differ only on pairs with at least one vertex in $W_{i,0}$ for some $i \in [t]$ and since $|W_{i,0}| \leq \lfloor \varepsilon^2 n/(5t) \rfloor + 1$ for every $i \in [t]$ we infer that

$$\operatorname{ind}(\mathscr{R}^*) - \operatorname{ind}(\mathscr{R}) \leq \binom{n}{2}^{-1} \sum_{i=1}^{t} \left(\frac{\varepsilon^2 n}{5t} + 1 \right) n \leq \frac{\varepsilon^2}{2} .$$

for sufficiently large n. Furthermore, since $\mathscr{R}^*$ is a refinement of $\mathscr{Q}$ it follows from the Cauchy-Schwarz inequality that $\operatorname{ind}(\mathscr{Q}) \leq \operatorname{ind}(\mathscr{R}^*)$ and, consequently,

$$\operatorname{ind}(\mathscr{R}) \geq \operatorname{ind}(\mathscr{R}^*) - \frac{\varepsilon^2}{2} \geq \operatorname{ind}(\mathscr{Q}) - \frac{\varepsilon^2}{2} \overset{(2.6)}{\geq} \operatorname{ind}(\mathscr{P}) + \frac{\varepsilon^2}{2} ,$$

which concludes the proof of the theorem. □

The proof of Theorem 2.1 shows that choosing

$$T_{\mathrm{FK}}(\varepsilon, t_0) = \max\{t_0, 1/\varepsilon\} \cdot (6/\varepsilon^2)^{2/\varepsilon^2} = t_0 2^{\mathrm{poly}(1/\varepsilon)}$$

suffices. In fact, in each refinement step we split the vertex classes V_i into at most $\lfloor 5/\varepsilon^2 + 1 \rfloor \leq 6/\varepsilon^2$ classes $W_{i,a}$, when we construct $\mathscr{R}$. Hence, each time property (*ii*) fails the number of vertex classes of the new partition increases by a factor of $6/\varepsilon^2$ and in total there are at most $2/\varepsilon^2$ iterations.

On the other hand, it was shown by Lovász and B. Szegedy [LS07] that for every ε with $0 < \varepsilon \leq 1/3$ there are graphs for which every partition into t classes satisfying property (*ii*) of Theorem 2.1 requires $t \geq 2^{1/(8\varepsilon)}/4$ and, hence, $t \gg 1/\varepsilon$. As a consequence Theorem 2.1 does not allow to obtain useful bounds for $e(U \cap V_i, U \cap V_j)$, since for such a graph $\varepsilon n^2 \gg n^2/t^2 = |V_i||V_j|$. Property (*ii*) of Theorem 2.1 only implies

$$e(U \cap V_i, U \cap V_j) \approx d(V_i, V_j)|U \cap V_i||U \cap V_j|$$

on average over all pairs $i < j$ for every "large" set U. However, Szemerédi's regularity lemma (which was proved long before Theorem 2.1) allows to control $e(U \cap V_i, U \cap V_j)$ for most $i < j$. The price of this is, however, a significantly larger upper bound for the number of partition classes t.

2.2 Szemerédi's regularity lemma

In this section we show how Szemerédi's regularity lemma from [Sze78] can be obtained from Theorem 2.1 by iterated applications. For that we consider the following simple corollary of Theorem 2.1, which was first considered by Tao [Tao06a].

Corollary 2.5. *For all ν, $\varepsilon > 0$, every function $\delta\colon \mathbb{N} \to (0,1]$, and every $t_0 \in \mathbb{N}$ there exist $T_0 = T_0(\nu, \varepsilon, \delta(\cdot), t_0)$ and n_0 such that for every graph $G = (V, E)$ with at least $|V| = n \geq n_0$ vertices the following holds. There exists a vertex partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ with $V_1 \cup \ldots \cup V_t = V$ and a refinement $\mathscr{Q} = \{W_{i,j} \colon i \in [t],\ j \in [s]\}$ with $W_{i,1} \cup \ldots \cup W_{i,s} = V_i$ for every $i \in [t]$ such that*

(*i*) *$\mathscr{P}$ is an (ε, t_0, T_0)-FK-partition,*

(*ii*) *$\mathscr{Q}$ is a $(\delta(t), t_0, T_0)$-FK-partition, and*

(*iii*) $\operatorname{ind}(\mathscr{Q}) \leq \operatorname{ind}(\mathscr{P}) + \nu$.

Before we deduce Corollary 2.5 from Theorem 2.1, we discuss property (*iii*). Roughly speaking, if two refining partitions $\mathscr{P}$ and $\mathscr{Q}$ satisfy property (*iii*), then this implies that $d(W_{i,a}, W_{j,b})$ and $d(V_i, V_j)$ are "relatively close" for "most" choices of $i < j$ and $a, b \in [s]$. More precisely, we have the following, which was already observed by Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00].

Lemma 2.6. *Let γ, $\nu > 0$, let $G = (V, E)$ be a graph with n vertices, and for some positive integers t and s let $\mathscr{P} = \{V_i \colon i \in [t]\}$ with $V_1 \cup \ldots \cup V_t = V$ be a vertex partition and let $\mathscr{Q} = \{W_{i,j} \colon i \in [t],\ j \in [s]\}$ be a refinement with $W_{i,1} \cup \ldots \cup W_{i,s} = V_i$ for every $i \in [t]$. If $\operatorname{ind}(\mathscr{Q}) \leq \operatorname{ind}(\mathscr{P}) + \nu$, then*

$$\sum_{1 \leq i < j \leq t} \sum_{a,b \in [s]} \Big\{ |W_{i,a}||W_{j,b}| \colon |d(W_{i,a}, W_{j,b}) - d(V_i, V_j)| \geq \gamma \Big\} \leq \frac{\nu}{\gamma^2} n^2 \,.$$

Proof. For $1 \leq i < j \leq t$ let

$$A_{ij}^{+} = \{(a,b) \in [s] \times [s] \colon d(W_{i,a}, W_{j,b}) - d(V_i, V_j) \geq \gamma\} \,.$$

Since

$$\begin{aligned} d(V_i, V_j)|V_i||V_j| &= \sum_{a,b \in [s]} d(W_{i,a}, W_{j,b})|W_{i,a}||W_{j,b}| \\ &= \sum_{(a,b) \in A_{ij}^{+}} d(W_{i,a}, W_{j,b})|W_{i,a}||W_{j,b}| + \sum_{(a,b) \notin A_{ij}^{+}} d(W_{i,a}, W_{j,b})|W_{i,a}||W_{j,b}| \,, \end{aligned}$$

we obtain from the defect form of Cauchy-Schwarz (Lemma 2.4), that

$$\sum_{a,b \in [s]} d^2(W_{i,a}, W_{j,b})|W_{i,a}||W_{j,b}| \geq d^2(V_i, V_j)|V_i||V_j| + \gamma^2 \sum_{(a,b) \in A_{ij}^{+}} |W_{i,a}||W_{j,b}| \,.$$

Summing over all $1 \leq i < j \leq t$ we get

$$\operatorname{ind}(\mathscr{Q}) \geq \operatorname{ind}(\mathscr{P}) + \frac{\gamma^2}{\binom{n}{2}} \sum_{1 \leq i < j \leq t} \sum_{(a,b) \in A_{ij}^{+}} |W_{i,a}||W_{j,b}| \,.$$

Since, by assumption $\mathrm{ind}(\mathscr{Q}) \leq \mathrm{ind}(\mathscr{P}) + \nu$, we have

$$\sum_{1\leq i<j\leq t} \sum_{(a,b)\in A^+_{ij}} |W_{i,a}||W_{j,b}| \leq \frac{\nu}{\gamma^2}\binom{n}{2} \leq \frac{\nu n^2}{2\gamma^2}\,.$$

Repeating the argument with the appropriate definition of A^-_{ij} yields the claim. □

Proof of Corollary 2.5. For the proof of the corollary we iterate Theorem 2.1. Without loss of generality we may assume that $\delta(t) \leq \varepsilon$ for every $t \in \mathbb{N}$. For given ν, ε, $\delta(\cdot)$, and t_0, we apply Theorem 2.1 and obtain an (ε, t_0, T_0)-FK-partition $\mathscr{P}$ with t classes. Since in the proof of Theorem 2.1 the initial partition was an arbitrary equitable partition, we infer that after another application of Theorem 2.1 with $\delta(t)$ (in place of ε) and t_0 we obtain an equitable refinement $\mathscr{Q}$ of $\mathscr{P}$ which is a $(\delta(t), t_0, T_0)$-FK-partition with st classes. In other words, $\mathscr{P}$ and $\mathscr{Q}$ satisfy properties (i) and (ii) of Corollary 2.5 and if (iii) also holds, then we are done. On the other hand, if (iii) fails, then we replace $\mathscr{P}$ by $\mathscr{Q}$ and iterate, i.e., we apply Theorem 2.1 with $\delta(ts)$ and $t_0 = ts$ to obtain an equitable refinement $\mathscr{Q}'$ of $\mathscr{P}' = \mathscr{Q}$. Since we only iterate as long as (iii) of Corollary 2.5 fails and since ν is fixed throughout the proof, this procedure must end after at most $1/\nu$ iterations. Therefore the upper bound T_0 on the number of classes is in fact independent of G and n and can be given by a recursive formula depending on ν, ε, $\delta(\cdot)$, and t_0. □

We now show that Corollary 2.5 applied with the right choice of parameters yields the following theorem, which is essentially Szemerédi's regularity lemma from [Sze78].

Theorem 2.7. *For every $\varepsilon > 0$ and every $t_0 \in \mathbb{N}$ there exist $T_{Sz} = T_{Sz}(\varepsilon, t_0)$ and n_0 such that for every graph $G = (V, E)$ with at least $|V| = n \geq n_0$ vertices the following holds. There exists a partition $\mathscr{P}$ of V such that*

(i) *$\mathscr{P} = \{V_i\colon\ i \in [t]\}$ is t-equitable, where $t_0 \leq t \leq t_{Sz}$, and*

(ii) *for all but at most εt^2 pairs (V_i, V_j) with $i < j$ we have that for all subsets $U_i \subseteq V_i$ and $U_j \subseteq V_j$*

$$\big|e(U_i, U_j) - d(V_i, V_j)|U_i||U_j|\big| \leq \varepsilon|V_i||V_j|\,. \tag{2.7}$$

We note that the usual statement of Szemerédi's regularity lemma (cf. Theorem 1.2) is slightly different from the one above. Usually $\varepsilon|V_i||V_j|$ on the right-hand side of (2.7) is replaced by $\varepsilon|U_i||U_j|$ and for (ii) it is assumed that $|U_i| \geq \varepsilon|V_i|$ and $|U_j| \geq \varepsilon|V_j|$. However, applying Theorem 2.7 with $\varepsilon' = \varepsilon^3$ would yield a partition with comparable regular properties (cf. definition of ε-regular pairs in (1.1)).

Definition 2.8. *Pairs (V_i, V_j) for which (2.7) holds for every $U_i \subseteq V_i$ and $U_j \subseteq V_j$ are called ε-*uniform*. Partitions satisfying all three properties (i)-(ii) of Theorem 2.7, we will refer to as $(\varepsilon, t_0, T_{\mathrm{Sz}})$-*Szemerédi-partition*. Again we may sometimes omit t_0 and T_{Sz} and simply refer to such partitions as ε-*Szemerédi-partitions*.

Below we deduce Theorem 2.7 from Corollary 2.5 and Lemma 2.6.

Proof of Theorem 2.7. For given $\varepsilon > 0$ and t_0, we apply Corollary 2.5 with

$$\nu' = \frac{\varepsilon^4}{36^2}, \quad \varepsilon' = 1, \quad \delta'(t) = \frac{\varepsilon}{36t^2}, \quad \text{and} \quad t_0' = t_0$$

and obtain constants T_0' and n_0' which define $T_{\mathrm{Sz}} = T_0'$ and $n_0 = n_0'$. (We remark that the choice for ε' has no bearing for the proof and therefore we set it equal to 1.) For a given graph $G = (V, E)$ with n vertices Corollary 2.5 yields two partitions $\mathscr{P} = \{V_i \colon i \in [t]\}$ and $\mathscr{Q} = \{W_{i,j} \colon i \in [t],\ j \in [s]\}$ satisfying properties (i)-(iii) of Corollary 2.5. We will show that, in fact, the coarser partition $\mathscr{P}$ also satisfies properties (i) and (ii) of Theorem 2.7. Since $\mathscr{P}$ is an $(\varepsilon', t_0', T_0')$-FK-partition by our choice of $t_0' = t_0$ and $T_{\mathrm{Sz}} = T_0'$ the partition $\mathscr{P}$ obviously satisfies property (i) of Theorem 2.7 and we only have to verify property (ii).

For that we consider for every $1 \le i < j \le t$ the set

$$A_{ij} = \left\{(a,b) \in [s] \times [s] \colon |d(W_{i,a}, W_{j,b}) - d(V_i, V_j)| \ge \frac{\varepsilon}{6}\right\}$$

and we let

$$I = \left\{\{i,j\} \colon 1 \le i < j \le t \text{ such that } \sum_{(a,b)\in A_{ij}} |W_{i,a}||W_{j,b}| \ge \frac{\varepsilon}{6}|V_i||V_j|\right\}.$$

We will first show that $|I| \le \varepsilon t^2$ and then we will verify that if $\{i,j\} \notin I$, then (2.7) holds. Indeed, due to property (iii) of Corollary 2.5 we have $\mathrm{ind}(\mathscr{Q}) \le \mathrm{ind}(\mathscr{P}) + \nu'$ and, consequently, it follows from Lemma 2.6 (applied with $\nu' = \varepsilon^4/36^2$ and $\gamma' = \varepsilon/6$) that

$$\frac{\varepsilon^2 n^2}{36} \ge \sum_{i<j} \sum_{(a,b)\in A_{ij}} |W_{i,a}||W_{j,b}| \ge \sum_{\{i,j\}\in I} \sum_{(a,b)\in A_{ij}} |W_{i,a}||W_{j,b}| \ge \frac{\varepsilon}{6} \sum_{\{i,j\}\in I} |V_i||V_j|\,.$$

Moreover, since $|V_i| \ge \lfloor n/t \rfloor \ge n/(2t)$ for every $i \in [t]$ we have $\varepsilon n^2/6 \ge |I| n^2/(4t^2)$ and, consequently,

$$|I| \le \frac{2}{3}\varepsilon t^2 < \varepsilon t^2\,. \tag{2.8}$$

Next we will show that if $\{i,j\} \notin I$ then the pair (V_i, V_j) is ε-regular, i.e., we show that (2.7) holds for every $U_i \subseteq V_i$ and $U_j \subseteq V_j$. For given sets $U_i \subseteq V_i$ and $U_j \subseteq V_j$ and $a,\, b \in [s]$ we set

$$U_{i,a} = U_i \cap W_{i,a} \quad \text{and} \quad U_{j,b} = U_j \cap W_{j,b}$$

and have

$$e(U_i, U_j) = \sum_{a,b\in[s]} e(U_{i,a}, U_{j,b})\,.$$

Appealing to the fact that $\mathscr{Q}$ is a $(\delta'(t), t_0', T_0')$-FK-partition and to (2.2) we obtain

$$e(U_i, U_j) = \sum_{a,b\in[s]} d(W_{i,a}, W_{j,b})|U_{i,a}||U_{j,b}| \pm 6\delta'(t)n^2 \,.$$

From the assumption $\{i, j\} \notin I$ we infer

$$\sum_{(a,b)\in A_{ij}} d(W_{i,a}, W_{j,b})|U_{i,a}||U_{j,b}| \leq \sum_{(a,b)\in A_{ij}} |W_{i,a}||W_{j,b}| \leq \frac{\varepsilon}{6}|V_i||V_j|$$

and, furthermore, for $(a, b) \notin A_{ij}$ we have

$$d(W_{i,a}, W_{j,b})|U_{i,a}||U_{j,b}| = \left(d(V_i, V_j) \pm \frac{\varepsilon}{6}\right)|U_{i,a}||U_{j,b}| \,.$$

Combining, those three estimates we infer

$$e(U_i, U_j) = \sum_{a,b\in[s]} d(V_i, V_j)|U_{i,a}||U_{j,b}| \pm \frac{\varepsilon}{6}|U_i||U_j| \pm \frac{\varepsilon}{6}|V_i||V_j| \pm 6\delta'(t)n^2 \,.$$

Hence from our choice of $\delta'(t)$ and $V_i \geq \lfloor n/t \rfloor \geq n/(2t)$ we deduce

$$\big|e(U_i, U_j) - d(V_i, V_j)|U_i||U_j|\big| \leq \frac{\varepsilon}{3}|V_i||V_j| + \frac{\varepsilon}{6}\left(\frac{n}{t}\right)^2 \leq \varepsilon|V_i||V_j| \,,$$

which concludes the proof of Theorem 2.7. □

In contrast to Theorem 2.1 the upper bound $T_{\mathrm{Sz}} = T_{\mathrm{Sz}}(\varepsilon, t_0)$ we obtain from the proof of Theorem 2.7 is not exponential, but of tower-type. In fact, we use Corollary 2.5 with $\nu = \varepsilon^4/36^2$ and $\delta(t) = \varepsilon/(36t^2)$. Due to the choice of ν we iterate Theorem 2.1 at most $36^2/\varepsilon^4$ times and each time the number of classes grows exponentially, i.e., t_i classes from the i-th iteration may split into $2^{O(t_i^4/\varepsilon^2)}$ classes for the next step. As a consequence, the upper bound $T_{\mathrm{Sz}} = T_{\mathrm{Sz}}(\varepsilon, t_0)$, which we obtain from this proof, is a tower of 4's of height $O(\varepsilon^{-4})$ with t_0 as the last exponent. The proof of Szemerédi's regularity lemma from [Sze78] yields a similar upper bound of a tower of 2's of height proportional to ε^{-5}. However, recall that the statement from [Sze78] is slightly different from the version proved here, by having a smaller error term in (2.7). A lower bound of similar type was obtained by Gowers [Gow97]. In fact, Gowers showed an example of a graph for which any partition satisfying even only a considerably weaker version of property (ii) requires at least t classes, where t is a tower of 2's of height proportional to $1/\varepsilon^{1/16}$.

2.3 The (ε, r)-regularity lemma

As we have just discussed in the previous section, the example of Gowers shows that we cannot prevent the situation when the number of parts t of a Szemerédi-partition is much larger than, say, $1/\varepsilon$. For several applications this presents an obstacle which one would like to overcome. More precisely one would like to obtain some control of the densities of subgraphs which are of size much smaller than, say, n/t^2. The (ε, r)-regularity lemma (Theorem 2.9), the regularity lemma of Alon, Fischer, Krivelevich, and M. Szegedy (Theorem 2.10), and the regular approximation lemma (Theorem 2.11), were partly developed to address such issues.

A version for 3-uniform hypergraphs of the following regularity lemma was obtained by Frankl and Rödl in [FR02] (see Chapter 4).

Theorem 2.9. *For every $\varepsilon > 0$, every function $r\colon \mathbb{N} \to \mathbb{N}$, and every $t_0 \in \mathbb{N}$ there exist $T_{FR} = T_{FR}(\varepsilon, r(\cdot), t_0)$ and n_0 such that for every graph $G = (V, E)$ with at least $|V| = n \geq n_0$ vertices the following holds. There exists a partition $\mathscr{P}$ of V such that*

(i) $\mathscr{P} = \{V_i\colon i \in [t]\}$ is t-equitable, where $t_0 \leq t \leq T_{FR}$, and

(ii) for all but at most εt^2 pairs (V_i, V_j) with $i < j$ we have that for all sequences of subsets $U_i^1, \dots, U_i^{r(t)} \subseteq V_i$ and $U_j^1, \dots, U_j^{r(t)} \subseteq V_j$

$$\Big| \big| \bigcup_{q=1}^{r(t)} E(U_i^q, U_j^q) \big| - d(V_i, V_j) \big| \bigcup_{q=1}^{r(t)} U_i^q \times U_j^q \big| \Big| \leq \varepsilon |V_i||V_j| \,. \tag{2.9}$$

Note that if $r(t) \equiv 1$ then Theorem 2.9 is identical to Theorem 2.7 and if $r(t) \equiv k$ for some constant $k \in \mathbb{N}$ (independent of t), then it is a direct consequence of Theorem 2.7. We remark that for arbitrary functions $r(\cdot)$, Theorem 2.9 can be proved along the lines of Szemerédi's proof of Theorem 2.7 from [Sze78]. Below we deduce Theorem 2.9, using a slightly different approach, namely we infer Theorem 2.9 from Corollary 2.5 in a similar way as we proved Theorem 2.7.

Proof. For given ε, $r(\cdot)$, and t_0 we follow the lines of the proof of Theorem 2.7. This time we apply Corollary 2.5 with a smaller choice of $\delta'(\cdot)$

$$\nu' = \frac{\varepsilon^4}{36^2}, \quad \varepsilon' = 1, \quad \delta'(t) = \frac{\varepsilon}{36t^2(4^{r(t)} - 3^{r(t)})}, \quad \text{and} \quad t_0' = t_0$$

and obtain T_0' and n_0', which determines T_{FR} and n_0. We define the sets A_{ij} and I identical as in the proof of Theorem 2.7, i.e., for $1 \leq i < j \leq t$ we set

$$A_{ij} = \left\{ (a, b) \in [s] \times [s]\colon\ |d(W_{i,a}, W_{j,b}) - d(V_i, V_j)| \geq \frac{\varepsilon}{6} \right\}$$

and we let

$$I = \left\{ \{i, j\}\colon\ 1 \leq i < j \leq t \text{ such that } \sum_{(a,b) \in A_{ij}} |W_{i,a}||W_{j,b}| \geq \frac{\varepsilon}{6}|V_i||V_j| \right\}.$$

Again we obtain (2.8) and the rest of the proof requires some small straightforward adjustments.

We set $r = r(t)$ and we will show that if $\{i, j\} \notin I$, then (2.9) holds for every sequence $\hat{U}_i^1, \dots, \hat{U}_i^r \subseteq V_i$ and $\hat{U}_j^1, \dots, \hat{U}_j^r \subseteq V_j$. For such given sequences we consider new sequences $U_i^1, \dots, U_i^R \subseteq V_i$ and $U_j^1, \dots, U_j^R \subseteq V_j$ satisfying the disjointness property (see (2.10) below). For that let $R = 4^r - 3^r$ and for a non-empty set $\varnothing \neq L \subseteq [r]$ let

$$\hat{U}_i(L) = \bigcap_{\ell \in L} \hat{U}_i^\ell \setminus \bigcup_{\ell \in L} \hat{U}_i^\ell \quad \text{and} \quad \hat{U}_j(L) = \bigcap_{\ell \in L} \hat{U}_j^\ell \setminus \bigcup_{\ell \in L} \hat{U}_j^\ell$$

and for two sets L, L' with non-empty intersection we set

$$U_i(L, L') = \hat{U}_i(L) \quad \text{and} \quad U_j(L, L') = \hat{U}_j(L').$$

Note that there are $R = 4^r - 3^r$ such pairs of sets L, L' and we can relabel the sequences $(U_i(L, L'))_{L \cap L' \neq \varnothing}$ and $(U_j(L, L'))_{L \cap L' \neq \varnothing}$ to $U_i^1, \dots, U_i^R \subseteq V_i$ and $U_j^1, \dots, U_j^R \subseteq V_j$. Note that for all $p \neq q$ the sets U_i^p and U_i^q may either be equal or disjoint. Moreover, due to this definition we obtain for all $1 \leq p < q \leq R$

$$(U_i^q \times U_j^q) \cap (U_i^p \times U_j^p) = \varnothing \quad \text{and} \quad \bigcup_{q \in [R]} U_i^q \times U_j^q = \bigcup_{q \in [r]} \hat{U}_i^q \times \hat{U}_j^q \,. \tag{2.10}$$

Furthermore, for $q \in [R]$ and a, $b \in [s]$ we set

$$U_{i,a}^q = U_i^q \cap W_{i,a} \quad \text{and} \quad U_{j,b}^q = U_j^q \cap W_{j,b}$$

and we get for every $q \in [R]$

$$e(U_i^q, U_j^q) = \sum_{a,b \in [s]} e(U_{i,a}^q, U_{j,b}^q)\,.$$

Appealing to the fact that $\mathscr{Q}$ is a $(\delta'(t), t_0', T_0')$-FK-partition and to (2.2) we obtain

$$e(U_i^q, U_j^q) = \sum_{a,b \in [s]} d(W_{i,a}, W_{j,b}) |U_{i,a}^q| |U_{j,b}^q| \pm 6\delta'(t) n^2\,.$$

From the assumption $\{i, j\} \notin I$ and the disjointness property from (2.10) we infer

$$\sum_{(a,b) \in A_{ij}} \sum_{q \in [R]} d(W_{i,a}, W_{j,b}) |U_{i,a}^q| |U_{j,b}^q| \leq \sum_{(a,b) \in A_{ij}} |W_{i,a}| |W_{j,b}| \leq \frac{\varepsilon}{6} |V_i| |V_j|$$

and, furthermore, for $(a, b) \notin A_{ij}$ we have

$$d(W_{i,a}, W_{j,b}) |U_{i,a}^q| |U_{j,b}^q| = \left(d(V_i, V_j) \pm \frac{\varepsilon}{6} \right) |U_{i,a}^q| |U_{j,b}^q|$$

for every $q \in [R]$. Combining, those three estimates we infer

$$\begin{aligned}\Big|\bigcup_{q\in[r]} E(\hat{U}_i^q, \hat{U}_j^q)\Big| &= \Big|\bigcup_{q\in[R]} E(U_i^q, U_j^q)\Big| \\ &= \Big(d(V_i, V_j) \pm \frac{\varepsilon}{6}\Big) \sum_{q\in[R]} \sum_{a,b\in[s]} |U_{i,a}^q||U_{j,b}^q| \pm \frac{\varepsilon}{6}|V_i||V_j| \pm 6R\delta'(t)n^2 \\ &= d(V_i, V_j)\Big|\bigcup_{q\in[R]} E(U_i^q, U_j^q)\Big| \pm \frac{\varepsilon}{3}|V_i||V_j| \pm 6R\delta'(t)n^2 .\end{aligned}$$

Hence from our choice of $\delta'(t)$, $R = (4^r - 3^r)$, and $V_i \geq \lfloor n/t \rfloor \geq n/(2t)$ we deduce from (2.10)

$$\Big||\bigcup_{q=1}^r E(\hat{U}_i^q, \hat{U}_j^q)| - d(V_i, V_j)|\bigcup_{q=1}^r \hat{U}_i^q \times \hat{U}_j^q|\Big| \leq \varepsilon |V_i||V_j| ,$$

which concludes the proof of Theorem 2.9. □

2.4 The regularity lemma of Alon et al.

In the last two sections we iterated the regularity lemma of Frieze and Kannan and obtained Corollary 2.5, from which we deduced Szemerédi's regularity lemma (Theorem 2.7) and the (ε, r)-regularity lemma (Theorem 2.9). From this point of view it seems natural to iterate these stronger regularity lemmas. This was indeed first carried out by Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00] who iterated Szemerédi's regularity lemma for an application in the area of property testing.

Theorem 2.10. *For every ν, $\varepsilon > 0$, every function $\delta\colon \mathbb{N} \to (0, 1]$, and every $t_0 \in \mathbb{N}$ there exist $T_{AFKS} = T_{AFKS}(\nu, \varepsilon, \delta(\cdot), t_0)$ and n_0 such that for every graph $G = (V, E)$ with at least $|V| = n \geq n_0$ vertices the following holds. There exists a vertex partition $\mathscr{P} = \{V_i\colon i \in [t]\}$ with $V_1 \cup \ldots \cup V_t = V$ and a refinement $\mathscr{Q} = \{W_{i,j}\colon i \in [t],\ j \in [s]\}$ with $W_{i,1} \cup \ldots \cup W_{i,s} = V_i$ for every $i \in [t]$ such that*

(*i*) *$\mathscr{P}$ is an $(\varepsilon, t_0, T_{AFKS})$-Szemerédi-partition,*

(*ii*) *$\mathscr{Q}$ is a $(\delta(t), t_0, T_{AFKS})$-Szemerédi-partition, and*

(*iii*) $\mathrm{ind}(\mathscr{Q}) \leq \mathrm{ind}(\mathscr{P}) + \nu$.

Proof. The proof is identical to the proof of Corollary 2.5 with the only adjustment that we iterate Theorem 2.7 instead of Theorem 2.1. □

The price for the stronger properties of the partitions $\mathscr{P}$ and $\mathscr{Q}$, in comparison to Szemerédi's regularity lemma, is again in the bound T_{AFKS}. In general T_{AFKS} can be expressed as a recursive formula in $\nu, \varepsilon, \delta(\cdot)$, and t_0, and for example, if $\delta(t)$ is given by a polynomial in $1/t$, then T_{AFKS} is an iterated tower-type function, which is sometimes referred to as a wowzer-type function.

Theorem 2.9 relates to Theorem 2.10 in the following way. It is a direct consequence of (2.9) that if (V_i, V_j) is not one of the exceptional pairs in (*ii*) of Theorem 2.9, then for any partition of V_i and V_j into at most $\sqrt{r(t)}$ parts of equal size, "most" of the $r(t)$ pairs have the density "close" (up to an error of $O(\sqrt{\varepsilon})$) to $d(V_i, V_j)$. Hence, if we set at the beginning $r(t) = (T_{\mathrm{Sz}}(\delta(t), t))^2$ and then apply Theorem 2.7 to obtain a $(\delta(t), t, T_{\mathrm{Sz}}(\delta(t), t))$-Szemerédi-partition $\mathscr{Q}$, which refines the given partition, then we arrive to a similar situation as in Theorem 2.10. In fact, we have two Szemerédi-partitions satisfying (*i*) and (*ii*) of Theorem 2.10 and (*iii*) would be replaced by the fact that $d(W_{i,a}, W_{j,b}) \approx d(V_i, V_j)$ for "most" pairs from the finer partition $\mathscr{Q}$.

2.5 The regular approximation lemma

The following lemma is another byproduct of the hypergraph generalizations of the regularity lemma for graphs and the general form will be presented in Chapter 4 (see Theorem 4.12). In a different context, Theorem 2.11 also appeared in the work of Lovász and B. Szegedy [LS07, Lemma 5.2].

Theorem 2.11. *For every $\nu > 0$, every function $\varepsilon\colon \mathbb{N} \to (0, 1]$, and every $t_0 \in \mathbb{N}$ there exist $T_0 = T_0(\nu, \varepsilon(\cdot), t_0)$ and n_0 such that for every graph $G = (V, E)$ with at least $|V| = n \geq n_0$ vertices the following holds. There exists a partition $\mathscr{P} = \{V_i\colon i \in [t]\}$ with $V_1 \cup \ldots \cup V_t = V$ and a graph $H = (V, E')$ on the same vertex set V as G such that*

(*a*) *$\mathscr{P}$ is an $(\varepsilon(t), t_0, T_0)$-Szemerédi-partition for H and*

(*b*) $|E \triangle E'| = |E \setminus E'| + |E' \setminus E| \leq \nu n^2$.

The main difference between Theorem 2.11 and Theorem 2.7 is in the choice of ε being a function of t. As already mentioned, it follows from the work of Gowers [Gow97] (or alternatively from the work of Lovász and B. Szegedy [LS07, Proposition 7.1]) that it is not possible to obtain a Szemerédi (or even a Frieze-Kannan) partition for certain graphs G with ε of order $1/t$. Property (*a*) of Theorem 2.11 asserts, however, that by adding and deleting at most νn^2 edges from/to G we can obtain another graph H which admits a "much more" regular partition, e.g., with $\varepsilon(t) \ll 1/t$.

Below we deduce Theorem 2.11 from the iterated regularity lemma of Alon, Fischer, Krivelevich and M. Szegedy (Theorem 2.10). The idea is to apply Theorem 2.10 with appropriate parameters to obtain Szemerédi-partitions $\mathscr{P} = \{V_i\colon i \in [t]\}$ and $\mathscr{Q} = \{W_{i,j}\colon i \in [t],\ j \in [s]\}$ for which $\mathscr{Q}$ refines $\mathscr{P}$ and $\mathrm{ind}(\mathscr{Q}) \leq \mathrm{ind}(\mathscr{P}) + \nu'$. The last condition and Lemma 2.6 imply that $d(W_{i,a}, W_{j,b}) \approx d(V_i, V_j)$ (with an error depending on ν') for "most" $i < j$ and $a, b \in [s]$. The strong regularity of the finer partition $\mathscr{Q}$ will then be used to adjust G (by adding and removing a few edges randomly) to obtain H for which $\mathscr{P}$ will have the desired properties. We now give the details of this outline.

Proof of Theorem 2.11. For given ν, $\varepsilon(\cdot)$, and t_0 we apply Theorem 2.10 with

$$\nu' = \frac{\nu^3}{16},$$

some arbitrary ε', say $\varepsilon' = 1$, $\delta'(t) = \min\{\varepsilon(t)/2, \nu/4\}$, and $t'_0 = t_0$. We also fix an auxiliary constant $\gamma' = \nu/2$. We then set $T_0 = T'_{\text{AFKS}}$ and $n_0 = n'_0$. After we apply Theorem 2.10 to the given graph $G = (V, E)$, we obtain an (ε', t_0, T_0)-Szemerédi-partition $\mathscr{P}$ and a $(\delta'(t), t_0, T_0)$-Szemerédi-partition $\mathscr{Q}$ which refines $\mathscr{P}$ such that

$$\text{ind}(\mathscr{Q}) \leq \text{ind}(\mathscr{P}) + \nu'.$$

Next we will change G and obtain the graph H, which will satisfy (*a*) and (*b*) of Theorem 2.11. For that:

(*A*) we replace every subgraph $G[W_{i,a}, W_{j,b}]$ which is not $\delta'(t)$-regular by a random bipartite graph of density $d(V_i, V_j)$ and

(*B*) for every $1 \leq i < j \leq t$ and a, $b \in [s]$ we add or remove edges randomly to change the density of $G[W_{i,a}, W_{j,b}]$ to $d(V_i, V_j) + o(1)$.

It follows from the Chernoff bound that the resulting graph $H = (V, E')$ has the property that for *every* $1 \leq i < j \leq t$ and a, $b \in [s]$ the induced subgraph $H[W_{i,a}, W_{j,b}]$ is $(\delta'(t) + o(1))$-uniform and $d_H(W_{i,a}, W_{j,b}) = d_H(V_i, V_j) + o(1)$, where $o(1) \to 0$ as $n \to \infty$. (Recall that for G from Lemma 2.6 we can only infer that $d_G(W_{i,a}, W_{j,b}) = d_G(V_i, V_j) \pm \gamma'$ for "most" pairs for some $\gamma' \gg \delta'(t)$.) Hence for every $1 \leq i < j \leq t$ and arbitrary sets $U_i \subseteq V_i$ and $U_j \subseteq V_j$ we have

$$\begin{aligned} e_H(U_i, U_j) &= \sum_{a,b \in [s]} \Big(d_H(W_{i,a}, W_{j,b})|U_i \cap W_{i,a}||U_j \cap W_{j,b}| \pm (\delta'(t) + o(1))|W_{i,a}||W_{j,b}| \Big) \\ &= d_H(V_i, V_j)|U_i||U_j| \pm 2\delta'(t)|V_i||V_j|. \end{aligned}$$

In other words, the partition $\mathscr{P}$ is a $(2\delta'(t) \leq \varepsilon(t), t_0, T_0)$-Szemerédi-partition for H, which is assertion (*a*) of Theorem 2.11. For part (*b*) we will estimate the symmetric difference of E and E'. Since $\mathscr{Q}$ is a $(\delta'(t), t_0, T_0)$-Szemerédi-partition for G the changes in Step (*A*) contributed at most

$$\delta'(t) t^2 s^2 \left\lceil \frac{n}{ts} \right\rceil^2 \leq \frac{\nu}{2} n^2 \tag{2.11}$$

to that difference.

In order to estimate the number of changed pairs introduced in Step (*B*) we appeal to Lemma 2.6. From that we infer that, since $\text{ind}(\mathscr{Q}) \leq \text{ind}(\mathscr{P}) + \nu'$, we "typically" changed only $\gamma'|W_{i,a}||W_{j,b}|$ pairs. More precisely, in Step (B) we changed at most

$$\sum_{i<j}\sum_{a,b\in[s]}\gamma'|W_{i,a}||W_{j,b}|+\sum_{i<j}\sum_{a,b\in[s]}\Big\{|W_{i,a}||W_{j,b}|\colon\ |d_G(W_{i,a},W_{j,b})-d_G(V_i,V_j)|\geq\gamma'\Big\}$$
$$\leq\left(\frac{\gamma'}{2}+\frac{\nu'}{(\gamma')^2}\right)n^2\leq\frac{\nu}{2}n^2\,. \quad (2.12)$$

Finally, from (2.11) and (2.12) we infer $|E\triangle E'|\leq\nu n^2$, which shows that H satisfies property (b) of Theorem 2.11. □

2.6 An early version of the regularity lemma

In this section we state an early version of Szemerédi's regularity lemma, which was introduced in [Sze75] and one of the key components in the proof of Theorem 1.1. Another application of that lemma lead to the upper bound for the Ramsey-Turán problem for K_4 due to Szemerédi [Sze72] and to the resolution of the $(6,3)$-problem, which was raised by Brown, Erdős and Sós [BES73, SEB73], and solved by Ruzsa and Szemerédi [RS78] (cf. Theorem 1.3).

Theorem 2.12. *For all positive ε_1, ε_2, δ, ϱ, and σ there exist T_0, S_0, M, and N such that for every bipartite graph $G=(X\cup Y,E)$ satisfying $|X|=m\geq M$ and $|Y|=n\geq N$ there exists a partition $X_0\cup X_1\cup\ldots\cup X_t=X$ with $t\leq T_0$ and for every $i=1,\ldots,t$ there exists a partition $Y_{i,0}\cup Y_{i,1}\cup\ldots\cup Y_{i,s_i}=Y$ with $s_i\leq S_0$ such that*

- *(a) $|X_0|\leq\varrho m$ and $|Y_{i,0}|\leq\sigma n$ for every $i=1,\ldots,t$, and*
- *(b) for every $i=1,\ldots,t$, every $j=1,\ldots,s_i$, and all sets $U\subseteq X_i$ and $W\subseteq Y_{i,j}$ with $|U|\geq\varepsilon_1|X_i|$ and $|W|\geq\varepsilon_2|Y_{i,j}|$ we have $d(U,W)\geq d(X_i,Y_{i,j})-\delta$.* □

Note that this lemma does not ensure such an elegant and easy to use structure of the partition as the later lemmas. More precisely, the partitions of Y may be very different for every $i=1,\ldots,t$. On the other hand, the upper bounds T_0 and S_0 are of similar type as those of Theorem 2.1, i.e., we have $T_0,S_0=2^{\mathrm{poly}(1/\min\{\varepsilon_1,\varepsilon_2,\delta,\varrho,\sigma\})}$. We also point out that for example in [RS78] Theorem 2.12 was applied iteratively, which in turn lead to a tower-type bound for the $(6,3)$-problem and up to now no better bound was found. A multipartite version of Theorem 2.12 was developed by Duke, Lefmann, and Rödl [DLR95] for efficiently approximating the subgraph frequencies in a given graph G on n vertices for subgraphs of up to $\Omega(\sqrt{\log\log(n)})$ vertices.

Theorem 2.13. *For every $\varepsilon>0$ and every integer $k\geq 2$ there exist $T_0=4^{k^2/\varepsilon^5}$ such that for every k-partite graph $G=(V,E)$ with vertex classes $V_1\cup\ldots\cup V_k=V$ and $|V_1|=\cdots=|V_k|=N$ there exists a partition $\mathscr{P}$ of $V_1\times\cdots\times V_k$ such that*

- *(i) the number of elements $W_1\times\cdots\times W_k$ in $\mathscr{P}$ is at most T_0,*
- *(ii) $|W_i|\geq\varepsilon^{k^2/\varepsilon^5}N$ for every $i=1,\ldots,k$ and every $W_1\times\cdots\times W_k$ in $\mathscr{P}$, and*

(*iii*) *we have*

$$\sum_{W_1\times\dots\times W_k\in\mathscr{P}_{irr}}\prod_{i=1}^{k}|W_i|\leq\varepsilon N^k\,.$$

for the subfamily $\mathscr{P}_{irr}\subseteq\mathscr{P}$ *containing those elements* $W_1\times\dots\times W_k$ *from* $\mathscr{P}$ *which contain an irregular pair* (W_i,W_j)*, i.e., a pair* (W_i,W_j) *with* $i<j$ *for which there exist subsets* $U_i\subseteq W_i$ *and* $U_j\subseteq W_j$ *with* $|U_i|\geq\varepsilon|W_i|$ *and* $|U_j|\geq\varepsilon|W_j|$ *such that* $|d(U_i,U_j)-d(W_i,W_j)|>\varepsilon$*.* □

The main advantage of Theorem 2.13, in comparison to Szemerédi's regularity lemma (Theorem 2.7), is the smaller upper bound T_0. The partition in Theorem 2.13 still conveys information if $1/\varepsilon$ and k tend slowly to infinity with $n=|V|$, for example, if $1/\varepsilon$ and k are of order $\log^c(n)$ for some small constant $c>0$. Due to the tower-type bound of Theorem 2.7 there $1/\varepsilon$ can be at most of order $\log^*(n)$, where $\log^*$ denotes the iterated logarithm function.

On the other hand, the upper bound T_0 in Theorem 2.13 is comparable to the one from Theorem 2.1 and as we will see in the next section Theorem 2.1 would be also well suited for the main application of Theorem 2.13 in [DLR95]. Moreover, the structure of the partition provided by Theorem 2.1 seems to be simpler and easier to work with.

2.7 Reduced graph and counting lemmas

In this section we show how regular properties of the partitions given by the regularity lemmas in the earlier sections can be applied to approximate the number of subgraphs of fixed isomorphism type of a given graph G. More precisely, for graphs G and F let $N_F(G)$ denote the number of labeled copies of F in G. Roughly speaking, we will show that $N_F(G)$ can be fairly well approximated by only studying the so-called *reduced graph* (or *cluster-graph*) of a regular partition.

Definition 2.14. *Let* $\varepsilon>0$*,* $G=(V,E)$ *be a graph, and let* $\mathscr{P}=\{V_i\colon i\in[t]\}$ *be a partition of* V*.*

(*i*) *For an* ε*-FK-partition* $\mathscr{P}$ *the* reduced graph $R=R_G(\mathscr{P})$ *is defined to be the weighted, complete, undirected graph with vertex set* $V(R)=[t]$ *and with edge weights* $w_R(i,j)=d(V_i,V_j)$*.*

(*ii*) *For an* ε*-Szemerédi-partition* $\mathscr{P}$ *the* reduced graph $R=R_G(\mathscr{P},\varepsilon)$ *is defined to be the weighted, undirected graph with vertex set* $V(R)=[t]$*, edge set* $E(R)=\{\{i,j\}\colon (V_i,V_j)$ *is* ε*-regular*$\}$*, and edge weights* $w_R(i,j)=d(V_i,V_j)$*.*

The reduced graph carries a lot of the structural information of the given graph G. In fact, in many applications of the regularity lemma, the original problem for G one is interested in can be turned into a "simpler" problem for the reduced graph.

Remark 2.15. Below we will consider (labeled) copies F_R of a given graph F in a reduced graph R. If R is the reduced graph of an FK-partition, then R is an edge-weighting of the complete graph and, consequently, any ordered set of $|V(F)|$ vertices of $V(R)$ spans a copy of F. On the other hand, if R is the reduced graph of an ε-Szemerédi-partition, then R is not a complete graph and for a labeled copy F_R of F in R with $V(F_R) = \{i_1, \dots, i_\ell\}$ we will have that (V_{i_j}, V_{i_k}) is ε-uniform for every edge $\{i_j, i_k\} \in E(F_R)$.

2.8 The global counting lemma

Here by a *counting lemma* we mean an assertion which enables us to deduce directly from the reduced graph some useful information on the number $N_F(G)$ of labeled copies of a fixed graph F in a large graph G. We will distinguish between two different settings here. The first counting lemma will yield an estimate on $N_F(G)$ in the context of Theorem 2.1. Since $N_F(G)$ concerns the total number of copies, we regard this result as a *global counting lemma*.

In contrast, for an ℓ-vertex graph F the *local counting lemma* (Theorem 2.18) will yield estimates on $N_F(G[V_{i_1}, \dots, V_{i_\ell}])$ for an induced ℓ-partite subgraph of G given by the regular partition $\mathscr{P}$. However, for such a stronger assertion we will require that $\mathscr{P}$ be a Szemerédi-partition.

Theorem 2.16. *Let F be a graph with vertex set $V(F) = [\ell]$. For every $\gamma > 0$ there exists $\varepsilon > 0$ such that for every $G = (V, E)$ with $|V| = n$ and every ε-FK-partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ with reduced graph $R = R_G(\mathscr{P})$ we have*

$$N_F(G) = \sum_{F_R} \prod_{\{i_j, i_k\} \in E(F_R)} w_R(i_j, i_k) \prod_{i_j \in V(F_R)} |V_{i_j}| \pm \gamma n^\ell\,, \tag{2.13}$$

where the sum runs over all labeled copies F_R of F in R (cf. Remark 2.15).

For a simpler notation we denote here and below the vertices $V(F_R)$ of a given copy of F_R of the ℓ-vertex graph F in R by $\{i_1, \dots, i_\ell\}$ and omit the dependence of F_R.

Proof. We follow an argument of Lovász and B. Szegedy from [LS06]. We prove Theorem 2.18 by induction on the number of edges of F. Clearly, the theorem holds for graphs with no edges and for graphs with one edge it follows from the definition of ε-FK-partition with $\varepsilon = \gamma$.

For given F and γ we let $\varepsilon \leq \gamma/12$ be sufficiently small, so that the statement for the induction assumption holds with $\gamma' = \gamma/2$. For two vertices $x, y \in V$ we set

$$d_{\mathscr{P}}(x, y) = \begin{cases} 0 & \text{if } x, y \in V_i \text{ for some } i \in [t], \\ d(V_i, V_j) & \text{if } x \in V_i \text{ and } y \in V_j \text{ for some } 1 \leq i < j \leq t \end{cases}$$

and we denote by $\mathbb{1}_E(x, y)$ the indicator function for E, i.e., $\mathbb{1}_E(x, y)$ equals 1 if $\{x, y\} \in E$ and it equals 0 otherwise. We consider the difference of the left-hand side and the main term of the right-hand side in (2.13) and obtain

$$\left|N_F(G) - \sum_{F_R} \prod_{\{i_j,i_k\}\in E(F_R)} w_R(i_j,i_k) \prod_{i_j\in V(F_R)} |V_{i_j}|\right| \\ = \left|\sum_{x_1,\dots,x_\ell\in(V)_\ell} \left(\prod_{\{i,j\}\in E(F)} \mathbb{1}_E(x_i,x_j) - \prod_{\{i,j\}\in E(F)} d_{\mathscr{P}}(x_i,x_j)\right)\right|, \quad (2.14)$$

where $x_1,\dots,x_\ell \in (V)_\ell$ is an arbitrary sequence of ℓ distinct vertices in V. Without loss of generality we may assume that $\{\ell-1,\ell\}$ is an edge in F and we denote by F^- the spanning subgraph of F with the edge $\{\ell-1,\ell\}$ removed. Then, applying the identity $\alpha_1\alpha_2 - \beta_1\beta_2 = \beta_2(\alpha_1-\beta_1) + \alpha_1(\alpha_2-\beta_2)$, we get the following upper bound for the right-hand side of the last equation

$$\left|\sum_{x_1,\dots,x_\ell\in(V)_\ell} d_{\mathscr{P}}(x_{\ell-1},x_\ell)\left(\prod_{\{i,j\}\in E(F^-)} \mathbb{1}_E(x_i,x_j) - \prod_{\{i,j\}\in E(F^-)} d_{\mathscr{P}}(x_i,x_j)\right)\right| \\ + \left|\sum_{x_1,\dots,x_\ell\in(V)_\ell} \left(\prod_{\{i,j\}\in E(F^-)} \mathbb{1}_E(x_i,x_j)\right)\left(\mathbb{1}_E(x_{\ell-1},x_\ell) - d_{\mathscr{P}}(x_{\ell-1},x_\ell)\right)\right|. \quad (2.15)$$

By the induction assumption we can bound the first term by $\gamma' n^\ell$, i.e., we have

$$\left|\sum_{x_1,\dots,x_\ell\in(V)_\ell} d_{\mathscr{P}}(x_{\ell-1},x_\ell)\left(\prod_{\{i,j\}\in E(F^-)} \mathbb{1}_E(x_i,x_j) - \prod_{\{i,j\}\in E(F^-)} d_{\mathscr{P}}(x_i,x_j)\right)\right| \\ \le \gamma' n^\ell. \quad (2.16)$$

We will verify a similar bound for the second term in (2.15). For that we will split the second term of (2.15) into two parts and rewrite each of the parts (see (2.17) and (2.18) below).

We consider the induced subgraph F^* of F, which we obtain by removing the vertices labeled $\ell-1$ and ℓ from F. For a copy $\widetilde{F}^*$ of F^* in G let $X_{\ell-1}(\widetilde{F}^*)$ and $X_\ell(\widetilde{F}^*)$ be those vertex sets such that for every pair $x_{\ell-1}\in X_{\ell-1}(\widetilde{F}^*)$ and $x_\ell \in X_\ell(\widetilde{F}^*)$ of distinct vertices, those two vertices extend $\widetilde{F}^*$ in G to a copy of F^-. More precisely, if $x_1,\dots,x_{\ell-2}$ is the vertex set of $\widetilde{F}^*$ then we set

$$X_{\ell-1}(\widetilde{F}^*) = \bigcap_{i\colon \{i,\ell-1\}\in E(F)} \Gamma_G(x_i)$$

and

$$X_\ell(\widetilde{F}^*) = \bigcap_{i\colon \{i,\ell\}\in E(F)} \Gamma_G(x_i)\,,$$

where $\Gamma_G(x)$ denotes the set of neighbours of x in G. To simplify the notation, below we will write $X_{\ell-1}$ or X_ℓ instead of $X_{\ell-1}(\widetilde{F}^*)$ or $X_\ell(\widetilde{F}^*)$ as $\widetilde{F}^*$ will be clear from the context. Since by definition edges contained in $X_{\ell-1} \cap X_\ell$ are counted twice in $e(X_{\ell-1}, X_\ell)$ (cf. (2.2)) we observe for the first part of the seond term in (2.15) that

$$\sum_{x_1,\dots,x_\ell \in (V)_\ell} \left(\prod_{\{i,j\} \in E(F^-)} \mathbb{1}_E(x_i, x_j) \right) \mathbb{1}_E(x_{\ell-1}, x_\ell) = \sum_{\widetilde{F}^*} e(X_{\ell-1}, X_\ell)\,, \tag{2.17}$$

Moreover, we have for the second part of the seond term in (2.15)

$$\begin{aligned} \sum_{x_1,\dots,x_\ell \in (V)_\ell} &\left(\prod_{\{i,j\} \in E(F^-)} \mathbb{1}_E(x_i, x_j) \right) d_{\mathscr{P}}(x_{\ell-1}, x_\ell) \\ &= \sum_{\widetilde{F}^*} \sum_{i \neq j \in [t]} d(V_i, V_j) |X_{\ell-1} \cap V_i| |X_\ell \cap V_j| \end{aligned} \tag{2.18}$$

and, consequently, we can bound the second term in (2.15) by

$$\sum_{\widetilde{F}^*} \left| e(X_{\ell-1}, X_\ell) - \sum_{i \neq j \in [t]} d(V_i, V_j) |X_{\ell-1} \cap V_i| |X_\ell \cap V_j| \right| \tag{2.19}$$

Finally, we can apply the fact that $\mathscr{P}$ is an ε-FK-partition in form of (2.2) and the fact that $N_{F^*}(G) \leq n^{\ell-2}$ to bound (2.19) by $n^{\ell-2} \cdot 6\varepsilon n^2$. Hence, from (2.14)–(2.19) we infer

$$\left| N_F(G) - \sum_{F_R} \prod_{\{i_j, i_k\} \in E(F_R)} w_R(i_j, i_k) \prod_{i_j \in V(F_R)} |V_{i_j}| \right| \leq (\gamma' + 6\varepsilon) n^\ell \leq \gamma n^\ell\,,$$

which concludes the proof of Theorem 2.18. □

A simple argument based on the principle of inclusion and exclusion yields an induced version of Theorem 2.16. Let $N_F^*(G)$ denote the number of labeled, induced copies of F in G.

Corollary 2.17. *Let F be a graph with vertex set $V(F) = [\ell]$. For every $\gamma > 0$ there exists $\varepsilon > 0$ such that for every $G = (V, E)$ with $|V| = n$ and every ε-FK-partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ with reduced graph $R = R_G(\mathscr{P})$ we have*

$$N_F^*(G) = \sum_{F_R} \prod_{\{i_j, i_k\} \in E(F_R)} w_R(i_j, i_k) \prod_{\{i_j, i_k\} \in E(\overline{F_R})} (1 - w_R(i_j, i_k)) \prod_{i_j \in V(F_R)} |V_{i_j}| \pm \gamma n^\ell,$$

where the sum runs over all labeled copies F_R of F in R and $\overline{F_R}$ denotes the complement graph of F_R on the same ℓ vertices $V(F_R)$.

Proof. Let F be a graph with $V(F) = [\ell]$ and let K^ℓ be the complete graph on the same vertex set. Let ε be sufficiently small, so that we can apply Theorem 2.16 with

$\gamma' = \gamma/2^{\binom{\ell}{2}-e(F)}$ for every graph $F' \subseteq K^\ell$ which contains F. Let G, an ε-FK-partition $\mathscr{P}$, and a reduced graph $R = R_G(\mathscr{P})$ be given.

Due to the principle of inclusion and exclusion we have

$$N_F^*(G) = \sum_{F \subseteq F' \subseteq K^\ell} (-1)^{e(F')-e(F)} N_{F'}(G)\,,$$

where we sum over all supergraphs F' of F contained in K^ℓ. Applying Theorem 2.16 for every such F' we obtain

$$N_F^*(G) = \sum_{F'} (-1)^{e(F')-e(F)} \left(\sum_{F'_R} \prod_{\{i_j,i_k\} \in E(F'_R)} w_R(i_j,i_k) \prod_{i_j \in V(F_R)} |V_{i_j}| \right) \pm \gamma n^\ell\,,$$

where the outer sum runs over all F' with $F \subseteq F' \subseteq K^\ell$ and the inner sum is indexed by all copies F'_R of F' in R. We can rewrite the main term by rearranging the sum in the following way: First we sum over all possible labeled copies F_R of F in R. Note that this fixes a unique labeled copy $K^\ell(F_R)$ of K^ℓ as well, and in the inner sum we consider all graphs F'_R in R "sandwiched" between F_R and $K^\ell(F_R)$. This way we obtain

$$\begin{aligned}
N_F^*(G) \pm \gamma n^\ell &= \sum_{F_R} \sum_{F_R \subseteq F'_R \subseteq K^\ell(F_R)} (-1)^{e(F'_R)-e(F_R)} \prod_{\{i_j,i_k\} \in E(F'_R)} w_R(i_j,i_k) \prod_{i_j \in V(F_R)} |V_{i_j}| \\
&= \sum_{F_R} \prod_{\{i_j,i_k\} \in E(F_R)} w_R(i_j,i_k) \prod_{i_j \in V(F_R)} |V_{i_j}| \\
&\qquad \times \sum_{F_R \subseteq F'_R \subseteq K^\ell(F_R)} (-1)^{e(F'_R)-e(F_R)} \prod_{\{i_j,i_k\} \in E(F'_R) \setminus E(F_R)} w_R(i_j,i_k) \\
&= \sum_{F_R} \prod_{\{i_j,i_k\} \in E(F_R)} w_R(i_j,i_k) \prod_{i_j \in V(F_R)} |V_{i_j}| \\
&\qquad \times \prod_{\{i_j,i_k\} \in E(K^\ell(F_R)) \setminus E(F_R)} (1 - w_R(i_j,i_k))\,,
\end{aligned}$$

which concludes the proof. □

2.9 The local counting lemma

For graphs F and G, a partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ of $V(G)$, and a labeled copy F_R of F in R with $V(F_R) = \{i_1, \ldots, i_\ell\}$ we denote by $N_F(G[F_R])$ the number of *partite isomorphic-copies* of F_R (and hence of F) in G induced on $V_{i_1} \cup \ldots \cup V_{i_\ell}$. In other words, $N_F(G[F_R])$ is the number of edge preserving mappings φ from $V(F_R)$ to $V_{i_1} \cup \ldots \cup V_{i_\ell}$ such that $\varphi(i_j) \in V_{i_j}$ for every $j = 1, \ldots, \ell$.

Roughly speaking, the global counting lemma from the last section asserts that if $\mathscr{P}$ is a sufficiently regular ε-FK-partition, then $N_G(F)$ can be estimated from the reduced graph $R_G(\mathscr{P})$. In fact, it follows that the average of $N_F(G[F_R])$ over all labeled copies F_R

of F in R is "close" to its expectation. The local counting lemma (Theorem 2.18), states that if $\mathscr{P}$ is, in fact, a sufficiently regular Szemerédi-partition, then this is not only true on average, but indeed for every copy F_R of F in R.

Recall, that by definition the edge set $E(R)$ of a reduced graph of a Szemerédi-partition $\mathscr{P}$ corresponds to the regular pairs of $\mathscr{P}$. Consequently, for a copy F_R of F in R we require that all edges of F_R correspond to regular pairs.

Theorem 2.18. *Let F be a graph with ℓ vertices. For every $\gamma > 0$ there exists $\varepsilon > 0$ such that for every $G = (V, E)$ with $|V| = n$ and every ε-Szemerédi-partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ with reduced graph $R = R_G(\mathscr{P}, \varepsilon)$ we have for every labeled copy F_R of F in R with $V(F_R) = \{i_1, \dots, i_\ell\}$*

$$N_F(G[F_R]) = \prod_{\{i_j,i_k\}\in E(F_R)} w_R(i_j, i_k) \prod_{i_j\in V(F_R)} |V_{i_j}| \pm \gamma \prod_{i_j\in V(F_R)} |V_{i_j}| \,.$$

Theorem 2.18 concerns the number of copies of a fixed graph F and will only give interesting bounds if we can assert $\gamma \ll \prod_{\{i_j,i_k\}\in E(F_R)} w_R(i_j, i_k)$. Moreover, it was shown by Chvátal, Rödl, Szemerédi, and Trotter [CRST83], that if H is a graph of bounded degree with cn/t vertices (for some appropriately small constant $c > 0$ which depends on $\min_{\{i_j,i_k\}\in E(F_R)} w_R(i_j, i_k)$ and $\Delta(H)$) and there exists a homomorphism from H into F_R, then, under the same assumptions as in Theorem 2.18, $G[F_R]$ contains a copy of H. A far reaching strengthening, the so-called *blow-up lemma*, was found by Komlós, Sárközy, and Szemerédi [KSS97]. The blow-up lemma allows, under some slightly more restrictive assumptions, to embed spanning graphs H of bounded degree.

Proof. We prove Theorem 2.18 by induction on the number of edges of F. Since the theorem is trivial for graphs with no edges and it follows from the definition of ε-Szemerédi-partition for $\varepsilon = \gamma$ for graphs with precisely one edge.

Let F be a graph with at least two edges and ℓ vertices. For given $\gamma > 0$ let $\varepsilon \le \gamma/2$ be sufficiently small, so that the theorem holds for F^- with $\gamma' = \gamma/2$. Let $G = (V, E)$ be given along with an ε-Szemerédi-partition $\mathscr{P} = \{V_i \colon i \in [t]\}$ and let F_R be a labeled copy of F in R. Without loss of generality we may assume that $V(F_R) = \{1, \dots, \ell\}$ and that $\{\ell - 1, \ell\}$ is an edge of F_R. We denote by F_R^- the subgraph of F_R which we obtain after deleting the edge $\{\ell - 1, \ell\}$ from F_R. We can express the number of partite isomorphic copies of F_R through

$$\begin{aligned} N_F(G[F_R]) &= \sum_{x_1\in V_1} \cdots \sum_{x_\ell\in V_\ell} \prod_{\{i,j\}\in E(F_R)} \mathbb{1}_E(x_i, x_j) \\ &= \sum_{x_1\in V_1} \cdots \sum_{x_\ell\in V_\ell} \prod_{\{i,j\}\in E(F_R^-)} \Big(\mathbb{1}_E(x_i, x_j) \times \\ &\qquad \times \Big(d(V_{\ell-1}, V_\ell) + \mathbb{1}_E(x_{\ell-1}, x_\ell) - d(V_{\ell-1}, V_\ell) \Big) \Big) \,. \end{aligned}$$

The last expression can be rewritten as

$$d(V_{\ell-1},V_\ell)\times N_{F^-}(G[F_R^-])+ \\ + \sum_{x_1\in V_1}\cdots\sum_{x_\ell\in V_\ell}\Big(\prod_{\{i,j\}\in E(F_R^-)}\mathbb{1}_E(x_i,x_j)\Big(\mathbb{1}_E(x_{\ell-1},x_\ell)-d(V_{\ell-1},V_\ell)\Big)\Big).$$

From the induction assumption we then infer

$$d(V_{\ell-1},V_\ell)N_F(G[F_R^-])=\prod_{\{i_j,i_k\}\in E(F_R)}w_R(i_j,i_k)\prod_{i_j\in V(F_R)}|V_{i_j}|\pm\frac{\gamma}{2}\prod_{i_j\in V(F_R)}|V_{i_j}|$$

and, therefore, it suffices to verify

$$\left|\sum_{x_1\in V_1}\cdots\sum_{x_\ell\in V_\ell}\Big(\prod_{\{i,j\}\in E(F_R^-)}\mathbb{1}_E(x_i,x_j)\Big(\mathbb{1}_E(x_{\ell-1},x_\ell)-d(V_{\ell-1},V_\ell)\Big)\Big)\right| \\ \leq\frac{\gamma}{2}\prod_{i_j\in V(F_R)}|V_{i_j}| \quad (2.20)$$

For that we will appeal to the regularity of $\mathscr{P}$. Let F_R^* be the induced subgraph of F_R which one obtains by removing the vertices $\ell-1$ and ℓ. For a partite isomorphic copy $\widetilde{F}^*$ of F_R^*, let $X_{\ell-1}(\widetilde{F}^*)\subseteq V_{\ell-1}$ and $X_\ell(\widetilde{F}^*)\subseteq V_\ell$ be those sets of vertices for which any choice of $x_{\ell-1}\in X_{\ell-1}(\widetilde{F}^*)$ and $x_\ell\in X_\ell(\widetilde{F}^*)$ complete $\widetilde{F}^*$ to a partite isomorphic copy of F_R^-. Consequently, summing over all partite isomorphic copies $\widetilde{F}^*$ of F_R^* in G we obtain

$$\begin{aligned}\left|\sum_{x_1\in V_1}\cdots\sum_{x_\ell\in V_\ell}\Big(\prod_{\{i,j\}\in E(F_R^-)}\mathbb{1}_E(x_i,x_j)\Big(\mathbb{1}_E(x_{\ell-1},x_\ell)-d(V_{\ell-1},V_\ell)\Big)\Big)\right| \\ =\left|\sum_{\widetilde{F}^*}e\big(X_{\ell-1}(\widetilde{F}^*),X_\ell(\widetilde{F}^*)\big)-d(V_{\ell-1},V_\ell)|X_{\ell-1}(\widetilde{F}^*)||X_\ell(\widetilde{F}^*)|\right| \\ \leq\sum_{\widetilde{F}^*}\left|e\big(X_{\ell-1}(\widetilde{F}^*),X_\ell(\widetilde{F}^*)\big)-d(V_{\ell-1},V_\ell)|X_{\ell-1}(\widetilde{F}^*)||X_\ell(\widetilde{F}^*)|\right| \\ \leq\prod_{i=1}^{\ell-2}|V_i|\times\varepsilon|V_{\ell-1}||V_\ell|,\end{aligned}$$

where in the last estimate we used the ε-regularity of $(V_{\ell-1},V_\ell)$ and the obvious upper bound on the number of partite isomorphic copies $\widetilde{F}^*$ of F_R^*. Since $\varepsilon\leq\gamma/2$ the assertion (2.20) follows and concludes the proof of Theorem 2.18. □

We close this section by noting that an induced version of Theorem 2.18 can be derived directly from Theorem 2.18 in a similar way as Corollary 2.17 (we omit the details).

Corollary 2.19. *Let F be a graph with ℓ vertices. For every $\gamma > 0$ there exists $\varepsilon > 0$ such that for every $G = (V, E)$ with $|V| = n$ and every ε-Szemerédi-partition $\mathscr{P} = \{V_i\colon i \in [t]\}$ with reduced graph $R = R_G(\mathscr{P}, \varepsilon)$ the following is true.*

For every labeled copy F_R of F contained in a clique $K_R^\ell \subseteq R$ with $V(F_R) = V(K_R^\ell) = \{i_1, \dots, i_\ell\}$

$$\prod_{\{i_j,i_k\}\in E(F_R)} w_R(i_j, i_k) \prod_{\{i_j,i_k\}\in\binom{V(F_R)}{2}\setminus E(F_R)} (1 - w_R(i_j, i_k)) \prod_{i_j\in V(F_R)} |V_{i_j}| = N_F^*(G[F_R]) \pm \gamma \prod_{i_j\in V(F_R)} |V_{i_j}|\,,$$

where $N_F^*(G[F_R])$ denotes the number of labeled, induced, partite isomorphic copies of F_R in the induced subgraph $G[F_R] = G[V_{i_1} \cup \ldots \cup V_{i_\ell}]$. □

Note that by assumption of Corollary 2.19 and the definition of the reduced graph for Szemerédi-partitions we require for $F_R \subseteq R$ with $V(F_R) = \{i_1, \dots, i_\ell\}$, that (V_{i_j}, V_{i_k}) is ε-uniform for every pair $\{i_j, i_k\}$ and not only for pairs corresponding to edges of F_R.

2.10 The removal lemma for graphs

A direct consequence of the local counting lemma is the so-called *removal lemma* for graphs, i.e., Theorem 1.4 for $k = 2$, which was first proved by Erdős, Frankl, and Rödl [EFR86]. For completeness we include the short proof of removal lemma for graphs based on Szemerédi's regularity lemma and the local counting lemma. We first restate Theorem 1.4 for $k = 2$.

Theorem 2.20 (Removal lemma for graphs (Theorem 1.4 for $k = 2$))**.** *For every graph F with ℓ vertices and every $\eta > 0$ there exists $c > 0$ and n_0 such that every graph $G = (V, E)$ on $n \geq n_0$ vertices with $N_F(G) < cn^\ell$, there exists a subgraph $H = (V, E')$ such that $N_F(H) = 0$ and $|E \setminus E'| \leq \eta\binom{n}{2}$.*

While the original proof of Ruzsa and Szemerédi was based on an iterated application of the early version of the regularity lemma, Theorem 2.12, the proof given in [EFR86] is based on Szemerédi's regularity lemma, Theorem 2.7. We remark that even in the triangle case both proofs give essentially the same tower-type dependency between c and η, i.e., c is a polynomial in $1/T$, where T is a tower of 2's of height polynomial in $1/\eta$. It is an intriguing open problem to find a proof which gives a better dependency between c and η.

Proof. Suppose that $G = (V, E)$ is a graph which even after the deletion of any set of at most $\eta\binom{n}{2}$ edges still contains a copy of F. We will show that such a graph G contains at least cn^ℓ copies of F. For that we apply Szemerédi's regularity lemma, Theorem 2.7, with

$$\varepsilon = \min\left\{\frac{\eta}{8\ell^2}, \frac{\varepsilon'}{3\ell^2}\right\}$$

and

$$t_0 = \frac{5}{\eta},$$

where ε' is given by the local counting lemma applied with F and

$$\gamma = \frac{1}{3}\left(\frac{\eta}{4}\right)^{e(F)},$$

and obtain an ε-Szemerédi-partition $\mathscr{P} = \{V_i\colon\ i \in [t]\}$ of V. Next we delete all edges $e \in E$ for which at least one of the following holds:

1. $e \subseteq V_i$ for some $i \in [t]$,
2. $e \in E(V_i, V_j)$ for some $1 \le i < j \le t$ such that (V_i, V_j) is not ε-regular,
3. $e \in E(V_i, V_j)$ for some $1 \le i < j \le t$ such that $d(V_i, V_j) \le \eta/2$.

Simple calculations show that we delete at most $\eta\binom{n}{2}$ edges in total. Let G' be the graph, which we obtain after the deletion of those edges. Due to the assumption on G, the graph G' must still contain a copy F_0 of F. Therefore the reduced graph $R = R_{G'}(\mathscr{P}, \varepsilon)$ must contain a copy of a homomorphic image F_R' of F for which $w_R(i_j, j_k) \ge \eta/2$ for all $\{i_j, j_k\} \in E(F_R')$.

If F_R' is a copy of F, then the local counting lemma, Theorem 2.18, implies that G' contains, for sufficiently large n at least

$$\left(\frac{\eta}{2}\right)^{e(F)} \left\lfloor\frac{n}{t}\right\rfloor^{\ell} - \gamma\left\lceil\frac{n}{t}\right\rceil^{\ell} \ge \frac{1}{2}\left(\frac{\eta}{2}\right)^{e(F)}\left(\frac{n}{t}\right)^{\ell}$$

copies of F. Consequently, $N_F(G) \ge N_F(G') \ge cn^{\ell}$, for some c only depending on η and $T_{\mathrm{Sz}}(\min\{\eta/(8\ell^2), \varepsilon'/(3\ell^2)\}, 1/\eta)$, where ε' only depends on F and η. In other words, there exists such a c which only depends on the graph F and η as claimed.

The case when F_R' is not isomorphic to F is very similar. For example, we may subdivide every vertex class V_i into ℓ classes, $V_{i,1} \cup \ldots \cup V_{i,\ell} = V_i$, and obtain a refinement $\mathscr{Q}$. It follows from the definition of ε-uniform pair, that if (V_i, V_j) is ε-regular, then $(V_{i,a}, V_{j,b})$ is $(3\ell^2\varepsilon)$-uniform for any $a, b \in [\ell]$ and $d(V_{i,a}, V_{j,b}) \ge d(V_i, V_j) - 2\ell^2\varepsilon$. Since F_R' was contained in R, the reduced graph $S = S_{G'}(\mathscr{Q}, 3\ell^2\varepsilon)$ must contain a full copy F_R of F for which $w_R(i_j, j_k) \ge \eta/2 - 2\ell^2\varepsilon \ge \eta/4$ for all $\{i_j, j_k\} \in E(F_R)$ and the local counting lemma yields $N_F(G) \ge cn^{\ell}$ for

$$c = \frac{1}{2\ell^{\ell}}\left(\frac{\eta}{4}\right)^{e(F)} T_{\mathrm{Sz}}\left(\min\left\{\frac{\eta}{8\ell^2}, \frac{\varepsilon'}{3\ell^2}\right\}, \frac{1}{\eta}\right)^{-\ell}.$$

□

As we discussed in Section 1.2.4 generalizations of Theorem 2.20 for graphs were obtained by several authors. In particular, the regularity lemma of Alon, Fischer, Krivelevich, and M. Szegedy, Theorem 2.10, was introduced to prove the natural analog of the removal lemma for induced copies of F. In fact, the proof of this statement is already considerably more involved. Later, Alon and Shapira [AS08b, AS08a] generalized those results by replacing the fixed graph F by a possibly infinite family of graphs $\mathcal{F}$.

Theorem 2.21. *For every (possibly infinite) family of graphs $\mathcal{F}$ and every $\eta > 0$ there exist constants $c > 0$, $C > 0$, and n_0 such that the following holds. Suppose $G = (V, E)$ is a graph on $n \geq n_0$ vertices. If for every $\ell = 1, \ldots, C$ and every $F \in \mathcal{F}$ on ℓ vertices we have $N^*_F(G) \leq cn^\ell$, then there exists a graph $H = (V, E')$ on the same vertex set as G such that $|E \triangle E'| \leq \eta\binom{n}{2}$ and $N^*_F(H) = 0$ for every $F \in \mathcal{F}$.* □

Theorem 2.21 is the special case of Theorem 1.19 for $k = 2$ and the proof of Alon and Shapira of Theorem 2.21 relied on Theorem 2.10. An alternative proof of Theorem 2.21 was found by Lovász and B. Szegedy in [LS05]. This new proof was based on the *limit approach* for sequences of dense graphs of those authors [LS06], which can be viewed as an infinitary iteration of Theorem 2.1. We will briefly explain this approach in the next section. The proof of the generalization of Theorem 2.21 to k-uniform hypergraphs presented in Chapter 5 followed similar ideas (see also [AT]).

2.11 Graph limits

We first introduced the (weak) regularity lemma of Frieze and Kannan and from an iterated version we deduced Szemerédi's regularity lemma and the (ε, r)-regularity lemma. Iterating Szemerédi's regularity lemma then resulted in the (strong) regularity lemma of Alon, Fischer, Krivelevich, and M. Szegedy, which was the key ingredient for the proof of Theorem 2.21.

It seems natural to further iterate any of those regularity lemmas. In fact, this was studied by Lovász and B. Szegedy [LS06]. Roughly speaking, those authors iterated the regularity lemma of Frieze and Kannan infinitely often. Below we will briefly outline some of their ideas. Note that due to the discussion above it does not matter which regularity lemma we iterate infinitely often, since we "pick up the other ones along the way".

Suppose $(G_i)_{i\in\mathbb{N}}$ is an infinite sequence of graphs with $|V(G_i)| \to \infty$ and $(\varepsilon_i)_{i\in\mathbb{N}}$ is a sequence of positive reals which tend to 0. Now we may apply Theorem 2.1 with ε_1 and $t_0 = 1$ to every sufficiently large graph G_i of the sequence. This way we obtain for every such graph G_i an ε_1-FK-partition $\mathscr{P}_{i,1}$ and a reduced graph $R_{i,1} = R_{G_i}(\mathscr{P}_{i,1})$. Note that all those partitions have at most $T_{\mathrm{FK}}(\varepsilon_1)$ parts. Hence, if we discretize the weights of the reduced graphs $R_{i,1}$ by quantities of up to ε_1, we note that there are only $\lceil 1/\varepsilon_1\rceil^{\binom{T_{\mathrm{FK}}(\varepsilon_1)}{2}}$ different possible reduced graphs. Consequently, there exists a weighted graph R_1 with at most $T_{\mathrm{FK}}(\varepsilon_1)$ vertices such that $R_{i,1} = R_1$ for infinitely many choices $i \in \mathbb{N}$. In other words, there exists an infinite subsequence $(G_{i_j})_{j\in\mathbb{N}}$ such that for every member there exists an ε_1-FK-partition, which yields R_1 as the reduced graph. We rename this sequence to $(G^1_i)_{i\in\mathbb{N}}$ and let $(\mathscr{P}^1_i)_{i\in\mathbb{N}}$ be the corresponding sequence of ε_1-FK-partitions.

We then repeat the above procedure with ε_2 for the infinite subsequence $(G^1_i)_{i\in\mathbb{N}}$, where the ε_2-FK-partitions should refine the ε_1-FK-partitions. This way we obtain a reduced graph R_2, an infinite subsequence $(G_i^2)_{i\in\mathbb{N}}$ of $(G_i^1)_{i\in\mathbb{N}}$, and a corresponding sequence of $(\mathscr{P}_i^2)_{i\in\mathbb{N}}$ of ε_2-FK-partitions. Repeating this step for every ε_j with $j\in\mathbb{N}$, we obtain a sequence of subsequences $(G_i^j)_{i\in\mathbb{N}}$ of graphs and a sequence of reduced graphs $(R_j)_{j\in\mathbb{N}}$. To avoid sequences of sequences of graphs we may pass to the diagonal sequence and let $(H_j)_{j\in\mathbb{N}}=(G_j^j)_{j\in\mathbb{N}}$ which is a subsequence of the original sequence of graphs $(G_i)_{i\in\mathbb{N}}$.

Summarizing the above, we have argued that for every infinite sequence of graphs $(G_i)_{i\in\mathbb{N}}$ with $|V(G_i)|\to\infty$ and every sequence of positive reals $(\varepsilon_i)_{i\in\mathbb{N}}$ there exists a subsequence $(H_j)_{j\in\mathbb{N}}$ of $(G_i)_{i\in\mathbb{N}}$, and a sequence of reduced graphs $(R_j)_{j\in\mathbb{N}}$ such that for every $j\in\mathbb{N}$ and every $k\in[j]$ the following holds:

(a) There exists an ε_k-FK-partition $\mathscr{P}_j^k$ of H_j such that $R_k=R_{H_j}(\mathscr{P}_j^k)$ and

(b) if $k<j$, then $\mathscr{P}_j^{k+1}$ refines $\mathscr{P}_j^k$.

In some sense the graphs in the sequence $(H_j)_{j\in\mathbb{N}}$ become more and more similar, since they have almost identical FK-partitions for smaller and smaller ε. On the other hand, they may have very different sizes, which makes it hard to compare them directly. In order to circumvent that we may scale them all to the same size, by viewing them as functions on $[0,1]^2$. We will now make this more precise.

Let R_j be a reduced graph with t_j vertices. We split $[0,1]$ into t_j intervals $I_{j,1}\cup\ldots\cup I_{j,t_j}=[0,1]$ each of size $1/t_j$. We then define the symmetric, step-function $\hat{R}_j\colon[0,1]^2\to[0,1]$ by setting

$$\hat{R}_j(x,y)=\begin{cases}w_R(k,\ell), & \text{if } (x,y) \text{ belongs to the interior of } I_{j,k}\times I_{j,\ell},\\ 0, & \text{otherwise.}\end{cases}$$

Recall that those reduced graphs came from refining partitions (see (b) above) and it will be important for us to assume that the partitions $I_{j,1}\cup\ldots\cup I_{j,t_j}$ and $I_{j+1,1}\cup\ldots\cup I_{j+1,t_{j+1}}$ refine each other in the "same" way. More precisely, we assume that the first t_{j+1}/t_j vertices of R_{j+1} correspond in the ε_{j+1}-FK-partitions to those classes which were all contained in the first class of the ε_j-FK-partitions, while the second set of t_{j+1}/t_j vertices of R_{j+1} correspond in the ε_{j+1}-FK-partitions to those classes which were all contained in the second class of the ε_j-FK-partitions and so on. This way we embedded the sequence of reduced graphs $(R_j)_{j\in\mathbb{N}}$ into the family of symmetric step-functions from $[0,1]^2\to[0,1]$. Similarly, we may embed the graphs from the sequence $(H_j)_{j\in\mathbb{N}}$. Here for a graph H_j on n_j vertices we split $[0,1]$ into n_j intervals $J_{j,1}\cup\ldots\cup J_{j,n_j}=[0,1]$ (identified by the vertices of H_j) and we set

$$\hat{H}_j(x,y)=\begin{cases}1, & \text{if } (x,y) \text{ is in the interior of } J_{j,u}\times J_{j,v} \text{ and } \{u,v\}\in E(H_j),\\ 0, & \text{otherwise.}\end{cases}$$

Again we suppose that the labeling of the vertices of H_j is "consistent", i.e., if u is a vertex contained in the k-th vertex class of the fixed ε_j-FK-partition of H_j, then we impose that $J_{j,u}\subseteq I_{j,k}$.

After this embedding we can rewrite the property that R_j is the reduced graph of an ε_j-FK-partition of H_j, by

$$\sup_{U\subseteq[0,1]}\left|\int_{U\times U}\hat{H}_j(x,y)-\hat{R}_j(x,y)\,\mathrm{d}x\mathrm{d}y\right|\le\hat{\varepsilon}_j\,, \tag{2.21}$$

for some $\hat{\varepsilon}_j$ which tends to 0 as ε_j tends to 0. (Note that $\hat{H}_j$ and $\hat{R}_j$ are piecewise linear and, hence, (Lebesgue) measurable on $[0,1]^2$.) Moreover, we can rephrase the global counting lemma, Theorem 2.16: Let F be a graph with $V(F)=[\ell]$ and let j be sufficiently large (so that ε_j is sufficiently small). Then

$$\frac{N_F(H_j)}{n_j^\ell}=\int_{(x_1,\dots,x_\ell)\in[0,1]^\ell}\prod_{\{p,q\}\in E(F)}\hat{R}_j(x_p,x_q)\,\mathrm{d}x_1\dots\mathrm{d}x_\ell\pm\hat{\gamma}_j\,, \tag{2.22}$$

where for fixed F we have $\hat{\gamma}_j\to 0$ as $\varepsilon_j\to 0$.

It was proved by Lovász and B. Szegedy in [LS06] that, due to property (*b*) above, the sequence $(\hat{R}_j)_{j\in\mathbb{N}}$ converges almost everywhere to a measurable, symmetric function $\hat{R}\colon[0,1]^2\to[0,1]$ and that (2.21) and (2.22) stay valid in the limit. The function $\hat{R}$ is called the limit of the sequence $(H_j)_{j\in\mathbb{N}}$.

Theorem 2.22. *For every sequence of graphs $(G_i)_{i\in\mathbb{N}}$ with $|V(G_i)|\to\infty$ there exists a subsequence $(H_j)_{j\in\mathbb{N}}$ and a sequence of reduced graphs $(R_j)_{j\in\mathbb{N}}$, and a measurable, symmetric function $\hat{R}\colon[0,1]^2\to[0,1]$ such that*

(*i*) *$\hat{R}_j$ converges pointwise almost everywhere to $\hat{R}$,*

(*ii*)

$$\lim_{j\to\infty}\sup_{U\subseteq[0,1]}\left|\int_{U\times U}\hat{H}_j(x,y)-\hat{R}(x,y)\,dx\,dy\right|=0\,,$$

and

(*iii*) *for every $\ell\in\mathbb{N}$ and every graph F with $V(F)=[\ell]$*

$$\lim_{j\to\infty}\frac{N_F(H_j)}{n_j^\ell}=\int_{(x_1,\dots,x_\ell)\in[0,1]^\ell}\prod_{\{p,q\}\in E(F)}\hat{R}(x_p,x_q)\,dx_1\dots dx_\ell\,.$$

□

The proof of Theorem 2.22 indicated above, essentially follows the lines of the proof of the implication (*a*) ⇒ (*b*) of Theorem 2.2 in [LS06] (see Lemma 5.1 and 5.2 in [LS06]).

3 The weak regularity lemma for hypergraphs

In this chapter we consider conditions which allow the embedding of linear hypergraphs of fixed size. In particular, we prove that any k-uniform hypergraph H of positive uniform density contains all linear k-uniform hypergraphs of a given size (Theorem 1.11).

The main ingredient in the proof of this result is a *counting lemma* for linear hypergraphs, which establishes that the straightforward extension of graph ε-regularity to hypergraphs suffices for counting linear hypergraphs.

3.1 Counting lemma for linear hypergraphs

A key tool we use in this chapter is the so-called *weak hypergraph regularity lemma*. This result is a straightforward extension of Theorem 1.2. Let $H^{(k)}$ be a k-uniform hypergraph and let $W_1, \ldots, W_k$ be mutually disjoint non-empty subsets of $V(H^{(k)})$. We denote by $d(W_1, \ldots, W_k)$ the *density* of the k-partite induced sub-hypergraph $H^{(k)}[W_1, \ldots, W_k]$ of $H^{(k)}$, defined by

$$d(W_1, \ldots, W_k) = \frac{e(W_1, \ldots, W_k)}{|W_1| \cdot \ldots \cdot |W_k|}\,.$$

We say the k-tuple $(V_1, \ldots, V_k)$ of mutually disjoint subsets $V_1, \ldots, V_k \subseteq V$ is (ε, d)-*regular*, for positive constants ε and d, if

$$|d_H(W_1, \ldots, W_k) - d| \leq \varepsilon$$

for all k-tuples of subsets $W_1 \subseteq V_1, \ldots, W_k \subseteq V_k$ satisfying $|W_1| \cdot \ldots \cdot |W_k| \geq \varepsilon |V_1| \cdot \ldots \cdot |V_k|$. Note, in particular, that if $(V_1, \ldots, V_k)$ is (ε, d)-regular, then

$$\big|H^{(k)}[W_1, \ldots, W_k] - d|W_1| \cdot \ldots \cdot |W_k|\big| \leq \varepsilon |V_1| \cdot \ldots \cdot |V_k| \tag{3.1}$$

holds for *any* $W_1 \subseteq V_1, \ldots, W_k \subseteq V_k$. We say the k-tuple $(V_1, \ldots, V_k)$ is ε-*regular* if it is (ε, d)-regular for some $d \geq 0$. The weak regularity lemma then states the following.

Theorem 3.1. *For all integers $k \geq 2$ and $t_0 \geq 1$, and every $\varepsilon > 0$, there exist $T_0 = T_0(k, t_0, \varepsilon)$ and $n_0 = n_0(k, t_0, \varepsilon)$ so that for every k-uniform hypergraph $H^{(k)}$ on $n \geq n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ so that the following hold:*

(i) $t_0 \leq t \leq T_0$,

(ii) $|V_0| \leq \varepsilon n$ *and* $|V_1| = \cdots = |V_t|$, *and*

(iii) for all but at most $\varepsilon\binom{t}{k}$ sets $\{i_1,\dots,i_k\}\subseteq[t]$, the k-tuple $(V_{i_1},\dots,V_{i_k})$ is ε-regular. □

The proof of Theorem 3.1 follows the lines of the original proof of Szemerédi [Sze78] (for details see e.g. [Chu91, FR92, Ste90]).

A key feature of the partition provided by Szemerédi's regularity lemma is the so-called *local counting lemma* (see Theorem 2.18). This lemma provides good estimates on the number of subgraphs of a fixed isomorphism type in an appropriate collection of ε-regular pairs. To be precise, let F be a graph (hypergraph) on the vertex set $[\ell]=\{1,\dots,\ell\}$ and let G be an ℓ-partite graph (hypergraph) with vertex partition $V(G)=V_1\cup\ldots\cup V_\ell$. A copy F_0 of F in G, on the vertices $v_1\in V_1,\dots,v_\ell\in V_\ell$, is said to be *partite-isomorphic* to F if $i\mapsto v_i$ defines a homomorphism. The counting lemma for graphs asserts that if (V_i,V_j) is (ε,d_{ij})-regular, where $d_{ij}^{\ell}\gg\varepsilon>0$ whenever $\{i,j\}\in E(F)$, then the number of labeled partite-isomorphic copies F_0 of F in G is within the interval $(1\pm\gamma)\prod_{\{i,j\}\in E(F)}d_{ij}\prod_{i\in[\ell]}|V_i|$, where $\gamma\to 0$ as $\varepsilon\to 0$. It is known that this fact does not extend to k-uniform hypergraphs ($k\geq 3$), and that stronger regularity lemmas are needed in that case (see, e.g., [Gow07, NRS06a, RS07b, RS07c, Tao06b] and Chapter 4). However, weak regularity is sufficient for estimating the number of linear sub-hypergraphs in an appropriately ε-regular environment.

Lemma 3.2 (Counting lemma for linear hypergraphs)**.** *For all integers $\ell\geq k\geq 2$ and every $\gamma,d_0>0$, there exist $\varepsilon=\varepsilon(\ell,k,\gamma,d_0)>0$ and $m_0=m_0(\ell,k,\gamma,d_0)$ so that the following holds.*

Let $F^{(k)}=([\ell],E(F^{(k)}))\in\mathscr{L}_\ell^{(k)}$ and let $H^{(k)}=(V_1\cup\ldots\cup V_\ell,E)$ be an ℓ-partite, k-uniform hypergraph where $|V_1|,\dots,|V_\ell|\geq m_0$. Suppose, moreover, that for all edges $f\in E(F^{(k)})$, the k-tuple $(V_i)_{i\in f}$ is (ε,d_f)-regular, where $d_f\geq d_0$. Then the number of partite-isomorphic copies of $F^{(k)}$ in $H^{(k)}$ is within the interval

$$(1\pm\gamma)\prod_{f\in E(F^{(k)})}d_f\prod_{i\in[\ell]}|V_i|\,.$$

Proof. The proof follows the lines of the proof of Theorem 2.18. Let integers $\ell\geq k\geq 2$ and $\gamma,d_0>0$ be fixed. We shall prove, by induction on $|E(F^{(k)})|$, the number of edges of $F^{(k)}$, that $\varepsilon=\gamma(d_0/2)^{|E(F^{(k)})|}$ will suffice to count copies of $F^{(k)}$ (with 'precision' γ), provided m_0 is large enough. (In this way, $\varepsilon=\gamma(d_0/2)^{\binom{\ell}{2}}$ works for all $F^{(k)}\in\mathscr{L}_\ell^{(k)}$.) If $|E(F^{(k)})|=0$ or $|E(F^{(k)})|=1$, the result is trivial. It is also easy to see that the result holds whenever $F^{(k)}$ consists of pairwise disjoint edges, since then the number of partite-isomorphic copies of $F^{(k)}$ in $H^{(k)}$ is within

$$\begin{aligned}\prod_{f\in E(F^{(k)})}(d_f\pm\varepsilon)\prod_{i\in[\ell]}|V_i|=(1\pm(\varepsilon/d_0))^{|E(F^{(k)})|}&\prod_{f\in E(F^{(k)})}d_f\prod_{i\in[\ell]}|V_i|\\ &=(1\pm\gamma)\prod_{f\in E(F^{(k)})}d_f\prod_{i\in[\ell]}|V_i|\,.\end{aligned}$$

Now, generally, take m_0 large enough so that we can apply the induction assumption on $|E(F^{(k)})|-1$ edges with precision $\gamma/2$ and d_0 (and note that $\varepsilon = \gamma(d_0/2)^{|E(F^{(k)})|} < (\gamma/2)(d_0/2)^{|E(F^{(k)})|-1}$). All copies of various sub-hypergraphs disussed below are tacitly assumed to be partite-isomorphic.

Let $F^{(k)} = ([\ell], E(F^{(k)})) \in \mathscr{L}_\ell^{(k)}$ have $|E(F^{(k)})| \geq 2$ edges and let $H^{(k)} = (V, E)$ be a k-uniform hypergraph satisfying the assumptions of Lemma 3.2. Fix an edge $e \in E(F^{(k)})$ and set $F_-^{(k)} = ([\ell], E(F^{(k)}) \setminus \{e\})$ to be the hypergraph obtained from $F^{(k)}$ by removing the edge e. Moreover, for a copy $T_-^{(k)}$ of $F_-^{(k)}$ in H, we denote by $e_{T_-^{(k)}}$ the unique k-tuple of vertices which together with $T_-^{(k)}$ forms a copy of $F^{(k)}$ in $H^{(k)}$. Furthermore, let $\mathbb{1}_E \colon \binom{V}{k} \to \{0,1\}$ be the indicator function of the edge set E of $H^{(k)}$. In this notation, a copy $T_-^{(k)}$ of $F_-^{(k)}$ in $H^{(k)}$ extends to a copy of $F^{(k)}$ if, and only if, $\mathbb{1}_E(e_{T_-^{(k)}}) = 1$. Consequently, summing over all copies $T_-^{(k)}$ of $F_-^{(k)}$ in $H^{(k)}$, we can count the number $\#\{F^{(k)} \subseteq H^{(k)}\}$ of copies of $F^{(k)}$ in $H^{(k)}$ by

$$\begin{aligned}
\#\{F^{(k)} \subseteq H^{(k)}\} &= \sum_{T_-^{(k)} \subseteq H^{(k)}} \mathbb{1}_E(e_{T_-^{(k)}}) \\
&= \sum_{T_-^{(k)} \subseteq H^{(k)}} (d_e + \mathbb{1}_E(e_{T_-^{(k)}}) - d_e) \\
&= d_e \times \#\{F_-^{(k)} \subseteq H^{(k)}\} + \sum_{T_-^{(k)} \subseteq H^{(k)}} (\mathbb{1}_E(e_{T_-^{(k)}}) - d_e) \\
&= (1 \pm \tfrac{\gamma}{2}) \prod_{f \in E(F^{(k)})} d_f \prod_{i \in [\ell]} |V_i| + \sum_{T_-^{(k)} \subseteq H^{(k)}} (\mathbb{1}_E(e_{T_-^{(k)}}) - d_e)\,, \qquad (3.2)
\end{aligned}$$

where we used the induction assumption for $F_-^{(k)}$ for the last estimate.

It is left to bound the error term $\sum_{T_-^{(k)} \subseteq H^{(k)}} (\mathbb{1}_E(e_{T_-^{(k)}}) - d_e)$ in (3.2). For that, we will appeal to the regularity of $(V_i)_{i \in e}$. Let $F_*^{(k)} = F^{(k)}[[\ell] \setminus e]$ be the induced sub-hypergraph of $F^{(k)}$ obtained by removing the vertices of e and all edges of $F^{(k)}$ intersecting e. For a copy $T_*^{(k)}$ of $F_*^{(k)}$ in $H^{(k)}$, let $\mathrm{ext}(T_*^{(k)})$ be the set of k-tuples $K \in \prod_{i \in e} V_i$ such that $V(T_*^{(k)}) \cup K$ spans a copy of $T_-^{(k)}$ in $H^{(k)}$. Since $F^{(k)}$ is a linear hypergraph, we have $|f \cap e| \leq 1$ for every edge f of $F_-^{(k)}$. Hence, for every $i \in e$, there exists a subset $W_i^{T_*^{(k)}} \subseteq V_i$ such that

$$\mathrm{ext}(T_*^{(k)}) = \prod_{i \in e} W_i^{T_*^{(k)}}\,.$$

Indeed, for every $i \in e$, the set $W_i^{T_*^{(k)}}$ consists of those vertices $v \in V_i$ with the property that $V(T_*^{(k)}) \cup \{v\}$ spans a copy of $F^{(k)}$ induced on $V(F_*^{(k)}) \cup \{i\}$ in $H^{(k)}$. With this notation, we can bound the error term in (3.2) as follows:

$$\begin{aligned}\Big|\sum_{T_-^{(k)}\subseteq H^{(k)}}(\mathbb{1}_E(e_{T_-^{(k)}})-d_e)\Big| &\le \sum_{T_*^{(k)}\subseteq H^{(k)}}\Big|\sum_{K\in\mathrm{ext}(T_*^{(k)})}(\mathbb{1}_E(K)-d_e)\Big|\\ &=\sum_{T_*^{(k)}\subseteq H^{(k)}}\Big|\sum\Big\{(\mathbb{1}_E(K)-d_e)\colon\ K\in\prod_{i\in e}W_i^{T_*^{(k)}}\Big\}\Big|\\ &\le \sum_{T_*^{(k)}\subseteq H^{(k)}}\varepsilon\prod_{i\in e}|V_i|\,,\end{aligned}$$

where the ε-regularity was used for the last estimate. Indeed, for a fixed copy $T_*^{(k)}\subseteq H^{(k)}$, we have

$$\Big|\sum\Big\{(\mathbb{1}_E(K)-d_e)\colon\ K\in\prod_{i\in e}W_i^{T_*^{(k)}}\Big\}\Big| = \Big||H^{(k)}\cap\prod_{i\in e}W_i^{T_*^{(k)}}|-d_e\prod_{i\in e}|W_i^{T_*^{(k)}}|\Big|,$$

so that we may appeal to (3.1). Now, because of the choice of ε we have

$$\begin{aligned}\Big|\sum_{T_-^{(k)}\subseteq H^{(k)}}(\mathbb{1}_E(e_{T_-})-d_e)\Big| &\le \varepsilon\sum_{T_*^{(k)}\subseteq H^{(k)}}\prod_{i\in e}|V_i|\\ &\le \varepsilon\prod_{i\in[\ell]}|V_i|\\ &\le \frac{\gamma}{2}\prod_{f\in E(F^{(k)})}d_f\prod_{i\in[\ell]}|V_i|\,,\end{aligned}$$

and Lemma 3.2 follows from (3.2). □

3.2 Quasi-random hypergraphs

In this section, we prove Theorem 1.8 according to the following outline. We first observe that a (ϱ,d)-quasi-random (k-uniform) hypergraph $H^{(k)}$ is (ε,d)-regular w.r.t. any disjoint family $U_1,\dots,U_k\subset V(H^{(k)})$ of large and equal-sized sets. As such, any partition $U_1\cup\ldots\cup U_\ell$ within $V(H^{(k)})$ of $\ell\ge k$ large equal-sized sets will satisfy the hypothesis of the counting lemma (Lemma 3.2), and will therefore contain the "right" number of copies of any hypergraph $F^{(k)}\in\mathscr{L}_\ell^{(k)}$. Applying this argument to a partition chosen at random then yields the "right" number of copies of $F^{(k)}$ in $H^{(k)}$.

Proof of Theorem 1.8. Let $k\ge 2$, $d,\gamma>0$ and $F\in\mathscr{L}^{(k)}$ on the vertex set $\{1,\dots,\ell\}$ be given. We set

$$\varepsilon=\varepsilon(\ell,k,\gamma/2,d)\quad\text{and}\quad\varrho=\frac{\varepsilon^2}{\ell(2k)^k}\tag{3.3}$$

and let $n\ge m_0(\ell,k,\gamma/2,d)/\varrho$ be sufficiently large, where the constants $\varepsilon(\ell,k,\gamma/2,d)$ and

$m_0(\ell, k, \gamma/2, d)$ are given by Lemma 3.2. Let $H^{(k)}$ be a (ϱ, d)-quasi-random k-uniform hypergraph on n vertices.

Following the outline (above), let $U_i \subset V$, $1 \le i \le k$, be mutually disjoint sets of size $|U_i| = m \ge \varrho n/\varepsilon$. We claim that $(U_1, \dots, U_k)$ is (ε, d)-regular w.r.t. $H^{(k)}$. Indeed, let $V_i \subseteq U_i$, $1 \le i \le k$, be given so that $|V_1| \cdot \ldots \cdot |V_k| \ge \varepsilon m^k$. (Note, in particular, that this implies $|V_i| \ge \varepsilon m \ge \varrho n$ for all $1 \le i \le k$.) To show that $|H^{(k)}[V_1, \dots, V_k]| = (d \pm \varepsilon)|V_1| \cdot \ldots \cdot |V_k|$, we observe, from inclusion-exclusion, that

$$|H^{(k)}[V_1, \dots, V_k]| = \sum_{I \subseteq [k]} (-1)^{|I|} \Big| H^{(k)}\Big[\bigcup_{j \in [k] \setminus I} V_j \Big] \Big|.$$

The (ϱ, d)-quasi-randomness of $H^{(k)}$ (together with $|V_i| \ge \varrho n$ for all $1 \le i \le k$) implies

$$\begin{aligned}
\Big|H^{(k)}[V_1, \dots, V_k]\Big| &= \sum_{I \subseteq [k]} (-1)^{|I|} (d \pm \varrho) \binom{|\bigcup_{j \in [k]\setminus I} V_j|}{k} \\
&= d \sum_{I \subseteq [k]} (-1)^{|I|} \binom{|\bigcup_{j \in [k]\setminus I} V_j|}{k} \pm \varrho \sum_{I \subseteq [k]} \binom{|\bigcup_{j \in [k]\setminus I} V_j|}{k} \\
&= d \sum_{I \subseteq [k]} (-1)^{|I|} \binom{|\bigcup_{j \in [k]\setminus I} V_j|}{k} \pm \varrho (2k)^k m^k \\
&= d|V_1| \cdot \ldots \cdot |V_k| \pm \varrho (2k)^k m^k \\
&= (d \pm \varrho (2k)^k/\varepsilon)|V_1| \cdot \ldots \cdot |V_k| \\
&= (d \pm \varepsilon)|V_1| \cdot \ldots \cdot |V_k| \,.
\end{aligned}$$

To finish the proof of Theorem 1.8, consider an ℓ-tuple of mutually disjoint sets $U_1, \dots, U_\ell$ with $|U_1| = \dots = |U_\ell| = m$, where m is a fixed integer satisfying $n/\ell \ge m \ge \varrho n/\varepsilon$. Then every k-tuple $I \in \binom{[\ell]}{k}$ satisfies that $(U_i)_{i \in I}$ is (ε, d)-regular (as shown above), and so by the choice of ε in (3.3), we can apply the counting lemma for linear hypergraphs (Lemma 3.2) to $U_1 \cup \ldots \cup U_\ell$. Consequently, $H^{(k)}[U_1, \dots, U_\ell]$ contains $(1 \pm \gamma/2) d^{e(F^{(k)})} m^\ell$ partite-isomorphic copies of $F^{(k)}$ (recall $V(F^{(k)}) = [\ell]$). Now, on the one hand, we note that there are $\binom{n}{m}\binom{n-m}{m} \dots \binom{n-(\ell-1)m}{m}$ choices for the partition $U_1 \cup \ldots \cup U_\ell$. On the other hand, for each ℓ-tuple of vertices $(u_1, \dots, u_\ell)$ in $V(H^{(k)})$, there are $\binom{n-\ell}{m-1}\binom{n-m-(\ell-1)}{m-1} \dots \binom{n-(\ell-1)m-1}{m-1}$ such partitions $U_1 \cup \ldots \cup U_\ell$ for which $(u_1, \dots, u_\ell) \in U_1 \times \dots \times U_\ell$. Consequently, the number of labeled copies of $F^{(k)}$ in $H^{(k)}$ is given by

$$\begin{aligned}
(1 \pm \gamma/2) d^{e(F^{(k)})} m^\ell & \frac{\binom{n}{m}\binom{n-m}{m} \dots \binom{n-(\ell-1)m}{m}}{\binom{n-\ell}{m-1}\binom{n-m-(\ell-1)}{m-1} \dots \binom{n-(\ell-1)m-1}{m-1}} \\
&= (1 \pm \gamma/2) d^{e(F^{(k)})} \frac{n!}{(n-\ell)!} = (1 \pm \gamma) d^{e(F^{(k)})} n^\ell,
\end{aligned}$$

where for the last step we use that n is sufficiently large. $\square$

3.3 Universal hypergraphs

In this section, we prove Theorem 1.11. The proof relies on the weak hypergraph regularity lemma, which allows us to locate a sufficiently dense and ε-regular ℓ-partite sub-hypergraph in any (ϱ, d)-dense hypergraph. The $(\xi, \mathscr{L}_\ell^{(k)})$-universality then follows from Lemma 3.2.

Proof of Theorem 1.11. Let integers $\ell \geq k \geq 2$ and $d > 0$ be given. To define the promised constants ϱ and ξ, we first consider a few auxiliary constants. Set $d_0 = d/(4k!)$ and $q = \lceil 1/d_0 \rceil$ and let $s = r_k(q, \ell)$ be the (k-uniform) Ramsey number for q and ℓ, i.e., s is the smallest integer s.t. any 2-coloring of $E(K_s^{(k)})$ yields a copy of $K_q^{(k)}$ in the first color, or a copy of $K_\ell^{(k)}$ in the second color. Set $\varepsilon = \min\{1/(2\binom{s}{k}),\ \varepsilon(\ell, k, 1/2, d_0)\}$, where $\varepsilon(\ell, k, 1/2, d_0)$ is given by Lemma 3.2 applied with ℓ, k, $\gamma = 1/2$, and d_0. Moreover, let $T_0 = T_0(k, s, \varepsilon)$ be given by Theorem 3.1 applied with k, $t_0 = s$, and ε. We now define the promised constants as

$$\varrho = \frac{q}{T_0} \qquad \text{and} \qquad \xi = \frac{d_0^{\binom{\ell}{2}}}{2T_0^\ell},$$

and let n_0 be sufficiently large.

Let $H^{(k)}$ be a (ϱ, d)-dense k-uniform hypergraph with vertex set V. The weak hypergraph regularity lemma yields a partition $V_0 \cup V_1 \cup \ldots \cup V_t$, $s \leq t \leq T_0$ (s and T_0 defined above) which satisfies properties (ii) and (iii) of Theorem 3.1 (with ε defined above). We consider the following auxiliary, so-called *reduced hypergraph*, $R^{(k)} = ([t], E_R)$, where $e \in \binom{[t]}{k}$ is an edge in E_R if, and only if, $(V_i)_{i \in e}$ is an ε-regular k-tuple. Hence,

$$|E_R| \geq (1 - \varepsilon)\binom{t}{k} > (1 - 1/\tbinom{s}{k})\binom{t}{k} \geq \mathrm{ex}(t, K_s^{(k)}),$$

where $\mathrm{ex}(t, K_s^{(k)})$ is the Turán number for $K_s^{(k)}$, i.e., the largest number of k-tuples among all $K_s^{(k)}$-free k-uniform hypergraphs on t vertices (the inequality we used above is well-known). Consequently, $R^{(k)}$ contains a copy of $K_s^{(k)}$, and we denote this copy by $R_s^{(k)} \subseteq R^{(k)}$. Now, we 2-color the edges of $R_s^{(k)}$ according to the density of the corresponding k-tuple. More precisely, we color the edge $e = \{i_1, \ldots, i_k\}$ “sparse” if $d(V_{i_1}, \ldots, V_{i_k}) \leq d_0$, and we color it “dense” otherwise. We now argue that $R_s^{(k)}$ does not contain a “sparse" copy of $K_q^{(k)}$.

Indeed, suppose $R_s^{(k)}$ does contain a “sparse” clique $K_q^{(k)}$. Let $i_1, \ldots, i_q$ be the vertices of this clique, and set $U = \bigcup_{j=1}^q V_{i_j}$. Since $i_1, \ldots, i_q$ spanned a “sparse” clique in $R_s^{(k)}$, the number of edges $e(U)$ of $H^{(k)}$ induced on U can be bounded from above by

$$\begin{aligned} e(U) &\le d_0 \binom{q}{k} \left(\frac{n}{t}\right)^k + q\binom{n/t}{2}\binom{qn/t}{k-2} \\ &< \left(d_0 + \frac{1}{q}\right) q^k \left(\frac{n}{t}\right)^k \\ &\le \frac{d(qn/t)^k}{2k!} \\ &< d\binom{|U|}{k}, \end{aligned} \tag{3.4}$$

where we used the choice of d_0 and q and the fact that n is sufficiently large. Clearly, the estimate in (3.4) violates the (ϱ, d)-denseness of $H^{(k)}$, and so $R_s^{(k)}$ contains no "sparse" clique $K_q^{(k)}$.

By the choice of $s = r_k(q, \ell)$, $R_s^{(k)}$ must contain a "dense" clique $K_\ell^{(k)}$. Let $i_1, \dots, i_\ell$ be the vertex set of that clique. From the preparation above, $H^{(k)}[V_{i_1}, \dots, V_{i_s}]$ satisfies the hypothesis of the counting lemma for linear hypergraphs (Lemma 3.2), and therefore, $H^{(k)} \supseteq H^{(k)}[V_{i_1}, \dots, V_{i_s}]$ contains at least

$$\frac{d_0^{e(S)}}{2}\left(\frac{n}{t}\right)^\ell \ge \frac{d_0^{\binom{\ell}{2}}}{2T_0^\ell} n^\ell = \xi n^\ell$$

copies of any $S \in \mathscr{L}_\ell^{(k)}$, making H $(\xi, \mathscr{L}_\ell^{(k)})$-universal. □

3.4 Non-universal hypergraphs

In this section, we deduce Corollary 1.12 from Theorem 1.11, according to the following outline. Since the given hypergraph $H^{(k)}$ is not universal (for linear hypergraphs), Theorem 1.11 implies that there must be a subset $U \subseteq V$, of linear size, containing only "few" edges. We apply this observation repeatedly, obtaining a partition $V_1 \cup \dots \cup V_t$ of nearly the entire vertex set, where $H^{(k)}[V_i]$ is "sparse" for every $i \in [t]$. This, however, implies that the number of edges of $H^{(k)}$ intersecting at least two classes from the partition must be slightly larger than expected. Finally, this "extra" density will "survive" when we distribute the remaining vertices of $H^{(k)}$ into $V_1, \dots, V_t$.

Proof of Corollary 1.12. Let integers $\ell \ge k \ge 2$ and $d > 0$ be fixed. To define the promised constants t, β and ξ, we first consider a few auxiliary constants. Set $c = d/4$. Theorem 1.11 yields constants $\varrho' = \varrho'(\ell, k, c)$, $\xi' = \xi'(\ell, k, c)$, and $n_0' = n_0'(\ell, k, c)$. Set

$$\sigma = \min\left\{(\varrho')^2, \frac{c^2}{16k^2}\right\}. \tag{3.5}$$

We now define the promised constants as

$$t = \left\lceil \frac{1-\sqrt{\sigma}}{\sigma} \right\rceil, \qquad \beta = \frac{d}{4t^{k-1}} \qquad \text{and} \qquad \xi = \xi' \sigma^{\ell/2}$$

and let $n_0 \geq \max\{n_0'/\sqrt{\sigma}, t/\sigma, 2kt\}$ be sufficiently large.

Note that it suffices to prove Corollary 1.12 for hypergraphs $H^{(k)}$ for which n is a multiple of t. Indeed, otherwise we could first remove constantly many ($x = n \pmod t$) vertices from $H^{(k)}$. For the resulting hypergraph H', we would obtain $\tau_t(\hat{H^{(k)}}) \geq d + \beta - o(1)$, and so distributing the removed x vertices appropriately into the corresponding cut of $\hat{H^{(k)}}$ implies $\tau_t(H^{(k)}) \geq d + \beta - o(1)$, where $o(1)$ tends to 0 as $n \to \infty$.

So, let $H^{(k)}$ be a k-uniform hypergraph with vertex set V and $|V| = n = mt \geq n_0$ (for some $m \in \mathbb{N}$) with at least $d\binom{n}{k}$ edges which is not $(\xi, \mathscr{L}_\ell^{(k)})$-universal. Because of the choice of ξ, we infer from Theorem 1.11 that no subset $W \subseteq V$ with $|W| \geq \sqrt{\sigma}n$ is $(\sqrt{\sigma}, c)$-dense. In other words, every such W contains a subset $W' \subseteq W$, $|W'| \geq \sqrt{\sigma}|W| \geq \sigma n$ such that $e(W') \leq c\binom{|W'|}{k}$. In fact, a simple averaging argument shows that there must be such a set W' with $|W'| = \lfloor \sigma n \rfloor$. Repeatedly selecting disjoint such W' yields a vertex partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ such that for all $i \in [t]$,

$$|V_i| = \lfloor \sigma n \rfloor \quad \text{and} \quad e(V_i) \leq c\binom{\sigma n}{k}, \quad \text{and} \quad |V_0| \leq (\sqrt{\sigma} + \sigma)n\,.$$

Indeed such a partition exists, since $(t-1)\lfloor \sigma n \rfloor < (1 - \sqrt{\sigma})n$ (owing to the choice of t) and $t\lfloor \sigma n \rfloor \geq t\sigma n - t \geq (1-\sqrt{\sigma})n - \sigma n$ (owing to the choices of t and n_0).

We now redistribute the vertices of V_0 among the classes $V_1, \ldots, V_t$ and obtain a partition $U_1 \cup \ldots \cup U_t = V$ such that, for each $i \in [t]$, $|U_i| = m = n/t$ and

$$e(U_i) \leq c\binom{\sigma n}{k} + \frac{|V_0|}{t}\binom{m}{k-1} \leq c\binom{m}{k} + (\sqrt{\sigma} + \sigma)m\binom{m}{k-1}\,.$$

Because of (3.5), we have $(\sqrt{\sigma} + \sigma)k \leq c/2$, and so

$$e_H(U_i) \leq \left(c + (\sqrt{\sigma} + \sigma)k\frac{m}{m-k+1}\right)\binom{m}{k} \leq 2c\binom{m}{k},$$

where we also used that $m = n/t \geq 2k$. Consequently, the number of edges which are not completely contained in any one of the sets U_i is at least $d\binom{n}{k} - 2ct\binom{m}{k}$, and so

$$\tau_t(H^{(k)}) \geq \frac{\left|E(H^{(k)}) \setminus \bigcup_{i=1}^t \binom{U_i}{k}\right|}{\binom{n}{k} - t\binom{m}{k}} \geq \frac{d\binom{n}{k} - 2ct\binom{m}{k}}{\binom{n}{k} - t\binom{m}{k}} \geq d + \beta\,, \tag{3.6}$$

where we used the choice of $c = d/4$ and $\beta = d/(4t^{k-1})$ and the fact that n is sufficiently large for the last inequality. □

4 Strong regular partitions of hypergraphs

In this chapter we consider two extensions of Theorem 1.2 to hypergraphs (see Theorem 4.12 and Theorem 4.15).

4.1 Statements of the regularity lemmas

For $U \subseteq V(H^{(k)})$, we denote by $H^{(k)}[U]$ the sub-hypergraph of $H^{(k)}$ induced on U (i.e. $H^{(k)}[U] = H^{(k)} \cap \binom{U}{k}$). A k-uniform *clique* of *order* j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \geq k$ vertices consisting of all $\binom{j}{k}$ different k-tuples.

In this chapter ℓ-partite, j-uniform hypergraphs play a special rôle, where $j \leq \ell$. Given pairwise disjoint vertex sets $V_1, \ldots, V_\ell$, we denote by $K_\ell^{(j)}(V_1, \ldots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph (i.e., the family of all j-element subsets $J \subseteq \bigcup_{i \in [\ell]} V_i$ satisfying $|V_i \cap J| \leq 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an (m, ℓ, j)-*hypergraph* $H^{(j)}$ on $V_1 \cup \cdots \cup V_\ell$ is any subset of $K_\ell^{(j)}(V_1, \ldots, V_\ell)$. Note that the vertex partition $V_1 \cup \cdots \cup V_\ell$ is an $(m, \ell, 1)$-hypergraph $H^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $j \leq i \leq \ell$ and set $\Lambda_i \in \binom{[\ell]}{i}$, we denote by $H^{(j)}[\Lambda_i] = H^{(j)}[\bigcup_{\lambda \in \Lambda_i} V_\lambda]$ the sub-hypergraph of the (m, ℓ, j)-hypergraph $H^{(j)}$ induced on $\bigcup_{\lambda \in \Lambda_i} V_\lambda$.

For an (m, ℓ, j)-hypergraph $H^{(j)}$ and an integer $j \leq i \leq \ell$, we denote by $\mathcal{K}_i(H^{(j)})$ the family of all i-element subsets of $V(H^{(j)})$ which span complete sub-hypergraphs in $H^{(j)}$ of order i. Note that $|\mathcal{K}_i(H^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $H^{(j)}$.

Given an $(m, \ell, j-1)$-hypergraph $H^{(j-1)}$ and an (m, ℓ, j)-hypergraph $H^{(j)}$ satisfying $V(H^{(j)}) \subseteq V(H^{(j-1)})$, we say an edge J of $H^{(j)}$ *belongs to* $H^{(j-1)}$ if $J \in \mathcal{K}_j(H^{(j-1)})$, i.e., J corresponds to a clique of order j in $H^{(j-1)}$. Moreover, $H^{(j-1)}$ *underlies* $H^{(j)}$ if $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$, i.e., every edge of $H^{(j)}$ belongs to $H^{(j-1)}$. This brings us to one of the main concepts of this chapter, the notion of a *complex*.

Definition 4.1 (**(m, ℓ, h)-complex**). *Let $m \geq 1$ and $\ell \geq h \geq 1$ be integers. An (m, ℓ, h)-complex $\boldsymbol{H}$ is a collection of (m, ℓ, j)-hypergraphs $\{H^{(j)}\}_{j=1}^h$ such that*

(a) $H^{(1)}$ is an $(m, \ell, 1)$-hypergraph, i.e., $H^{(1)} = V_1 \cup \cdots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$, and

(b) $H^{(j-1)}$ underlies $H^{(j)}$ for $2 \leq j \leq h$, i.e., $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$.

Remark 4.2. We may also define hypergraphs and complexes in the same way for underlying vertex sets $V_1, \ldots, V_\ell$ with different cardinalities. In such a case we will drop the m and say $H^{(j)}$ is an (ℓ, j)-hypergraph or $\boldsymbol{H}$ is an (ℓ, h)-complex.

4.1.1 Regular complexes and partitions

We begin with a notion of relative density of a j-uniform hypergraph w.r.t. $(j-1)$-uniform hypergraph on the same vertex set.

Definition 4.3 (**relative density**). *Let $H^{(j)}$ be a j-uniform hypergraph and let $H^{(j-1)}$ be a $(j-1)$-uniform hypergraph on the same vertex set. We define the density of $H^{(j)}$ w.r.t. $H^{(j-1)}$ as*

$$d(H^{(j)}|H^{(j-1)}) = \begin{cases} \frac{|H^{(j)}\cap\mathcal{K}_j(H^{(j-1)})|}{|\mathcal{K}_j(H^{(j-1)})|} & \text{if } \left|\mathcal{K}_j(H^{(j-1)})\right| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We now define a notion of regularity of an (m,j,j)-hypergraph with respect to an $(m,j,j-1)$-hypergraph.

Definition 4.4. *Let reals $\varepsilon > 0$ and $d_j \geq 0$ be given along with an (m,j,j)-hypergraph $H^{(j)}$ and an underlying $(m,j,j-1)$-hypergraph $H^{(j-1)}$. We say $H^{(j)}$ is (ε,d_j)-regular w.r.t. $H^{(j-1)}$ if whenever $Q^{(j-1)} \subseteq H^{(j-1)}$ satisfies*

$$|\mathcal{K}_j(Q^{(j-1)})| \geq \varepsilon|\mathcal{K}_j(H^{(j-1)})|,$$

then

$$d(H^{(j)}|Q^{(j-1)}) = d_j \pm \varepsilon.$$

Next we extend the notion of (ε,d_j)-regularity from (m,j,j)-hypergraphs to (m,ℓ,j)-hypergraphs $H^{(j)}$.

Definition 4.5 (**(ε,d_j)-regular hypergraph**). *A (m,ℓ,j)-hypergraph $H^{(j)}$ is (ε,d_j)-regular w.r.t. an $(m,\ell,j-1)$-hypergraph $H^{(j-1)}$ if for every $\Lambda_j \in \binom{[\ell]}{j}$ the restriction $H^{(j)}[\Lambda_j] = H^{(j)}[\bigcup_{\lambda\in\Lambda_j} V_\lambda]$ is (ε,d_j)-regular w.r.t. to the restriction $H^{(j-1)}[\Lambda_j] = H^{(j-1)}[\bigcup_{\lambda\in\Lambda_j} V_\lambda]$.*

We sometimes write ε-regular to mean $(\varepsilon, d(H^{(j)}|H^{(j-1)}))$-regular.

Finally, we close this section with the notion of a regular complex.

Definition 4.6 (**$(\varepsilon,\boldsymbol{d})$-regular complex**). *Let $\varepsilon > 0$ and $\boldsymbol{d} = (d_2,\dots,d_h)$ be a vector of non-negative reals. We say an (m,ℓ,h)-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ is $(\varepsilon,\boldsymbol{d})$-regular if $H^{(j)}$ is (ε,d_j)-regular w.r.t. $H^{(j-1)}$ for every $j = 2,\dots,h$.*

The regularity lemmas for k-uniform hypergraphs which we prove in this chapter provide a structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)},\dots,\mathscr{P}^{(k-1)}\}$ of vertices, pairs, …, and $(k-1)$-tuples of some vertex set. We now discuss the structure of these partitions following the approach of [RS04]. First we define the *refinement* of a partition.

Definition 4.7 (**refinement**). *Suppose $A \supseteq B$ are sets, $\mathscr{A}$ is a partition of A, and $\mathscr{B}$ is a partition of B. We say $\mathscr{A}$ refines $\mathscr{B}$ and write $\mathscr{A} \prec \mathscr{B}$ if for every $A \in \mathscr{A}$ there either exists a $B \in \mathscr{B}$ such that $A \subseteq B$ or $A \subseteq A \setminus B$.*

Let k be a fixed integer and V be a set of vertices. Throughout this chapter we require a family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ on V to satisfy properties which we are going to describe below (see Definition 4.8).

Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. For every $1 \le j \le k$ let $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing j-tuples J, i.e., the set of j-tuples which satisfy $|J \cap V_i| \le 1$ for every $V_i \in \mathscr{P}^{(1)}$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ into (i,i)-hypergraphs have been defined for $1 \le i \le j-1$. Then for every $(j-1)$-tuple I in $\mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$ there exist a unique $P^{(j-1)} = P^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in P^{(j-1)}$. Moreover, for every j-tuple J in $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ we define the *polyad of* J

$$\hat{P}^{(j-1)}(J) = \bigcup \left\{ P^{(j-1)}(I) \colon I \in \binom{J}{j-1} \right\}.$$

In other words, $\hat{P}^{(j-1)}(J)$ is the unique collection of j partition classes of $\mathscr{P}^{(j-1)}$ in which J spans a copy of $K_j^{(j-1)}$. Observe that $\hat{P}^{(j-1)}(J)$ can be viewed as a $(j, j-1)$-hypergraph, i.e., a j-partite, $(j-1)$-uniform hypergraph. More generally, for $1 \le i < j$, we set

$$\hat{P}^{(i)}(J) = \bigcup \left\{ P^{(i)}(I) \colon I \in \binom{J}{i} \right\} \quad \text{and} \quad \boldsymbol{P}(J) = \{\hat{P}^{(i)}(J)\}_{i=1}^{j-1}. \tag{4.1}$$

Next, we define $\hat{\mathscr{P}}^{(j-1)}$ the family of all polyads

$$\hat{\mathscr{P}}^{(j-1)} = \{\hat{P}^{(j-1)}(J) \colon J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\}.$$

Note that $\hat{P}^{(j-1)}(J)$ and $\hat{P}^{(j-1)}(J')$ are not necessarily distinct for different j-tuples J and J'. However, we view $\hat{\mathscr{P}}^{(j-1)}$ as a set and, consequently, $\{\mathcal{K}_j(\hat{P}^{(j-1)}) \colon \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ we have in this chapter is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{P}^{(j-1)}) \colon \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}. \tag{4.2}$$

In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is subpartitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$. Note, that due to (4.2) we inductively infer that $\boldsymbol{P}(J)$ defined in (4.1) is a $(j, j-1)$-complex.

Throughout this chapter we also want to have control over the number of partition classes in $\mathscr{P}^{(j)}$, and more specifically, over the number of classes contained in $\mathcal{K}_j(\hat{P}^{(j-1)})$ for a fixed polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. We render this precisely in the following definition.

Definition 4.8 (family of partitions). *Suppose V is a set of vertices, $k \ge 2$ is an integer and $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is a family of partitions on V, if it satisfies the following:*

(*i*) *$\mathscr{P}^{(1)}$ is a partition of V into a_1 classes,* (*ii*) $\mathscr{P}^{(j)}$ *is a partition of* $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ *satisfying:*

$$\mathscr{P}^{(j)} \quad \text{refines} \quad \{\mathcal{K}_j(\hat{P}^{(j-1)})\colon\ \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\} \text{ and}$$
$$\big|\{P^{(j)} \in \mathscr{P}^{(j)}\colon\ P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}\big| = a_j \text{ for every } \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\,.$$

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *is t-bounded, if* $\max\{a_1, \dots, a_{k-1}\} \le t$.

It is easy to see that for a t-bounded family of partitions $\mathscr{P}(k-1, \boldsymbol{a})$ and an integer j, $2 \le j \le k-1$, we have

$$|\hat{\mathscr{P}}^{(j-1)}| = \binom{a_1}{j} \prod_{h=2}^{j-1} a_h^{\binom{j}{h}} \le t^{2^k}. \tag{4.3}$$

We now combine Definition 4.7 and Definition 4.8 and define the *refinement of a family of partitions.*

Definition 4.9 (**refinement of families**)**.** *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ *and* $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}^{\mathscr{R}})$ *are families of partitions on the same vertex set* V*. We say* $\mathscr{P}$ *refines* $\mathscr{R}$ *and write* $\mathscr{P} \prec \mathscr{R}$*, if* $\mathscr{P}^{(j)} \prec \mathscr{R}^{(j)}$ *(cf. Definition 4.7) for every* $j \in [k-1]$.

4.1.2 Hypergraph regularity lemmas

In this chapter we prove two hypergraph regularity lemmas (and corresponding counting lemmas), which may be viewed as strengthened versions of the hypergraph regularity lemma from [RS04]. Those new lemmas were already applied in [ARS07, CFKO, NRS06b, NORS08, RSST07]. As in Szemerédi's regularity lemma, such hypergraph regularity lemmas should ensure the existence of partitions of the edge set of a k-uniform hypergraph which satisfy certain properties. Besides the structural conditions discussed in the last section the partitions ensured by the main theorems in this chapter will satisfy two more properties which we define below. More specifically, the family of partitions $\mathscr{P}$ have to satisfy properties analogous to (*i*) and (*ii*) of Theorem 1.2. We first extend the notion of equitability.

Definition 4.10 (($\eta, \varepsilon, \boldsymbol{a}$)**-equitable**)**.** *Suppose* V *is a set of* n *vertices,* η *and* ε *are positive reals,* $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ *is a vector of positive integers, and* a_1 *divides* n.

We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *on* V *(as defined in Definition 4.8) is* $(\eta, \varepsilon, \boldsymbol{a})$*-equitable if it satisfies the following:*

(*a*) $|\binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})| \le \eta \binom{n}{k}$,

(*b*) $\mathscr{P}^{(1)} = \{V_i\colon\ i \in [a_1]\}$ *is an equitable vertex partition, i.e.,* $|V_i| = |V|/a_1$ *for* $i \in [a_1]$*, and*

(*c*) *for every k-tuple* $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ *the following holds:* $\boldsymbol{P}(K) = \{\hat{P}^{(j)}\}_{j=1}^{k-1}$ *(cf. (4.1)) is an* $(\varepsilon, \boldsymbol{d})$*-regular* $(n/a_1, k, k-1)$*-complex, where* $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$.

Next, we extend (*ii*) of Theorem 1.2. In this chapter we consider two possible extensions, which give rise to the two different regularity lemmas below.

Definition 4.11 (**perfectly ε-regular**)**.** *Suppose ε is some positive real. Let $G^{(k)}$ be a k-uniform hypergraph with vertex set V and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on V. We say $G^{(k)}$ is perfectly ε-regular w.r.t. $\mathscr{P}$, if for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ we have that $G^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ is ε-regular w.r.t. $\hat{P}^{(k-1)}$.*

The following theorem is one of the two main results in this chapter.

Theorem 4.12 (Regular approximation lemma)**.** *Let $k \geq 2$ be a fixed integer. For all positive constants η and ν, and every function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ there are integers $t_{Thm.4.12}$ and $n_{Thm.4.12}$ so that the following holds.*

For every k-uniform hypergraph $H^{(k)}$ with $|V(H^{(k)})| = n \geq n_{Thm.4.12}$ and $(t_{Thm.4.12})!$ dividing n there exist a k-uniform hypergraph $G^{(k)}$ on the same vertex set and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(i) $\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{Thm.4.12}$-bounded,

(ii) $G^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular w.r.t. $\mathscr{P}$, and

(iii) $|G^{(k)} \triangle H^{(k)}| \leq \nu n^k$.

Let us briefly compare Theorem 4.12 for $k = 2$ with Theorem 1.2. Note that as discussed in [KS96, Section 1.8] there are graphs with irregular pairs in any partition. Therefore, due to the "perfectness" in (*ii*) of Theorem 4.12 one has to alter $H = H^{(2)}$ to obtain $G = G^{(2)}$.

The main difference between Theorem 4.12 for $k = 2$ and Theorem 1.2, however, is in the choice of ε being a function of $a_1^{\mathscr{P}}$. It follows from the work of Gowers in [Gow97] that it is not possible to regularize a graph H with an ε in such a way that, e.g., $\varepsilon < 1/a_1^{\mathscr{P}}$ can be ensured, where $a_1^{\mathscr{P}} = |\mathscr{P}^{(1)}|$ is the number of vertex classes. Properties (*i*) and (*iii*) of Theorem 4.12 assert, however, that by adding or deleting at most νn^2 edges from H one can obtain a graph G which admits an $\varepsilon(a_1^{\mathscr{P}})$ regular partition, with $\varepsilon(a_1^{\mathscr{P}}) < 1/a_1^{\mathscr{P}}$. Such a lemma for graphs can be also deduced from the iterated regularity lemma in [AFKS00].

The other result of this chapter, Theorem 4.15, concerns the case in which we do not change the given hypergraph $H^{(k)}$. Due to the discussion above such a lemma needs to allow exceptional pairs (or polyads for $k \geq 3$) in the partition $\mathscr{P}$. Moreover, the measure of regularity of $H^{(k)}$ w.r.t. $\mathscr{P}$ (called δ_k here) cannot depend on $a_1^{\mathscr{P}}$. In fact, in our proof of Theorem 4.15 δ_k is a constant independent of each $a_1^{\mathscr{P}}, \dots, a_{k-1}^{\mathscr{P}}$. On the other hand, as in [FR02, RS04] we will infer that $H^{(k)}$ is $(\delta_k, *, r)$-regular (defined below), where r may depend on $a_1^{\mathscr{P}}, \dots, a_{k-1}^{\mathscr{P}}$. We first extend Definition 4.5.

Definition 4.13 (**(δ_k, d_k, r)-regular hypergraph**)**.** *Let δ_k and d_k be positive reals and r be a positive integer. Suppose $H^{(k-1)}$ is an $(m,k,k-1)$-hypergraph spanning at least one $K_k^{(k-1)}$.*

We say an (m,k,k)-hypergraph $H^{(k)}$ is (δ_k, d_k, r)-regular w.r.t. $H^{(k-1)}$ if for every collection $\boldsymbol{Q}^{(k-1)} = \{Q_1^{(k-1)}, \dots, Q_r^{(k-1)}\}$ of not necessarily disjoint sub-hypergraphs of $H^{(k-1)}$ which satisfy

$$\left| \bigcup_{i \in [r]} \mathcal{K}_k(Q_i^{(k-1)}) \right| \geq \delta_k \left| \mathcal{K}_k(H^{(k-1)}) \right| > 0 \,,$$

we have

$$\frac{|H^{(k)} \cap \bigcup_{i\in[r]} \mathcal{K}_k(Q_i^{(k-1)})|}{|\bigcup_{i\in[r]} \mathcal{K}_k(Q_i^{(k-1)})|} = d_k \pm \delta_k\,.$$

We write $(\delta_k, *, r)$-regular to mean $(\delta_k, d(H^{(k)}|H^{(k-1)}), r)$-regular. Moreover, if $r = 1$, then a $(\delta_k, d_k, 1)$-regular hypergraph is (ε, d_k)-regular with $\varepsilon = \delta_k$ (cf. Definition 4.5) and vice versa.

Finally, we give the second extension of (ii) of Theorem 1.2, which will be ensured by Theorem 4.15.

Definition 4.14 (**$(\delta_k, *, r)$-regular w.r.t. $\mathscr{P}$**)**.** *Suppose δ_k is a positive real and r is a positive integer. Let $H^{(k)}$ be a k-uniform hypergraph with vertex set V and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on V. We say $H^{(k)}$ is $(\delta_k, *, r)$-regular w.r.t. $\mathscr{P}$, if*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \\ \text{and } H^{(k)} \text{ is not } (\delta_k, *, r)\text{-regular w.r.t. } \hat{P}^{(k-1)}\Big\}\Big| \le \delta_k |V|^k\,.$$

The following theorem is a strengthening of the main result of [RS04].

Theorem 4.15 (Regularity lemma)**.** *Let $k \ge 2$ be a fixed integer. For all positive constants η and $\delta_k > 0$, and all functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0, 1]$ there exist integers $t_{Thm.4.15}$ and $n_{Thm.4.15}$ so that the following holds.*

For every k-uniform hypergraph $H^{(k)}$ with $|V(H^{(k)})| = n \ge n_{Thm.4.15}$ vertices, where $(t_{Thm.4.15})!$ divides n, there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(i) $\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{Thm.4.15}$-bounded and

(ii) $H^{(k)}$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.

4.1.3 Hypergraph counting lemmas

In this chapter we prove the (local) counting lemmas corresponding to Theorem 4.12 and Theorem 4.15. Similarly as Theorem 2.18 such a lemma should ensure the “right” number of copies of a given k-uniform hypergraph in an appropriate collection of dense and regular polyads provided by the corresponding regularity lemma. Here the “right” number means that the number of copies is approximately the same as in the random object of the same density. In order to avoid some technical details, for the hypergraph case we restrict our attention to the lower bound only. We now first state the counting lemma for Theorem 4.12. For that we use the following notation.

Definition 4.16 (**ν-close**)**.** *Let m and $\ell \ge k \ge 2$ be integers and $\nu > 0$. Furthermore, let $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ be an $(m, \ell, k-1)$-complex, and let $H^{(k)}$ and $G^{(k)}$ be k-uniform sub-hypergraphs of $\mathcal{K}_k(R^{(k-1)})$. We say $H^{(k)}$ and $G^{(k)}$ are ν-close w.r.t. $\boldsymbol{R}$, if for every*

$\Lambda_k \in \binom{[\ell]}{k}$ *we have*

$$\left|\left(H^{(k)} \cap \mathcal{K}_k(R^{(k-1)}[\Lambda_k])\right) \triangle \left(G^{(k)} \cap \mathcal{K}_k(R^{(k-1)}[\Lambda_k])\right)\right| \leq \nu |\mathcal{K}_k(R^{(k-1)})| .$$

The counting lemma for Theorem 4.12 estimates the number of cliques in a hypergraph $H^{(k)}$, which is ν-close to an ε-regular hypergraph $G^{(k)}$.

Theorem 4.17 (Counting lemma for Theorem 4.12)**.** *For all integers $\ell \geq k \geq 2$ and all constants $\gamma > 0$ and $d_k > 0$ there is some $\nu > 0$ such that for every $d_0 > 0$ there is $\varepsilon > 0$ and m_0 so that the following holds.*

Suppose

(*i*) $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ *is an $(\varepsilon, (d_2, \dots, d_{k-1}))$-regular $(m, \ell, k-1)$-complex with $d_i \geq d_0$ for every $i = 2, \dots, k-1$ and $m \geq m_0$,*

(*ii*) $G^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ *is (ε, d_k)-regular w.r.t. $R^{(k-1)}[\Lambda_k]$ for all $\Lambda_k \in \binom{[\ell]}{k}$ and*

(*iii*) $H^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ *is ν-close to $G^{(k)}$ w.r.t. $\boldsymbol{R}$.*

Then

$$|\mathcal{K}_\ell(H^{(k)})| \geq (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell .$$

We give the details of the proof of Theorem 4.17 in Section 4.6. Basically, it will follow from the "closeness" of $H^{(k)}$ and $G^{(k)}$ (cf. (*iii*)) that the number of $\mathcal{K}_\ell^{(k)}$'s in $G^{(k)} \cap H^{(k)}$ will be essentially the same as in $G^{(k)}$. Therefore, in order to prove Theorem 4.17 it suffices to find a lower bound on the number of such cliques in $G^{(k)}$. For that we will make use of the so-called *dense counting lemma* (see Theorem 4.19 below) which was proved by Kohayakawa, Rödl, and Skokan [KRS02]. The dense counting lemma estimates the number of $\mathcal{K}_\ell^{(k)}$'s in a "densely regular" complex such as $\{R^{(1)}, \dots, R^{(k-1)}, G^{(k)}\}$. Here "densely regular" means that the measure of regularity is much smaller then the densities of the complex in which one wants to count, i.e., $\varepsilon \ll d_i$ for all $i = 2, \dots, k$. In other words, compared to the measure of regularity the complex is relatively dense in every layer.

Note that such a "densely regular" environment cannot be enforced by an application of the regularity lemma, Theorem 4.15, since δ_k is independent of $a_2, \dots, a_{k-1}$. Consequently, a counting lemma useful in conjunction with Theorem 4.15 has to allow the following hierarchy of the constants

$$d_k \gg \delta_k \gg d_{k-1} = a_{k-1}^{-1}, d_{k-2} = a_{k-2}^{-1}, \dots, d_2 = a_2^{-1} \geq \delta, \frac{1}{r} . \tag{4.4}$$

The methods developed in this chapter allow a simple proof of the following theorem, which matches the hierarchy in (4.4).

Theorem 4.18 (Counting lemma for Theorem 4.15)**.** *For all integers $\ell \geq k \geq 2$ and positive constants $\gamma > 0$ and $d_k > 0$, there exist $\delta_k > 0$ such that for every $d_{k-1}, \dots, d_2 >$*

0 with $\frac{1}{d_i} \in \mathbb{N}$ for every $i = 2, \ldots, k-1$ there are constants $\delta > 0$ and positive integers r and m_0 so that the following holds.

Suppose

(i) $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ is an $(\delta, (d_2, \ldots, d_{k-1}))$-regular $(m, \ell, k-1)$-complex with $m \geq m_0$, and

(ii) $H^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ is (δ_k, d_k, r)-regular w.r.t. $R^{(k-1)}[\Lambda_k]$ for all $\Lambda_k \in \binom{[\ell]}{k}$.

Then

$$|\mathcal{K}_\ell(H^{(k)})| \geq (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell \,.$$

We note that the restriction that $\frac{1}{d_i} \in \mathbb{N}$ for $i = 2, \ldots, k-1$ in (i) is not essential, since the hypergraph regularity lemma, Theorem 4.15, provides a partition $\mathscr{P}$ in which all densities of the underlying structure are reciprocal of integers, i.e., $d_i = \frac{1}{a_i}$ for $i = 2, \ldots, k-1$.

4.2 Auxiliary results

In this section we review a few results that are essential for our proofs of Theorem 4.12, Theorem 4.15, Theorem 4.17, and Theorem 4.18.

4.2.1 The dense counting and extension lemma

The following theorem can be used to estimate the number of copies of $K_\ell^{(h)}$ in an appropriate collection of dense and regular blocks within a regular partition provided by the regular approximation lemma, Theorem 4.12. Moreover, it can be applied to count the number of $K_k^{(k-1)}$'s in the polyads of the partitions obtained by Theorem 4.12 and Theorem 4.15.

Theorem 4.19 (Dense counting lemma)**.** *For all integers $2 \leq h \leq \ell$ and all $\gamma > 0$ and $d_0 > 0$ there exist $\varepsilon_{DCL} = \varepsilon_{DCL}(h, \ell, \gamma, d_0) > 0$ and an integer $m_{DCL} = m_{DCL}(h, \ell, \gamma, d_0)$ so that if $\boldsymbol{d} = (d_2, \ldots, d_h) \in \mathbb{R}^{h-1}$ satisfying $d_j \geq d_0$ for $2 \leq j \leq h$ and $m \geq m_{DCL}$, and if $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ is an $(\varepsilon_{DCL}, \mathbf{d})$-regular (m, ℓ, h)-complex, then*

$$\left|\mathcal{K}_\ell(H^{(h)})\right| = (1 \pm \gamma) \prod_{j=2}^{h} d_j^{\binom{\ell}{j}} \times m^\ell \,.$$

This theorem was proved by Kohayakawa, Rödl, and Skokan in [KRS02, Theorem 6.5]. For completeness we give a short proof of a generalization of Theorem 4.19 below. The generalization of Theorem 4.19 allows us to estimate the number of copies of an arbitrary hypergraph $F^{(h)}$ with vertices $\{1, \ldots, \ell\}$ in an (m, ℓ, k)-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ satisfying that $H^{(j)}[\Lambda_j]$ is regular w.r.t. $H^{(j-1)}[\Lambda_j]$ whenever $\Lambda_j \subseteq e$ for some edge e of $F^{(h)}$. Rather than counting copies of K_ℓ in an “everywhere” regular complex, this lemma counts copies of $F^{(h)}$ in $H^{(h)}$ satisfying the less restrictive assumptions above. We introduce some more notation before we give the precise statement below (see Theorem 4.22).

For a fixed h-uniform hypergraph $F^{(h)}$, we define the j-th shadow for $j \in [h]$ by

$$\Delta_j(F^{(h)}) = \{J \colon |J| = j \text{ and } J \subseteq f \text{ for some edge } f \in F^{(h)}\}.$$

We extend the notion of an $(\varepsilon, \boldsymbol{d})$-regular complex (cf. Definition 4.6) to $(\varepsilon, \boldsymbol{d}, F^{(h)})$-regular complex.

Definition 4.20 (**$(\varepsilon, \boldsymbol{d}, F^{(h)})$-regular complex**). *Let $\varepsilon > 0$ and let $\boldsymbol{d} = (d_2, \ldots, d_h)$ be a vector of non-negative reals. Let $F^{(h)}$ be an h-uniform hypergraph with vertex set $V(F^{(h)}) = [\ell]$.*

We say an (m, ℓ, h)-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ with vertex partition $H^{(1)} = V_1 \cup \cdots \cup V_\ell$ is $(\varepsilon, \boldsymbol{d}, F^{(h)})$-regular if for every $2 \le j \le h$ the following holds

(a) for every $\Lambda_j \in \Delta_j(F^{(h)})$ the (m, j, j)-hypergraph $H^{(j)}[\Lambda_j]$ is (ε, d_j)-regular w.r.t. $H^{(j-1)}[\Lambda_j]$ and

(b) for every $\Lambda_j \notin \Delta_j(F^{(h)})$ the (m, j, j)-hypergraph $H^{(j)}[\Lambda_j]$ is empty.

Definition 4.20 imposes only a regular structure on those (m, h, h)-sub-complexes of $\boldsymbol{H}$ which naturally correspond to edges of the hypergraph $F^{(h)}$ (i.e., on a subcomplex induced on $V_{\lambda_1}, \ldots, V_{\lambda_h}$, where $\{\lambda_1, \ldots, \lambda_h\}$ forms an edge in $F^{(h)}$). We need one more definition before we can state the generalization of Theorem 4.19.

Definition 4.21 (**partite isomorphic**). *Suppose $F^{(h)}$ is an h-uniform hypergraph with $V(F^{(h)}) = [\ell]$ and $H^{(h)}$ is an (m, ℓ, h)-hypergraph with vertex partition $V(H^{(h)}) = V_1 \cup \cdots \cup V_\ell$. We say a copy $F_0^{(h)}$ of $F^{(h)}$ in $H^{(h)}$ is partite isomorphic to $F^{(h)}$ if there is a labeling of $V(F_0^{(h)}) = \{v_1, \ldots, v_\ell\}$ such that*

(i) $v_i \in V_i$ for every $i \in [\ell]$, and

(ii) $v_i \mapsto i$ is a hypergraph isomorphism (edge preserving bijection of the vertex sets) between $F_0^{(h)}$ and $F^{(h)}$.

Moreover, for every edge $e \in H^{(h)}$ we denote by $\operatorname{ext}(e; F^{(h)})$ the number of partite isomorphic copies of $F^{(h)}$ in $H^{(h)}$ which contain e.

Theorem 4.22 (General dense counting lemma). *For all integers $2 \le h \le \ell$, every h-uniform hypergraph $F^{(h)}$ on ℓ vertices, and all positive constants γ and d_0 there exist $\varepsilon_{GDCL} = \varepsilon_{GDCL}(F^{(h)}, \gamma, d_0) > 0$ and an integer $m_{GDCL} = m_{GDCL}(F^{(h)}, \gamma, d_0)$ such that the following holds.*

If $\boldsymbol{d} = (d_2, \ldots, d_h) \in \mathbb{R}^{h-1}$ satisfies $d_j \ge d_0$ for $2 \le j \le h$ and $m \ge m_{GDCL}$, and if $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ is an $(\varepsilon_{GDCL}, \boldsymbol{d}, F^{(h)})$-regular (m, ℓ, h)-complex, then the number of partite isomorphic copies of $F^{(h)}$ in $H^{(h)}$ is

$$(1 \pm \gamma) \prod_{j=2}^{h} d_j^{|\Delta_j(F^{(h)})|} \times m^\ell .$$

Theorem 4.22 is a consequence of [KRS02, Corollary 6.11]. The proof presented there was based on a double induction over the uniformity h and the number of vertices of $F^{(h)}$. As it turned out a double induction over h and the number of edges in $F^{(h)}$ allows a somewhat simpler argument and we will follow this idea. In that sense the proof presented here is similar to the proof of the counting lemma in [Tao06b]. Due to the induction we prove a slightly more general statement (see Theorem 4.25 below).

First we extend the notion of an $(\varepsilon, \boldsymbol{d}, F^{(h)})$-regular complex (cf. Definition 4.20) to $(\varepsilon, \boldsymbol{d}, \boldsymbol{F})$-regular complex, where we replace the given h-uniform hypergraph $F^{(h)}$ by a $(1, \ell, h)$-complex $\boldsymbol{F}$.

Definition 4.23 (**$(\varepsilon, \boldsymbol{d}, \boldsymbol{F})$-regular complex**). *Let $\varepsilon > 0$ and let $\boldsymbol{d} = (d_2, \dots, d_h)$ be a vector of non-negative reals. Let $\boldsymbol{F} = \{F^{(j)}\}_{j=1}^h$ be a $(1, \ell, h)$-complex on ℓ vertices $\{1, \dots, \ell\}$. We say an (m, ℓ, h)-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ with vertex partition $H^{(1)} = V_1 \cup \dots \cup V_\ell$ is $(\varepsilon, \boldsymbol{d}, \boldsymbol{F})$-regular if for every $2 \le j \le h$ the following holds*

- (*a*) *for every edge $\Lambda_j \in F^{(j)}$ the (m, j, j)-hypergraph $H^{(j)}[\Lambda_j]$ is (ε, d_j)-regular w.r.t. $H^{(j-1)}[\Lambda_j]$ and*
- (*b*) *for every $\Lambda_j \notin F^{(j)}$ the (m, j, j)-hypergraph $H^{(j)}[\Lambda_j]$ is empty.*

Definition 4.23 is a slight generalization of Definition 4.23. When $F^{(j)} = \Delta_j(F^{(h)})$, then the notion of a $(\varepsilon, \boldsymbol{d}, \boldsymbol{F})$-regular complex coincides with that of a $(\varepsilon, \boldsymbol{d}, F^{(h)})$-regular complex. However, Definition 4.23 allows to chose $F^{(j)} \supseteq \Delta_j F^{(h)}$. Finally we adjust Definition 4.21 in a straight forward manner.

Definition 4.24. *Let $\boldsymbol{F} = \{F^{(j)}\}_{j=1}^h$ be a $(1, \ell, h)$-complex with vertex set $V(F^{(1)}) = [\ell]$ and let $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ be a (m, ℓ, h)-complex with vertex partition $V(H^{(1)}) = V_1 \cup \dots \cup V_\ell$. We say a copy $\boldsymbol{F}_0$ of $\boldsymbol{F}$ in $\boldsymbol{H}$ is partite isomorphic to $\boldsymbol{F}$ if there is a labeling of $V(F_0^{(1)}) = \{v_1, \dots, v_\ell\}$ such that*

- (*i*) *$v_i \in V_i$ for every $i \in [\ell]$, and*
- (*ii*) *$v_i \mapsto i$ is a hypergraph isomorphism (edge preserving bijection of the vertex sets) between $F_0^{(j)}$ and $F^{(j)}$ for every $j = 1, \dots, h$.*

Theorem 4.25. *For all integers $1 \le h \le \ell$, every $(1, \ell, h)$-complex $\boldsymbol{F} = \{F^{(j)}\}_{j=1}^h$, and all positive constants γ and d_0 there exist $\varepsilon = \varepsilon(\boldsymbol{F}, \gamma, d_0) > 0$ and an integer $m_0 = m_0(\boldsymbol{F}, \gamma, d_0)$ such that if $\boldsymbol{d} = (d_2, \dots, d_h) \in \mathbb{R}^{h-1}$ satisfies $d_j \ge d_0$ for $2 \le j \le h$ and $m \ge m_0$, and if $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ is an $(\varepsilon, \boldsymbol{d}, \boldsymbol{F})$-regular (m, ℓ, h)-complex, then the number of partite isomorphic copies of $\boldsymbol{F}$ in $\boldsymbol{H}$ is*

$$(1 \pm \gamma) \prod_{j=2}^{h} d_j^{|F^{(j)}|} \times m^\ell .$$

Clearly, Theorem 4.25 is a generalization of Theorem 4.22 and Theorem 4.19.

Proof. Theorem 4.25 is trivial if $h = 1$. (Alternatively, we could start the induction with $h = 2$, for which Theorem 4.25 reduces to the well-known counting lemma for graphs (see, e.g., [KSSS02])).

Let $h \geq 2$. If $F^{(h)} = \varnothing$, then Theorem 4.25 follows from the induction assumption for $h-1$. So let $|F^{(h)}| \geq 1$ and positive constants γ and d_0 be given. Fix some arbitrary edge $e \in F^{(h)}$ and let $F_-^{(h)} = F^{(h)} \setminus e$ and $\boldsymbol{F}_- = \{F^{(1)}, \ldots, F^{(h-1)}, F_-^{(h)}\}$. We set

$$\varepsilon = \min \left\{ \varepsilon_{\text{Thm.4.25}}(\boldsymbol{F}_-, \gamma/2, d_0) \, , \frac{\gamma}{2} d_0^{\sum_{j=2}^{h} |F^{(j)}|} \right\}$$

and let m_0 be sufficiently large.

Let $\boldsymbol{H}$ be a $(\varepsilon, \boldsymbol{d}, \boldsymbol{F})$-regular (m, ℓ, h)-complex. Set $H_-^{(h)} = H^{(h)} \setminus H^{(h)}[e]$, i.e., we obtain $H_-^{(h)}$ from $H^{(h)}$ by removing those edges which are spanned by the vertex classes $V_{i_1} \cup \cdots \cup V_{i_h}$ indexed by elements of $e = \{i_1, \ldots, i_h\} \in \binom{[\ell]}{h}$. Moreover, let $\boldsymbol{H}_- = \{H^{(1)}, \ldots, H^{(h-1)}, H_-^{(h)}\}$. Clearly, $\boldsymbol{H}_-$ is a $(\varepsilon, \boldsymbol{d}, \boldsymbol{F}_-)$-regular (m, ℓ, h)-complex and due to the choice of ε and the induction assumption on the number edges in $F_-^{(h)}$, the number $\#\{\boldsymbol{F}_- \subseteq \boldsymbol{H}_-\}$ of partite isomorphic copies of $\boldsymbol{F}_-$ in $\boldsymbol{H}_-$ is

$$\#\{\boldsymbol{F}_- \subseteq \boldsymbol{H}_-\} = \left(1 \pm \frac{\gamma}{2}\right) \prod_{j=2}^{h-1} d_j^{|F^{(j)}|} \times d_h^{|F^{(h)}|-1} \times m^{\ell} \, . \tag{4.5}$$

For a partite isomorphic copy $\boldsymbol{F}_{-,0} = \{F_0^{(1)}, \ldots, F_0^{(h-1)}, F_{-,0}^{(h)}\}$ of $\boldsymbol{F}$ in $\boldsymbol{H}$, let $\eta(\boldsymbol{F}_{-,0})$ be the unique set of those h vertices for which

$$\{F_0^{(1)}, \ldots, F_0^{(h-1)}, F_{-,0}^{(h)} \cup \eta(\boldsymbol{F}_{-,0})\}$$

is a partite isomorphic copy of $\boldsymbol{F}$. Note that $\eta(\boldsymbol{F}_{-,0})$ does not necessarily span an edge in $H^{(h)}$. We denote by $1_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0}))\colon H^{(h)} \to \{0,1\}$ the indicator function, indicating if the edge is present or not, i.e., $\mathbb{1}_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0})) = 1$ if and only if $\eta(\boldsymbol{F}_{-,0}) \in H^{(h)}$. Hence, the number $\#\{\boldsymbol{F} \subseteq \boldsymbol{H}\}$ of partite isomorphic copy of $\boldsymbol{F}$ in $\boldsymbol{H}$ equals

$$\begin{aligned} \#\{\boldsymbol{F} \subseteq \boldsymbol{H}\} &= \sum \Big\{ \mathbb{1}_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0}))\colon \\ &\qquad\qquad \boldsymbol{F}_{-,0} \text{ is partite isomorphic copy of } \boldsymbol{F}_- \text{ in } \boldsymbol{H}_- \Big\} \\ &= \sum_{\boldsymbol{F}_{-,0}} (d_h + \mathbb{1}_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0})) - d_h) \\ &= \#\{\boldsymbol{F}_- \subseteq \boldsymbol{H}_-\} \times d_h \pm \left| \sum_{\boldsymbol{F}_{-,0}} \mathbb{1}_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0})) - d_h \right| . \end{aligned} \tag{4.6}$$

Due to (4.5) we have good control of the first term in (4.6) and we will bound the contribution of the "±-term" using the regularity of $\boldsymbol{H}$. For that, consider the induced sub-complexes $\boldsymbol{F}_*$ and $\boldsymbol{H}_*$ on $X = [\ell] \setminus e \subseteq F^{(1)}$ and $Y = H^{(1)} \setminus \bigcup_{i_j \in e} V_{i_j}$, i.e.,

$$\boldsymbol{F}_* = \boldsymbol{F}[X] := \Big\{F^{(1)} \setminus e, F^{(2)}[X], \dots, F^{(h)}[X]\Big\}$$

and

$$\boldsymbol{H}_* = \boldsymbol{H}[Y] := \Big\{H^{(1)} \setminus \bigcup_{i_j \in e} V_{i_j}, H^{(2)}[Y], \dots, H^{(h)}[Y]\Big\}.$$

For a partite isomorphic copy $\boldsymbol{F}_{0,*}$ of $\boldsymbol{F}_*$ in $\boldsymbol{H}_*$, let $\mathrm{EXT}(\boldsymbol{F}_{0,*})$ be the set of all crossing h-tuples $\eta \in \bigcup_{i_j \in e} V_{i_j}$ such that $V(F^{(1)}_{0,*}) \cup \eta$ spans a partite isomorphic copy of $\boldsymbol{F}_-$ in $\boldsymbol{H}_-$, which extends $\boldsymbol{F}_{0,*}$. Since $F^{(h)} \subseteq \mathcal{K}_{h-1}(F^{(h-1)})$, e induces a $K_h^{(h-1)}$ in $F^{(h-1)}$ and hence $\mathrm{EXT}(\boldsymbol{F}_{0,*}) \subseteq \mathcal{K}_h(H^{(h-1)}[\bigcup_{i_j \in e} V_{i_j}])$. Set

$$\begin{aligned} Q^{(h-1)}(\boldsymbol{F}_{0,*}) &= \Delta_{h-1}\big(\mathrm{EXT}(\boldsymbol{F}_{0,*})\big) \\ &= \{\eta' \subset \eta \colon |\eta'| = h-1 \text{ and } \eta \in \mathrm{EXT}(\boldsymbol{F}_{0,*})\}\,. \end{aligned}$$

Clearly, $Q^{(h-1)}(\boldsymbol{F}_{0,*}) \subseteq H^{(h-1)}[\bigcup_{i_j \in e} V_{i_j}]$ and $\mathcal{K}_h(Q^{(h-1)}(\boldsymbol{F}_{0,*})) \supseteq \mathrm{EXT}(\boldsymbol{F}_{0,*})$. A moment's thought shows that, in fact, $\mathcal{K}_h(Q^{(h-1)}(\boldsymbol{F}_{0,*})) = \mathrm{EXT}(\boldsymbol{F}_{0,*})$[1]. Hence the regularity of $\boldsymbol{H}$ yields

$$\begin{aligned} \Big|\sum_{\boldsymbol{F}_{-,0}} \mathbb{1}_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0})) - d_h\Big| &= \sum_{\boldsymbol{F}_{*,0}} \Big|\sum_{\eta \in \mathrm{EXT}(\boldsymbol{F}_{0,*})} \mathbb{1}_{H^{(h)}}(\eta(\boldsymbol{F}_{-,0})) - d_h\Big| \\ &\le \#\{\boldsymbol{F}_* \subseteq \boldsymbol{H}_*\} \times \varepsilon \Big|\mathcal{K}_h\Big(H^{(h-1)}\Big[\bigcup_{i_j \in e} V_{i_j}\Big]\Big)\Big| \\ &\le m^{\ell-h} \times \varepsilon m^h \\ &\le \varepsilon m^\ell\,. \end{aligned} \tag{4.7}$$

Combining (4.5)–(4.7) and recalling the choice of ε, we infer

$$\begin{aligned} \#\{\boldsymbol{F} \subseteq \boldsymbol{H}\} &= d_h \times \Big(1 \pm \frac{\gamma}{2}\Big) \prod_{j=2}^{h-1} d_j^{|F^{(j)}|} \times d_h^{|F^{(h)}|-1} \times m^\ell \pm \varepsilon m^\ell \\ &= \Big(1 \pm \frac{\gamma}{2}\Big) \prod_{j=2}^{h} d_j^{|F^{(j)}|} \times m^\ell \pm \varepsilon m^\ell \\ &= (1 \pm \gamma) \prod_{j=2}^{h} d_j^{|F^{(j)}|} \times m^\ell\,. \end{aligned}$$

[1] Indeed the existence of a clique $K \in \mathcal{K}(Q^{(h-1)}(\boldsymbol{F}_{0,*})) \setminus \mathrm{EXT}(\boldsymbol{F}_{0,*})$ implies that for some disjoint sets $J \subsetneq K$ and $I \subseteq V(F^{(1)}_{0,*})$, say $J = \{v_{i_1}, \dots, v_{i_j}\}$ and $I = \{v_{i_{j+1}}, \dots, v_{i_h}\}$, we have $J \cup I \notin H^{(h)}$, while $\{i_1, \dots, i_h\} \in F^{(h)}$. On the other hand, for any $(h-1)$-tuple $\widetilde{H} \in Q^{(h-1)}(\boldsymbol{F}_{0,*})$, with $\widetilde{H} \supseteq J$ there exists $H \in \mathrm{EXT}(\boldsymbol{F}_{0,*})$ with $\widetilde{H} \subset H$, yielding a contradiction.

□

Theorem 4.22 yields the following corollary, Corollary 4.26, which states that "most" edges of the h-uniform layer of an $(\varepsilon, \boldsymbol{d}, F^{(h)})$-regular complex belong to the "right" number of partite isomorphic copies of $F^{(h)}$.

Corollary 4.26 (Dense extension lemma). *For all integers ℓ and h with $2 \le h \le \ell$, for every h-uniform hypergraph $F^{(h)}$ on ℓ vertices, and for all $\gamma > 0$ and $d_0 > 0$ there exist $\varepsilon_{DEL} = \varepsilon_{DEL}(F^{(h)}, \gamma, d_0) > 0$ and an integer $m_{DEL} = m_{DEL}(F^{(h)}, \gamma, d_0)$ so that if $\boldsymbol{d} = (d_2, \dots, d_h) \in \mathbb{R}^{h-1}$ satisfying $d_j \ge d_0$ for $2 \le j \le h$ and $m \ge m_{DEL}$, and if $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{h}$ is an $(\varepsilon_{DEL}, \boldsymbol{d}, F^{(h)})$-regular (m, ℓ, h)-complex, then*

$$|H^{(h)}| = |F^{(h)}| \times (1 \pm \gamma) \prod_{j=2}^{h} d_j^{\binom{h}{j}} \times m^h\,, \tag{4.8}$$

and for all but at most $\gamma |H^{(h)}|$ edges $e \in H^{(h)}$ we have

$$ext(e; F^{(h)}) = (1 \pm \gamma) \prod_{j=2}^{h} d_j^{|\Delta_j(F^{(h)})| - \binom{h}{j}} \times m^{\ell - h}\,. \tag{4.9}$$

Proof of Corollary 4.26. The proof is based on the following useful consequence of the Cauchy–Schwarz inequality.

Fact 4.27. *For every real $\gamma > 0$, there is some $\beta > 0$ such that if $x_1, \dots, x_N$ are non-negative real numbers which for some $A \in \mathbb{R}$ satisfy*

$$\sum_{i=1}^{N} x_i = (1 \pm \beta) N A \quad \text{and} \quad \sum_{i=1}^{N} x_i^2 = (1 \pm \beta) N A^2\,,$$

then for all but at most γN indices $i \in [N]$ we have $x_i = (1 \pm \gamma) A$. □

Let an h-uniform hypergraph $F^{(h)}$ with vertex set $V(F^{(h)}) = [\ell]$ and positive reals γ and d_0 be given. We have to find appropriate constants $\varepsilon_{\mathrm{DEL}}$ and m_{DEL}.

First for every edge f in $F^{(h)}$, let $D(F^{(h)}, f)$ be the h-uniform hypergraph on $2\ell - h$ vertices constructed from two copies of $F^{(h)}$ by identifying corresponding vertices of the edge f. Now let $\beta \le \gamma$ be given by Fact 4.27 applied with γ. We fix promised constants $\varepsilon_{\mathrm{DEL}}$ and m_{DEL} by setting

$$\varepsilon_{\mathrm{DEL}} = \min\Big\{\varepsilon_{\mathrm{DCL}}(h, h, \tfrac{\beta}{3}, d_0), \varepsilon_{\mathrm{GDCL}}(F^{(h)}, \tfrac{\beta}{3}, d_0), \min_{f \in F^{(h)}} \{\varepsilon_{\mathrm{GDCL}}(D(F^{(h)}, f), \tfrac{\beta}{3}, d_0)\}\Big\},$$

where $\varepsilon_{\mathrm{DCL}}$ and $\varepsilon_{\mathrm{GDCL}}$ are given by Theorem 4.19 and Theorem 4.22, respectively.

Similarly, set

$$m_{\mathrm{DEL}} = \max\Big\{ m_{\mathrm{DCL}}(h,h,\tfrac{\beta}{3},d_0),\, m_{\mathrm{GDCL}}(F^{(h)},\tfrac{\beta}{3},d_0),\\ \max_{f\in F^{(h)}}\{m_{\mathrm{GDCL}}(D(F^{(h)},f),\tfrac{\beta}{3},d_0)\}\Big\}.$$

After we fixed all constants, let $\boldsymbol{H}=\{H^{(j)}\}_{j=1}^h$ be an $(\varepsilon_{\mathrm{DEL}},\boldsymbol{d},F^{(h)})$-regular (m,ℓ,h)-complex with vertex partition $V_1\cup\dots\cup V_h$, $m\geq m_{\mathrm{DEL}}$, and $\boldsymbol{d}=(d_2,\dots,d_h)$ satisfying $d_j\geq d_0$ for every $j=2,\dots,h$. From the choice of $\varepsilon_{\mathrm{DEL}}\leq\varepsilon_{\mathrm{DCL}}(h,h,\frac{\beta}{3},d_0)$ and since $m\geq m_{\mathrm{DEL}}\geq m_{\mathrm{DCL}}(h,h,\frac{\beta}{3},d_0)$, Theorem 4.19 (applied to the (m,h,h)-complex $\boldsymbol{H}[\Lambda_h]=\{H^{(j)}[\Lambda_h]\}_{j=1}^h$ for every $\Lambda_h\in\binom{[\ell]}{h}$ that is an edge in $F^{(h)}$) yields

$$|H^{(h)}| = |F^{(h)}|\times\left(1\pm\frac{\beta}{3}\right)\prod_{j=2}^{h}d_j^{\binom{h}{j}}\times m^h\,, \tag{4.10}$$

which implies (4.8). Moreover, since $\varepsilon_{\mathrm{DEL}}\leq\varepsilon_{\mathrm{GDCL}}(F^{(h)},\frac{\beta}{3},d_0)$ and $m\geq m_{\mathrm{DEL}}\geq m_{\mathrm{GDCL}}(F^{(h)},\frac{\beta}{3},d_0)$ we can apply Theorem 4.22 to estimate the number of partite isomorphic copies of $F^{(h)}$ in $H^{(h)}$ by

$$\left(1\pm\frac{\beta}{3}\right)\prod_{j=2}^{h}d_j^{|\Delta_j(F^{(h)})|}\times m^\ell\,. \tag{4.11}$$

Consequently,

$$\begin{aligned}\sum_{e\in H^{(h)}}\mathrm{ext}(e;F^{(h)}) &\overset{(4.11)}{=} |F^{(h)}|\times\left(1\pm\frac{\beta}{3}\right)\prod_{j=2}^{h}d_j^{|\Delta_j(F^{(h)})|}\times m^\ell\\ &\overset{(4.10)}{=}\frac{1\pm\frac{\beta}{3}}{1\pm\frac{\beta}{3}}\times|H^{(h)}|\times\prod_{j=2}^{h}d_j^{|\Delta_j(F^{(h)})|-\binom{h}{j}}\times m^{\ell-h}\\ &= (1\pm\beta)|H^{(h)}|A\,,\end{aligned}\tag{4.12}$$

for

$$A=\prod_{j=2}^{h}d_j^{|\Delta_j(F^{(h)})|-\binom{h}{j}}\times m^{\ell-h}\,. \tag{4.13}$$

In view of (4.12) and Fact 4.27 it is only left to verify

$$\sum_{e\in H^{(h)}}\left(\mathrm{ext}(e;F^{(h)})\right)^2=(1\pm\beta)|H^{(h)}|A^2 \tag{4.14}$$

for showing Corollary 4.26. For that let Λ_h be an edge in $F^{(h)}$. Consider, the complex $\boldsymbol{DC}(\boldsymbol{H},\Lambda_h)$ which we obtain by taking two copies $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ of $\boldsymbol{H}$ and identifying those vertices with its copy which belong to a vertex class indexed by some $\lambda\in\Lambda_h$.

More explicitely, for $1 \le i \le \ell$ let $V_i = \{v_{1,i}, \dots, v_{m,i}\}$ be the vertex classes of $\boldsymbol{H}$. Suppose $W_i = \{w_{i,1}, \dots, w_{i,m}\}$ and $U_i = \{u_{i,1}, \dots, u_{i,m}\}$ are the vertex classes of the copies $\boldsymbol{H}_1 = \{H_1^{(j)}\}_{j=1}^h$ and $\boldsymbol{H}_2 = \{H_2^{(j)}\}_{j=1}^h$ of $\boldsymbol{H}$ so that $w_{i,r} \mapsto v_{i,r}$ (respectively, $u_{i,r} \mapsto v_{i,r}$) for every $1 \le i \le \ell$ and $1 \le r \le m$ is an hypergraph isomorphism between $H_1^{(j)}$ (resp. $H_2^{(j)}$) and $H^{(j)}$ for every $j = 2, \dots, h$. Then, $\boldsymbol{DC}(\boldsymbol{H}, \Lambda_h)$ is the complex which we obtain from $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ by identifying $w_{\lambda,r}$ with $u_{\lambda,r}$ for every $\lambda \in \Lambda_h$ and $1 \le r \le m$.

It follows from the assumptions on $\boldsymbol{H}$, that for every edge $\Lambda_h \in F^{(h)}$ the complex $\boldsymbol{DC}(\boldsymbol{H}, \Lambda_h)$ is an $(\varepsilon_{\mathrm{DEL}}, \boldsymbol{d}, D(F^{(h)}, \Lambda_h))$-regular $(m, 2\ell - h, h)$-complex. Consequently, the earlier choice of $\varepsilon_{\mathrm{DEL}}$ and m_{DEL} allows us to apply Theorem 4.22 to $\boldsymbol{DC}(\boldsymbol{H}, \Lambda_h)$ to estimate the number of partite isomorphic copies of $D(F^{(h)}, \Lambda_h)$ in $\boldsymbol{DC}(\boldsymbol{H}, \Lambda_h)$ by

$$\left(1 \pm \frac{\beta}{3}\right) \prod_{j=2}^{h} d_j^{|\Delta_j(D(F^{(h)}, \Lambda_h))|} \times m^{2\ell - h} . \tag{4.15}$$

On the other hand, the number of partite isomorphic copies of $D(F^{(h)}, \Lambda_h)$ in the complex $\boldsymbol{DC}(\boldsymbol{H}, \Lambda_h)$ coincides with $\sum\{(\mathrm{ext}(e; F^{(h)}))^2 \colon\ e \in H^{(h)}[\Lambda_h]\}$. Since

$$|\Delta_j(D(F^{(h)}, \Lambda_h))| = 2|\Delta_j(F^{(h)})| - \binom{h}{j}$$

for every $j = 2, \dots, h$ we have

$$\sum_{e \in H^{(h)}[\Lambda_h]} \left(\mathrm{ext}(e; F^{(h)})\right)^2 = \left(1 \pm \frac{\beta}{3}\right) \prod_{j=2}^{h} d_j^{2|\Delta_j(F^{(h)})| - \binom{h}{j}} \times m^{2\ell - h} .$$

Repeating the same argument for every edge $\Lambda_h \in F^{(h)}$ yields

$$\sum_{e \in H^{(h)}} \left(\mathrm{ext}(e; F^{(h)})\right)^2 = |F^{(h)}| \times \left(1 \pm \frac{\beta}{3}\right) \prod_{j=2}^{h} d_j^{2|\Delta_j(F^{(h)})| - \binom{h}{j}} \times m^{2\ell - h} .$$

Hence, in view of (4.13) and (4.10) we have

$$\sum_{e \in H^{(h)}} \left(\mathrm{ext}(e; F^{(h)})\right)^2 = \frac{1 \pm \frac{\beta}{3}}{1 \pm \frac{\beta}{3}} \times |H^{(h)}| \times A^2 = (1 \pm \beta)|H^{(h)}|A^2 ,$$

which gives (4.14) and concludes the proof of Corollary 4.26. □

4.2.2 Facts concerning regular hypergraphs

In this section we state some facts about regular hypergraphs which are useful for the proofs in this chapter. The first assertion roughly says that the complement of a regular hypergraph is also regular.

Proposition 4.28. *Let $j \geq 2$, m, $r \geq 1$ be fixed integers and let δ_1, δ_2 and $d_1 \geq d_2$ be positive reals. If $P_1^{(j)}$ is a (δ_1, d_1, r)-regular (m, j, j)-hypergraph w.r.t. some underlying $(m, j, j-1)$-hypergraph $\hat{P}^{(j-1)}$ and $P_2^{(j)} \subseteq P_1^{(j)}$ is (δ_2, d_2, r)-regular w.r.t. $\hat{P}^{(j-1)}$, then $P_1^{(j)} \setminus P_2^{(j)}$ is $(\delta_1 + \delta_2, d_1 - d_2, r)$-regular w.r.t. $\hat{P}^{(j-1)}$. Moreover, if $P_1^{(j)} = \mathcal{K}_j(\hat{P}^{(j-1)})$, then $P_1^{(j)} \setminus P_2^{(j)}$ is $(\delta_2, 1 - d_2, r)$-regular w.r.t. $\hat{P}^{(j-1)}$.* □

The proof of Proposition 4.28, as well as the proof of the next proposition, is straightforward from the definition of (δ, d, r)-regularity and we therefore omit both of them.

Proposition 4.29. *Let $j \geq 2$, m, $r \geq 1$ be fixed integers and let δ and d be positive reals. If $P^{(j)}$ is a (δ, d, r)-regular (m, j, j)-hypergraph w.r.t. some underlying $(m, j, j-1)$-hypergraph $\hat{P}^{(j-1)}$ and $\hat{Q}^{(j-1)} \subseteq \hat{P}^{(j-1)}$ such that $|\mathcal{K}_j(\hat{Q}^{(j-1)})| > 0$, then $P^{(j)} \cap \mathcal{K}_j(\hat{Q}^{(j-1)})$ is $(\alpha\delta, d, r)$-regular w.r.t. $\hat{Q}^{(j-1)}$ for $\alpha = |\mathcal{K}_j(\hat{P}^{(j-1)})|/|\mathcal{K}_j(\hat{Q}^{(j-1)})|$.* □

The next two facts regard regularity properties of the union of regular hypergraphs. The first of those two propositions states that the union of regular (m, j, j)-hypergraphs which share the same underlying $(m, j, j-1)$-hypergraph is regular. Again the proof is straightforward and we refrain from presenting it here.

Proposition 4.30. *Let $j \geq 2$, m, t, $r \geq 1$ be fixed integers and let δ and $d(1), \ldots, d(t)$ be positive reals. Suppose $P_1^{(j)}, \ldots, P_t^{(j)}$ is a family of pairwise edge disjoint (m, j, j)-hypergraphs with the same underlying $(m, j, j-1)$-hypergraph $\hat{P}^{(j-1)}$.*

If $P_\tau^{(j)}$ is $(\delta, d(\tau), r)$-regular w.r.t. $\hat{P}^{(j-1)}$ for every $\tau \in [t]$, then $P^{(j)}$ is $(t\delta, d, r)$-regular w.r.t. $\hat{P}^{(j-1)}$, where $P^{(j)} = \bigcup_{\tau \in [t]} P_\tau^{(j)}$ and $d = \sum_{\tau \in [t]} d(\tau)$. □

The next proposition gives us control when we union hypergraphs having different underlying polyads. Before we make this precise, we define the setup for our proposition.

Setup 4.31. Let $j \geq 2$, m, $t \geq 1$ be fixed integers and let δ and d be positive reals. Let $\{\hat{P}_\tau^{(j-1)}\}_{\tau \in [t]}$ be a family of $(m, j, j-1)$-hypergraphs such that

$$\bigcup_{\tau \in [t]} \hat{P}_\tau^{(j-1)} \text{ is a } j\text{-partite } (j-1)\text{-uniform hypergraph,}$$

$$\mathcal{K}_j\Big(\bigcup_{\tau \in [t]} \hat{P}_\tau^{(j-1)}\Big) = \bigcup_{\tau \in [t]} \mathcal{K}_j(\hat{P}_\tau^{(j-1)}), \tag{4.16}$$

and

$$\mathcal{K}_j(\hat{P}_\tau^{(j-1)}) \cap \mathcal{K}_j(\hat{P}_{\tau'}^{(j-1)}) = \varnothing \quad \text{for} \quad 1 \leq \tau < \tau' \leq t\,.$$

Let $\{P_\tau^{(j)}\}_{\tau \in [t]}$ be a family of (m, j, j)-hypergraphs such that $\hat{P}_\tau^{(j-1)}$ underlies $P_\tau^{(j)}$ for any $\tau \in [t]$. Set $\hat{P}^{(j-1)} = \bigcup_{\tau \in [t]} \hat{P}_\tau^{(j-1)}$ and $P^{(j)} = \bigcup_{\tau \in [t]} P_\tau^{(j)}$.

Proposition 4.32. *Let $r \geq 1$ be a fixed integer and let $\{P_\tau^{(j)}\}_{\tau \in [t]}$ and $\{\hat{P}_\tau^{(j-1)}\}_{\tau \in [t]}$ satisfy Setup 4.31. If $P_\tau^{(j)}$ is (δ, d, r)-regular w.r.t. $\hat{P}_\tau^{(j-1)}$ for every $\tau \in [t]$, then $P^{(j)}$ is $(2\sqrt{\delta}, d, r)$-regular w.r.t. $\hat{P}^{(j-1)}$.* □

For $r = 1$ a proof of Proposition 4.32 appeared in [NRS06a] and the proof presented there works verbatim for general $r \geq 1$.

The proof of the following lemma is based on Chernoff's inequality and the fact that randomly chosen sub-hypergraphs of a regular hypergraph are regular. Similar statements were proved in [FR02, RS04] and we will omit the technical details here.

Proposition 4.33 (Slicing lemma). *Let $j \geq 2$, s_0, $r \geq 1$ be integers and let δ_0, ϱ_0, and p_0 be positive real numbers. There is an integer $m_{SL} = m_{SL}(j, s_0, r, \delta_0, \varrho_0, p_0)$ so that the following holds. If $m \geq m_{SL}$,*

(i) $\hat{P}^{(j-1)}$ *is a $(m, j, j-1)$-hypergraph satisfying $|\mathcal{K}_j(\hat{P}^{(j-1)})| \geq m^j / \ln m$ and*

(ii) $P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})$ *is an (δ, ϱ, r)-regular (m, j, j)-hypergraph with $\varrho \geq \varrho_0 \geq 2\delta \geq 2\delta_0$.*

Then for any positive integer $1 \leq s \leq s_0$ and all positive reals $p_1, \dots, p_s$ satisfying

(iii) $\sum_{\sigma \in [s]} p_\sigma \leq 1$ *and $p_\sigma \geq p_0$ for $\sigma \in [s]$*

there exists a partition $\{T_0^{(j)}, T_1^{(j)}, \dots, T_s^{(j)}\}$ of $P^{(j)}$ so that $T_\sigma^{(j)}$ is $(3\delta, p_\sigma \varrho, r)$-regular w.r.t. $\hat{P}^{(j-1)}$ for every $\sigma = 1, \dots, s$.

Moreover, $T_0^{(j)}$ is $(3\delta, (1 - \sum_{\sigma \in [s]} p_\sigma)\varrho, r)$-regular with respect to $\hat{P}^{(j-1)}$ and $T_0^{(j)} = \varnothing$ if $\sum_{\sigma \in [s]} p_\sigma = 1$. □

4.3 Outline of the proofs

Roughly speaking, our proof of both theorems, Theorem 4.12 and Theorem 4.15, is based on the following induction scheme

$$\text{Theorem 4.15 for } k \Longrightarrow \text{Theorem 4.12 for } k \Longrightarrow \text{Theorem 4.15 for } k+1\,.$$

To carry out the technical details for such an induction scheme, we need to strengthen the statements of Theorem 4.15 (**r**egularity **l**emma) and of Theorem 4.12 (**r**egularity **a**pproximation **l**emma) to more general, but, unfortunately, less esthetically pleasing statement RL(k), Lemma 4.34, and RAL(k), Lemma 4.36.

Before we start to discuss these more general statements we will briefly outline why they are needed. While the proof of the implication Theorem 4.12 for $k \Longrightarrow$ Theorem 4.15 for $k+1$ could follow the lines of [FR02, RS04] (now using Theorem 4.12 for k to regularize the witnesses, which provides the cleaner partition $\mathscr{P}$), the need for generalizing the statements comes from the implication Theorem 4.15 for $k \Longrightarrow$ Theorem 4.12 for k. In our proof of this implication we need to apply Theorem 4.15 for k twice. After the first application we obtain an $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable partition $\mathscr{P}$ which is bounded. However, the hypergraph H will only be δ_k-regular w.r.t. $\mathscr{P}$, where δ_k is a constant independent of $\boldsymbol{a}^{\mathscr{P}}$, and not $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular, as required by part (*ii*) of Theorem 4.12. To obtain such an $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular hypergraph $G^{(k)}$, which will be "ν close to $H^{(k)}$" (cf. (*iii*) of Theorem 4.12) we need to apply Theorem 4.15 again. It will be essential for us that the partition obtained in the second application of Theorem 4.15 will refine $\mathscr{P}$, the partition obtained in the first application. This is the reason why we will strengthen the statement of Theorem 4.15 (see Lemma 4.34). This change is due to the induction scheme requiring a corresponding strengthening of Theorem 4.12 (see Lemma 4.36).

We now state the strengthened variant of Theorem 4.15. It allows us to enter the regularity lemma with an initial equitable family of partitions $\mathscr{O}$ and a family of k-uniform hypergraphs $H_1^{(k)}, \dots, H_s^{(k)}$. It then guarantees the existence of an equitable refinement $\mathscr{P}$ of $\mathscr{O}$ for which each $H_i^{(k)}$ is regular. (Since it might not be completely obvious that Theorem 4.15 follows from Lemma 4.34 stated below, we give the formal reduction after Remark 4.35.)

Lemma 4.34 (RL(k))**.** *For all positive integers o and s, all positive reals η and δ_k, and all functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ there is a positive real μ_{RL} and positive integers t_{RL} and n_{RL} such that the following holds.*

Suppose

- *(a) V is a set of cardinality $n \geq n_{RL}$ and $(t_{RL})!$ divides n,*
- *(b) $\mathscr{O} = \mathscr{O}(k-1, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{RL}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and o-bounded family of partitions on V, and*
- *(c) $\mathscr{H}^{(k)} = \{H_1^{(k)}, \dots, H_s^{(k)}\}$ is a partition of $\binom{V}{k}$.*

Then there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

- *(P1) $\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_{RL}-bounded,*
- *(P2) $\mathscr{P} \prec \mathscr{O}$,*

and for every $i \in [s]$

- *(H) $H_i^{(k)}$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.*

Remark 4.35. In the inductive proof we will apply Lemma 4.34 twice. In the second application in Section 4.4.2 it will be convenient to use a variant of Lemma 4.34, where assumptions (*a*) and (*b*) are replaced by

- (a') $V = V_1 \cup \dots \cup V_k$, $|V_i| = m \geq n_{\mathrm{RL}}/k$ and $t_{\mathrm{RL}}!$ divides m,
- (b') $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ is a $(\mu_{\mathrm{RL}}/3, \boldsymbol{d})$-regular $(m, k, k-1)$-complex, where the vertex set $R^{(1)} = V_1 \cup \dots \cup V_k$ and $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$, $a_i \in \mathbb{N}$ and $a_i \leq o$ for $2 \leq i < k$.

Moreover, we weaken conclusion (*P2*) in this context, insisting only that $\mathscr{P}$ "refines" the given complex $\boldsymbol{R}$, more precisely

- (*P2'*) $\mathscr{P}^{(1)} \prec R^{(1)} = V_1 \cup \dots \cup V_k$ and for every $2 \leq j < k$ and every $P^{(j)} \in \mathscr{P}^{(j)}$ we have either $P^{(j)} \subseteq R^{(j)}$ or $P^{(j)} \cap R^{(j)} = \varnothing$.

We note that this version of Lemma 4.34 is in fact a consequence of Lemma 4.34. Indeed, (a') clearly implies (a). Moreover any complex $\boldsymbol{R}$ satisfying (b') can be extended to a family of partitions $\mathscr{O} = \mathscr{O}(k-1, \boldsymbol{a}^{\mathscr{O}})$ on V satisfying (b) of Lemma 4.34 with $\boldsymbol{a}^{\mathscr{O}} = (a_2, \ldots, a_{k-1})$. The proof of this observation is quite straightforward but tedious and we give a sketch only.

First set $\mathscr{O}^{(1)} = \{V_1, \ldots, V_k\}$. Then we use the slicing lemma, Proposition 4.33, for every pair $1 \le i < j \le k$ to partition the bipartite graph $K(V_i, V_j) \setminus R^{(2)}$ into $a_2 - 1$ distinct $(\mu_{\mathrm{RL}}, 1/a_2)$-regular graphs. This is possible, since due to Proposition 4.28, $K(V_i, V_j) \setminus R^{(2)}$ is $(\mu_{\mathrm{RL}}/3, (a_2-1)/a_2)$-regular and since $a_2 \in \mathbb{N}$. The $a_2 - 1$ graphs obtained this way together with $R^{(2)}[V_i, V_j]$ define the partition $\mathscr{O}^{(2)}$ on $V_i \cup V_j$.

Iterating this procedure inductively for triples, $\ldots, (k-1)$-tuples based on Proposition 4.33 yields the family of partitions $\mathscr{O}(k-1, \boldsymbol{a}^{\mathscr{O}} = (a_2, \ldots, a_{k-1}))$ on V satisfying (b) of Lemma 4.34 for $o = \max\{k, a_2, \ldots, a_{k-1}\}$.

Finally, we note that conclusion ($P2$) of Lemma 4.34, then yields ($P2'$) due to the construction of $\mathscr{O}^{(k)}$ from $\boldsymbol{R}$ above.

We now verify that Lemma 4.34 implies Theorem 4.15 for the same k.

Proof: $RL(k) \Longrightarrow$ Theorem 4.15 for k. Let k be a fixed integer and let constants η and δ_k and functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ be given by Theorem 4.15. We want to apply Lemma 4.34. For that we will define an auxiliary family of partitions $\mathscr{O}$. In fact any sufficiently equitable partition would do. In order to avoid trivial cases we are going to split the vertex set into k parts of the same size and any part of the partition $\mathscr{O}^{(j)}$ will be isomorphic to the complete j-partite j-uniform hypergraph of the appropriate order for $2 \le j \le k-1$ (see (4.17) below). With this in mind we apply Lemma 4.34 with $o = k$, $s = 2$, and the given constants η and δ_k, and functions r and δ to obtain μ_{RL}, t_{RL} and n_{RL}. We then set $t_{Thm.4.15} = t_{\mathrm{RL}}$ and $n_{Thm.4.15} = n_{\mathrm{RL}}$.

Now let $n \ge n_{Thm.4.15}$ be a multiple of $t_{Thm.4.15} = (t_{\mathrm{RL}})!$ and $H^{(k)}$ be a hypergraph with vertex set V, where $|V| = n$. Set $a_1^{\mathscr{O}} = k$, $a_j^{\mathscr{O}} = 1$ for $j = 2, \ldots, k-1$, $\boldsymbol{a}^{\mathscr{O}} = (a_1^{\mathscr{O}}, \ldots, a_{k-1}^{\mathscr{O}})$ and let $V = V_1 \cup \cdots \cup V_{a_1^{\mathscr{O}}} = \mathscr{O}^{(1)}$ be some arbitrary equitable vertex partition. Moreover, set

$$\mathscr{O}^{(j)} = \{K_j^{(j)}(V_{i_1}, \ldots, V_{i_j})\colon 1 \le i_1 < \cdots < i_j \le a_1^{\mathscr{O}} = k\} \tag{4.17}$$

and

$$\mathscr{H}^{(k)} = \left\{ H^{(k)}, \binom{V}{k} \setminus H^{(k)} \right\}.$$

Clearly, the partition $\mathscr{O}$ constructed this way is $(\eta^{\mathscr{O}}, \mu, \boldsymbol{a}^{\mathscr{O}})$-equitable for some constant $\eta^{\mathscr{O}} > 0$ and every $\mu > 0$. Consequently, V, $\mathscr{O}$ and $\mathscr{H}^{(k)}$ satisfy the assumptions (a)–(c) of Lemma 4.34 for n_{RL}, t_{RL}, $o = a_1^{\mathscr{O}} = k$, $s = 2$ and any μ_{RL}. Then, ($P1$) and (H) yield conclusions (i) and (ii) of Theorem 4.15. □

Next we state a similarly strengthened version of Theorem 4.12.

Lemma 4.36 (RAL(k)). *For all positive integers o and s, all positive reals η and ν, and every function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ there is a positive real μ_{RAL} and positive integers t_{RAL} and n_{RAL} such that the following holds.*

Suppose

(*a*) *V is a set of cardinality $n \geq n_{RAL}$ and $(t_{RAL})!$ divides n,*

(*b*) *$\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ is a $(\eta^{\mathscr{O}}, \mu_{RAL}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and o-bounded family of partitions on V, and*

(*c*) *$\mathscr{H}^{(k)} = \{H_1^{(k)}, \dots, H_s^{(k)}\}$ is a partition of $\binom{V}{k}$ so that $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}$.*

Then there exist a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(*P1*) *$\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_{RAL}-bounded and*

(*P2*) *$\mathscr{P} \prec \mathscr{O}(k-1) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$.*

Furthermore, there exists a partition $\mathscr{G}^{(k)} = \{G_1^{(k)}, \dots, G_s^{(k)}\}$ of $\binom{V}{k}$ such that for every $i \in [s]$ the following holds

(*G1*) *$G_i^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular w.r.t. $\mathscr{P}$,*

(*G2*) *$|G_i^{(k)} \triangle H_i^{(k)}| \leq \nu n^k$, and*

(*G3*) *if $H_i^{(k)} \subseteq Cross_k(\mathscr{O}^{(1)})$ then $G_i^{(k)} \subseteq Cross_k(\mathscr{O}^{(1)})$ and $\mathscr{G}^{(k)} \prec \mathscr{O}^{(k)}$.*

Lemma 4.36 yields Theorem 4.12 for the same k in a similar way as Lemma 4.34 implies Theorem 4.15. We give the formal reduction below.

Proof: RAL(k) $\Longrightarrow$ Theorem 4.12 for k. Let k be a fixed integer and let positive reals η, ν, and a function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ be given. We want to apply RAL(k). For that we set

$$\begin{aligned} o_{\text{RAL}} = k+1\,, \quad s_{\text{RAL}} = 4\,, \quad \eta_{\text{RAL}} = \eta\,, \\ \nu_{\text{RAL}} = \tfrac{\nu}{2}\,, \quad \text{and} \quad \varepsilon_{\text{RAL}}(\cdot,\dots,\cdot) = \tfrac{1}{2}\varepsilon(\cdot,\dots,\cdot)\,. \end{aligned} \tag{4.18}$$

Lemma 4.36 then yields positive constants μ_{RAL}, t_{RAL}, and n_{RAL}. For Theorem 4.12 we set $t_{Thm.4.12} = t_{\text{RAL}}$ and $n_{Thm.4.12} = n_{\text{RAL}}$. Then, let $H^{(k)}$ be a k-uniform hypergraph on $n \geq n_{Thm.4.12}$ vertices where n is a multiple of $(t_{Thm.4.12})! = (t_{\text{RAL}})!$. In view of Lemma 4.36 we construct an auxiliary family of partitions $\mathscr{O}$. For that set $a_1^{\mathscr{O}} = k+1$, $a_j^{\mathscr{O}} = 1$ for $j = 2, \dots, k$, $\boldsymbol{a}^{\mathscr{O}} = (a_1^{\mathscr{O}}, \dots, a_k^{\mathscr{O}})$ and let $V(H^{(k)}) = V_1 \cup \dots \cup V_{a_1^{\mathscr{O}}} = \mathscr{O}^{(1)}$ be some arbitrary equitable vertex partition. Moreover, for $j = 2, \dots, k$ set

$$\mathscr{O}^{(j)} = \{K_j^{(j)}(V_{i_1}, \dots, V_{i_j})\colon 1 \leq i_1 < \dots < i_j \leq a_1^{\mathscr{O}}\}\,.$$

Clearly, $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ defined that way is $(\eta^{\mathscr{O}}, \mu, \boldsymbol{a}^{\mathscr{O}})$-equitable for some $\eta^{\mathscr{O}} > 0$ and any fixed $\mu > 0$. Moreover, $\mathscr{O}$ is $(a_1^{\mathscr{O}} = k+1 = o)$-bounded and, hence, assumption (*b*) of RAL(k) holds.

We now define a partition $\mathscr{H}^{(k)}$ of $\binom{V}{k}$. For that we set $\bar{H}^{(k)} = \binom{V}{k} \setminus H^{(k)}$ and define

$$\mathscr{H}^{(k)} = \Big\{ H^{(k)}_{\mathscr{O}} = H^{(k)} \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}), \qquad H^{(k)}_{\bar{\mathscr{O}}} = H^{(k)} \setminus H^{(k)}_{\mathscr{O}}, \\ \bar{H}^{(k)}_{\mathscr{O}} = \bar{H}^{(k)} \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}), \qquad \bar{H}^{(k)}_{\bar{\mathscr{O}}} = \bar{H}^{(k)} \setminus \bar{H}^{(k)}_{\mathscr{O}} \Big\}.$$

Clearly, $\mathscr{H}^{(k)}$ satisfies (c) of RAL(k) with $s_{\mathrm{RAL}} = 4$.

Lemma 4.36 now yields the existence of $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ satisfying $(P1)$ and, consequently by (4.18), (i) of Theorem 4.12. Moreover, Lemma 4.36 yields a partition $\mathscr{G}^{(k)} = \{G^{(k)}_{\mathscr{O}}, G^{(k)}_{\bar{\mathscr{O}}}, \bar{G}^{(k)}_{\mathscr{O}}, \bar{G}^{(k)}_{\bar{\mathscr{O}}}\}$ of $\binom{V}{k}$ satisfying $(G1)$–$(G3)$.

We set $G^{(k)} = G^{(k)}_{\mathscr{O}} \cup G^{(k)}_{\bar{\mathscr{O}}}$. Then $(G1)$, Proposition 4.30 (applied to $G^{(k)}_{\mathscr{O}}$ and $G^{(k)}_{\bar{\mathscr{O}}}$ for every polyad in $\hat{\mathscr{P}}^{(k-1)}$), and (4.18) imply part (ii) of Theorem 4.12.

Finally, it is easy to see that (iii) of Theorem 4.12 follows from $(G2)$ and the choice of ν_{RAL} in (4.18), since

$$G^{(k)} \triangle H^{(k)} \subseteq (G^{(k)}_{\mathscr{O}} \triangle H^{(k)}_{\mathscr{O}}) \cup (G^{(k)}_{\bar{\mathscr{O}}} \triangle H^{(k)}_{\bar{\mathscr{O}}}).$$

□

Due to the implications proved above we note that it suffices to show

$$\mathrm{RL}(2) \quad \text{and} \quad \mathrm{RL}(k) \Longrightarrow \mathrm{RAL}(k) \Longrightarrow \mathrm{RL}(k+1) \quad \text{for} \quad k \geq 2,$$

in order to establish Theorem 4.15 and Theorem 4.12 inductively.

We outline the basis of the induction, the proof of RL(2), in the next section. The proofs of each of the two implications establishing the induction step are the content of Section 4.4 and Section 4.5, respectively.

Sketch of the proof of RL(2)

Observe that in the statement of RL(2), Lemma 4.34 for $k = 2$, the constant μ and the function δ have no bearing. Consequently, RL(2) reduces to the following statement.

Lemma 4.37 (RL(2))**.** *For all positive integers o and s, all positive reals η and δ_2, and any function $r\colon \mathbb{N} \to \mathbb{N}$ there are positive integers t_{RL} and n_{RL} such that the following holds.*

Suppose

(a) *V is a set of cardinality $n \geq n_{RL}$ and $(t_{RL})!$ divides n,*

(b) *$\mathscr{O}^{(1)}$ is a vertex partition $V_1 \cup \dots \cup V_{a_1^{\mathscr{O}}}$ of V, where $|V_1| = \dots = |V_{a_1^{\mathscr{O}}}|$ and $a_1^{\mathscr{O}} \leq o$*

(c) *$\mathscr{H} = \{H_1, \dots, H_s\}$ is a partition of $\binom{V}{2}$ the complete graph on n vertices.*

Then there exists a partition $\mathscr{P}^{(1)} = \{W_1, \dots, W_{a_1^{\mathscr{P}}}\}$ of V so that

(*P1*) $|W_1| = \dots = |W_{a_1^{\mathscr{P}}}|$, $\mathrm{Cross}_2(\mathscr{P}^{(1)}) \geq (1-\eta)\binom{n}{2}$, *and* $a_1^{\mathscr{P}} \leq t_{RL}$,

(*P2*) *for every* $i \in [a_1^{\mathscr{P}}]$ *we have* $W_i \subseteq V_j$ *for some* $j \in [a_1^{\mathscr{O}}]$

and for every $i \in [s]$

(*H*) H_i *is* $(\delta_2, *, r(a_1^{\mathscr{P}}))$*-regular w.r.t.* $\mathscr{P}^{(1)}$.

The proof of RL(2) follows closely the lines of the proof of Szemerédi's regularity lemma [Sze78], Theorem 1.2. There are three differences, however. The first and the last of them are standard.

(*1*) Rather than one graph we have a fixed number of graphs $H_1, \dots, H_s$ to regularize. Such a regularity lemma was used in a number of applications and is discussed for example in [KS96, Section 1.9].

(*2*) This difference which regards the concept of regularity in (*H*) is perhaps most significant. Instead of a single pair $A' \subseteq A$, $B' \subseteq B$, $|A'||B'| \geq \varepsilon|A||B|$ that witnesses the irregularity of a bipartite graph with vertex classes A and B, we consider here a more complicated witness; namely an r-tuple of pairs (A_i, B_i) of sets where $A_1, \dots, A_r \subseteq A$, $B_1, \dots, B_r \subseteq B$ and $|\bigcup_{i\in[r]} A_i \times B_i| \leq \varepsilon|A||B|$ (cf. Definition 4.13 with $k = 2$ and $H^{(1)} = (A, B)$).

We recall that the proof of Szemerédi's regularity lemma [Sze78] is based on a procedure in which, having an initial partition $\mathscr{P}_0^{(1)}$, one constructs a sequence $\mathscr{P}_0^{(1)}, \mathscr{P}_1^{(1)}, \dots$ of partitions. To each partition a quantity (called *index*) is associated which is known to satisfy $\mathrm{ind}(\mathscr{P}^{(1)}) \leq 1$ for every vertex partition $\mathscr{P}^{(1)}$. On the other hand, one proves that if $\mathscr{P}_i^{(1)}$ is irregular, then

$$\mathrm{ind}(\mathscr{P}_{i+1}^{(1)}) \geq \mathrm{ind}(\mathscr{P}_i^{(1)}) + \frac{\delta_2^4}{10}.$$

Consequently, one infers that after at most $10/\delta_2^4$ iterations one arrives to a partition which is δ_2-regular.

While in [Sze78], if $\mathscr{P}_i^{(1)}$ was partition into $a_1^{\mathscr{P}_i}$ parts implied that $\mathscr{P}_{i+1}^{(1)}$ is a partition into at most $4^{a_1^{\mathscr{P}_i}}$ parts, in our proof (due to the fact that the witness has $r(a_1^{\mathscr{P}_i})$ parts for each pair) we may have as many as $4^{r(a_1^{\mathscr{P}_i}) \times a_1^{\mathscr{P}_i}}$ partition classes in $\mathscr{P}_{i+1}^{(1)}$. Consequently, t_{RL} (which is an upper bound for the number of classes in the final partition) depends not only on δ_2, but also on the function $r(\cdot)$. It is independent, however, of the cardinality of the vertex set V.

(*3*) In order to avoid the exceptional class V_0 we assume that the cardinality of V is divisible by $(t_{\mathrm{RL}})!$. This allows us to redistribute all the vertices in V_i which would remain from subdividing the witnesses. Such a lemma was considered, e.g., in [RS04].

4.4 Proof of: $\mathrm{RL}(k) \Longrightarrow \mathrm{RAL}(k)$

In order to simplify the presentation we break the proof into two parts. In the first part we deduce RAL(k) from RL(k) and the following lemma.

Lemma 4.38. *For every positive integer s, all positive reals ν and ε, and every vector $\boldsymbol{d} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \le i \le k-1$, there exist positive reals $\delta_{4.38}$ and $\xi_{4.38}$ and integers $t_{4.38}$ and $m_{4.38}$ such that the following holds.*

Suppose

(*a*) *$m \ge m_{4.38}$ and $(t_{4.38})!$ divides m,*

(*b*) *$\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{4.38}, \boldsymbol{d})$-regular $(m, k, k-1)$-complex,*

(*c*) *$F^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ is $\xi_{4.38}$-regular w.r.t. $R^{(k-1)}$, and*

(*d*) *$\{H_1^{(k)}, \dots, H_s^{(k)}\}$ is a partition of $F^{(k)}$, where $H_i^{(k)}$ is $(\nu/12, *, t_{4.38}^{2^k})$-regular with respect to $R^{(k-1)}$ for every $i \in [s]$.*

Then there exists a partition $\{G_1^{(k)}, \dots, G_s^{(k)}\}$ of $F^{(k)}$ so that for every $i \in [s]$ the following holds

(*i*) *$G_i^{(k)}$ is $(\varepsilon, d(H_i^{(k)} | R^{(k-1)}))$-regular w.r.t. $R^{(k-1)}$ and*

(*ii*) *$|G_i^{(k)} \triangle H_i^{(k)}| \le \nu |\mathcal{K}_k(R^{(k-1)})|$.*

In Section 4.4.1 we derive RAL(k) from Lemma 4.38 and RL(k), then, in Section 4.4.2, we give the proof of Lemma 4.38 which is based on another application of RL(k).

4.4.1 Lemma 4.38 and $\mathrm{RL}(k)$ imply $\mathrm{RAL}(k)$

The idea of this reduction is as follows. Let $\mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ and $\mathscr{H}^{(k)}$ be given by RAL(k). We apply RL(k) to $\mathscr{O}(k-1) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$ and $\mathscr{H}^{(k)}$. The constants will be chosen in such a way that after that application of RL(k) a "typical" polyad $\hat{P}^{(k-1)}$ with its underlying complex $\boldsymbol{P} = \{\hat{P}^{(j)}\}_{j=1}^{k-1}$ matches the assumptions of Lemma 4.38 for $\boldsymbol{R} = \boldsymbol{P}$, $F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{k-1})$ (where $O^{(k)} \in \mathscr{O}^{(k)}$), and

$$\{\widetilde{H}_h^{(k)} = H_h^{(k)} \cap F^{(k)} \colon \ H_h^{(k)} \in \mathscr{H}^{(k)} \text{ and } H_h^{(k)} \subseteq O^{(k)}\}\,.$$

Lemma 4.38 then yields hypergraphs $\widetilde{G}_h^{(k)}$ satisfying (*i*) and (*ii*) of Lemma 4.38. Repeating this for all "typical" polyads $\hat{P}^{(k-1)}$ and $O^{(k)} \in \mathscr{O}^{(k)}$ and taking appropriate care of the "untypical" case, then yields the promised hypergraphs $G_1^{(k)} \dots G_s^{(k)}$ with properties (*G1*)–(*G3*) of RAL(k). We give the technical details of this outline below.

Proof: RL(k) $\wedge$ Lemma 4.38$\Longrightarrow$ RAL(k). Let constants o_{RAL}, s_{RAL}, η_{RAL}, and ν_{RAL}, and a function $\varepsilon_{\mathrm{RAL}} \colon \mathbb{N}^{k-1} \to (0,1]$ be given (w.l.o.g. we may assume that $\varepsilon_{\mathrm{RAL}}$ is monotone in every coordinate). We have to determine μ_{RAL}, t_{RAL}, and n_{RAL} (see (4.25)).

Our proof relies on an application of RL(k) followed by an application of Lemma 4.38. In order to match the assumptions of Lemma 4.38 the parameters for the application of RL(k) have to match these assumptions. Consequently, "constant-wise" we first apply Lemma 4.38 to foresee what is needed for its application, which will be provided by RL(k). With this in mind we set

$$s_{4.38} = s_{\mathrm{RAL}}\,, \qquad \nu_{4.38} = \frac{\nu_{\mathrm{RAL}}}{2}\,. \tag{4.19}$$

For every choice of ε and $d_1, \dots, d_{k-1}$ (satisfying $1/d_i \in \mathbb{N}$), Lemma 4.38 yields constants $\delta_{4.38}$, $\xi_{4.38}$, $t_{4.38}$, and $m_{4.38}$. Accordingly, we define functions δ_{aux}, $\xi_{\mathrm{aux}}\colon \mathbb{N}^{k-1} \to (0,1]$ and t_{aux}, $m_{\mathrm{aux}}\colon \mathbb{N}^{k-1} \to \mathbb{N}$ mapping any given $\boldsymbol{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$ to the corresponding constant from Lemma 4.38 with $\varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a})$ and $d_2 = 1/a_2, \dots, d_{k-1} = 1/a_{k-1}$. More precisely, we set for $x \in \{\delta, \xi, t, m\}$

$$x_{aux}(\boldsymbol{a}) = x_{L.4.38}(s = s_{4.38}, \nu = \nu_{4.38}, \varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}), d_2 = \tfrac{1}{a_2}, \dots, d_{k-1} = \tfrac{1}{a_{k-1}}) \tag{4.20}$$

where $x_{L.4.38}(s, \nu, \varepsilon, d_2, \dots, d_{k-1})$ is given by Lemma 4.38 applied with constants s, ν, ε, and $d_2, \dots, d_{k-1}$. Without loss of generality we assume that the functions defined in (4.20) are monotone in every coordinate.

We now choose the parameters for the application of RL(k). For that we set

$$o_{\mathrm{RL}} = o_{\mathrm{RAL}}\,, \quad s_{\mathrm{RL}} = s_{\mathrm{RAL}}\,, \quad \eta_{\mathrm{RL}} = \eta_{\mathrm{RAL}}\,, \quad \text{and} \quad \delta_{k,\mathrm{RL}} = \min\left\{\frac{\nu_{4.38}}{12}, \frac{\nu_{\mathrm{RAL}}}{2 s_{\mathrm{RAL}}}\right\} \tag{4.21}$$

and consider functions $r_{\mathrm{RL}}\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta_{\mathrm{RL}}\colon \mathbb{N}^{k-1} \to (0,1]$ defined for every integer vector $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ by

$$r_{\mathrm{RL}}(\boldsymbol{a}) = \big(t_{\mathrm{aux}}(\boldsymbol{a})\big)^{2^k} \quad \text{and} \tag{4.22}$$

$$\delta_{\mathrm{RL}}(\boldsymbol{a}) = \min\Big\{\varepsilon_{\mathrm{RAL}}(\boldsymbol{a})\,,\ \delta_{\mathrm{aux}}(\boldsymbol{a})\,,\ \varepsilon_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = \tfrac{1}{2}, d_0 = \min_{2 \le i < k} a_i^{-1})\Big\}, \tag{4.23}$$

where $\varepsilon_{\mathrm{DCL}}(h, \ell, \gamma, d_0)$ is given by Theorem 4.19.

Having defined all parameters for RL(k), Lemma 4.34, in (4.21), (4.22) and (4.23), Lemma 4.34 now yields positive constants μ_{RL}, t_{RL}, and n_{RL}. We use t_{RL} to establish "worst case" estimates on the functions ξ_{aux}, t_{aux}, and m_{aux} and set

$$\xi_{\mathrm{worst}} = \xi_{\mathrm{aux}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}})\,,\ t_{\mathrm{worst}} = t_{\mathrm{aux}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}})\,,\ \text{and} \quad m_{\mathrm{worst}} = m_{\mathrm{aux}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}}) \tag{4.24}$$

Finally, we define μ_{RAL}, t_{RAL}, and n_{RAL} promised by RAL(k). For that we set

$$\begin{aligned}
\mu_{\mathrm{RAL}} &= \min\left\{\mu_{\mathrm{RL}}\,, \frac{\varepsilon_{\mathrm{RAL}}(t_{\mathrm{RL}},\dots,t_{\mathrm{RL}})}{2t_{\mathrm{RL}}^{2^k}}\,, \frac{\xi_{\mathrm{worst}}}{2t_{\mathrm{RL}}^{2^k}}\right\}\,,\\
t_{\mathrm{RAL}} &= t_{\mathrm{RL}} + t_{\mathrm{worst}}\,,\\
\text{and}&\\
n_{\mathrm{RAL}} &= \max\Big\{n_{\mathrm{RL}}\,, t_{\mathrm{RL}}m_{\mathrm{worst}}\,,\\
&\qquad t_{\mathrm{RL}}m_{\mathrm{DCL}}(h=k-1, \ell=k, \gamma=\tfrac{1}{2}, d_0=t_{\mathrm{RL}}^{-1})\Big\}\,.
\end{aligned} \tag{4.25}$$

Note that for given input parameters o_{RAL}, s_{RAL}, η_{RAL}, and ν_{RAL}, and a given function $\varepsilon_{\mathrm{RAL}}\colon \mathbb{N}^{k-1} \to (0,1]$ of RAL(k), finally in (4.25) we defined the corresponding output parameters. Now we need to show, that with this choice we will be able to verify RAL(k), Lemma 4.36.

Let V, $\mathscr{O}_{\mathrm{RAL}}$, and $\mathscr{H}^{(k)}$ satisfying (a)–(c) of RAL(k), Lemma 4.36 be given, i.e.,

(RAL.a) $|V| = n \geq n_{\mathrm{RAL}}$ and $(t_{\mathrm{RAL}})!$ divides n,

(RAL.b) $\mathscr{O}_{\mathrm{RAL}} = \mathscr{O}_{\mathrm{RAL}}(k, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RAL}}}) = \{\mathscr{O}_{\mathrm{RAL}}^{(j)}\}_{j=1}^{k}$ is $(\eta^{\mathscr{O}_{\mathrm{RAL}}}, \mu_{\mathrm{RAL}}, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RAL}}})$-equitable (for some $\eta^{\mathscr{O}_{\mathrm{RAL}}} > 0$) and o_{RAL}-bounded, and

(RAL.c) $|\mathscr{H}^{(k)}| = s_{\mathrm{RAL}}$ and $\mathscr{H}^{(k)} \prec \mathscr{O}_{\mathrm{RAL}}^{(k)}$.

Our objective is to find a family of partitions $\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RAL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$ on V and a partition $\mathscr{G}^{(k)} = \{G_1^{(k)}, \dots, G_{s_{\mathrm{RAL}}}^{(k)}\}$ of $\binom{V}{k}$ so that

(RAL.$P1$) $\mathscr{P}_{\mathrm{RAL}}$ is $(\eta_{\mathrm{RAL}}, \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$-equitable, t_{RAL}-bounded,

(RAL.$P2$) $\mathscr{P}_{\mathrm{RAL}} \prec \mathscr{O}_{\mathrm{RAL}}(k-1) = \{\mathscr{O}_{\mathrm{RAL}}^{(j)}\}_{j=1}^{k-1}$,

(RAL.$G1$) $G_i^{(k)}$ is perfectly $\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$-regular w.r.t. $\mathscr{P}_{\mathrm{RAL}}$ for every $i \in [s_{\mathrm{RAL}}]$,

(RAL.$G2$) $|G_i^{(k)} \triangle H_i^{(k)}| \leq \nu_{\mathrm{RAL}} n^k$ for every $i \in [s_{\mathrm{RAL}}]$, and

(RAL.$G3$) if $H_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)})$ then $G_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)})$ for every $i \in [s_{\mathrm{RAL}}]$ and $\mathscr{G}^{(k)} \prec \mathscr{O}_{\mathrm{RAL}}^{(k)}$.

Without loss of generality we may assume that

$$H_i^{(k)} \neq \varnothing \quad \text{for every} \quad i \in [s_{\mathrm{RAL}}]\,. \tag{4.26}$$

Otherwise we simply set $G_i^{(k)} = \varnothing$ for every $i \in [s_{\mathrm{RAL}}]$ for which $H_i^{(k)} = \varnothing$ and obviously (RAL.$G1$)–(RAL.$G3$) holds for those $G_i^{(k)}$ for any family of partitions $\mathscr{P}$.

As we already mentioned we are going to apply RL(k) to V, to the family of partitions $\mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RAL}}(k-1) = \{\mathscr{O}_{\mathrm{RAL}}^{(j)}\}_{j=1}^{k-1}$, to the vector $\boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}} = (a_1^{\mathscr{O}_{\mathrm{RAL}}}, \dots, a_{k-1}^{\mathscr{O}_{\mathrm{RAL}}})$, and to the partition $\mathscr{H}^{(k)}$ with constants o_{RL}, s_{RL}, η_{RL}, and $\delta_{k,\mathrm{RL}}$ defined in (4.21) and functions r_{RL} and δ_{RL} defined in (4.22) and (4.23). For that we have to verify that

(RL.a) $|V| = n \geq n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides n,

(RL.b) $\mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}}) = \{\mathscr{O}_{\mathrm{RL}}^{(j)}\}_{j=1}^{k-1}$ is $(\eta^{\mathscr{O}_{\mathrm{RL}}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}})$-equitable (for some $\eta^{\mathscr{O}_{\mathrm{RL}}} > 0$) and o_{RL}-bounded, and

(RL.c) $|\mathscr{H}^{(k)}| = s_{\mathrm{RL}}$.

We note that (RL.a) is an easy consequence of the choice of $n_{\mathrm{RAL}} \geq n_{\mathrm{RL}}$ and $t_{\mathrm{RAL}} \geq t_{\mathrm{RL}}$ in (4.25) and (RAL.a). Similarly, (RL.b) follows from the choice of $\mu_{\mathrm{RAL}} \leq \mu_{\mathrm{RL}}$ in (4.25) and (RAL.b), while (RL.c) is a consequence of (RAL.c) and the choice of $s_{\mathrm{RL}} = s_{\mathrm{RAL}}$ in (4.21). Having verified that (RL.a)–(RL.c) hold, we reason that there is a family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ on V which satisfies properties (*P1*), (*P2*), and (*H*) of Lemma 4.34

(RL.*P1*) $\mathscr{P}_{\mathrm{RL}}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-equitable and t_{RL}-bounded,

(RL.*P2*) $\mathscr{P}_{\mathrm{RL}} \prec \mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RAL}}(k-1)$, and

(RL.*H*) $H_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathscr{P}_{\mathrm{RL}}$ for every $i \in [s_{\mathrm{RL}}]$.

We set

$$\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RL}} \quad \text{and} \quad \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}} = \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}\,. \tag{4.27}$$

It then follows from (RL.*P1*) and (RL.*P2*) and the choices of $\eta_{\mathrm{RL}} = \eta_{\mathrm{RAL}}$ in (4.21), $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leq \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ in (4.23), and $t_{\mathrm{RAL}} \geq t_{\mathrm{RL}}$ in (4.25), that

$$\mathscr{P}_{\mathrm{RAL}} \text{ satisfies (RAL.}P1\text{) and (RAL.}P2\text{)}\,. \tag{4.28}$$

It is left to ensure the existence of the partition $\mathscr{G}^{(k)}$ of $\binom{V}{k}$ which satisfies (RAL.*G1*)–(RAL.*G3*).

Before we prove the existence of $\mathscr{G}^{(k)}$ we make some preparations, which simplify the presentation. We complete the partition $\mathscr{O}_{\mathrm{RAL}}^{(k)}$ (which partitions $\mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)})$), to a partition of $\binom{V}{k}$. For that we set

$$\widetilde{\mathscr{O}}^{(k)} = \mathscr{O}_{\mathrm{RAL}}^{(k)} \cup \left(\binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)})\right)\,. \tag{4.29}$$

We also define for every $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$

$$I(O^{(k)}) = \{i \in [s_{\mathrm{RAL}}] \colon\ H_i^{(k)} \subseteq O^{(k)} \text{ and } H_i^{(k)} \neq \varnothing\}\,. \tag{4.30}$$

Note that due to (RAL.c), (4.26), and (4.29) the family

$$\{I(O^{(k)}) \colon\ O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}\}$$

forms a partition of $[s_{\mathrm{RAL}}]$. Before we continue we make the observation.

Claim 4.39. *For every* $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ *and* $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{RL}$ *the following holds. Set* $F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$*, then* $F^{(k)}$ *is* $(2t^{2k}_{RL}\mu_{RAL})$*-regular w.r.t.* $\hat{P}^{(k-1)}$.

Proof. The claim is trivial if $F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) = \varnothing$ and, hence, we assume that

$$F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \neq \varnothing\,. \tag{4.31}$$

We distinguish two cases. From, $\mathscr{P}_{\mathrm{RL}} \prec \mathscr{O}_{\mathrm{RAL}}(k-1)$ (cf. (RL.$P2$)) we infer that either $\hat{P}^{(k-1)}$ is contained in some polyad $\hat{O}^{(k-1)} \in \hat{\mathscr{O}}^{(k-1)}_{\mathrm{RAL}}$ or $\mathcal{K}_k(\hat{P}^{(k-1)}) \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}}) = \varnothing$. If $\mathcal{K}_k(\hat{P}^{(k-1)}) \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}}) = \varnothing$, then we have $O^{(k)} = \binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$ (using (4.31)) and, consequently, $F^{(k)} = \mathcal{K}_k(\hat{P}^{(k-1)})$. Hence, $F^{(k)}$ is ξ-regular w.r.t. $\hat{P}^{(k-1)}$ for every $\xi > 0$ which yields the claim in that case.

On the other hand, if $\hat{P}^{(k-1)} \subseteq \hat{O}^{(k-1)}$ for some $\hat{O}^{(k-1)} \in \hat{\mathscr{O}}^{(k-1)}_{\mathrm{RAL}}$, then we have due to (4.31) and the fact that $\mathscr{O}_{\mathrm{RAL}}$ is a family of partitions (cf. Definition 4.8) that $O^{(k)} \subseteq \mathcal{K}_k(\hat{O}^{(k-1)})$. Therefore, (RAL.$b$) and Proposition 4.29 (applied with $j = k$, m, $r = 1$, $\delta = \mu_{\mathrm{RAL}}$, $d = d(O^{(k)}|\hat{O}^{(k-1)})$, $P^{(k)} = O^{(k)}$, $\hat{P}^{(k-1)} = \hat{O}^{(k-1)}$, and $\hat{Q}^{(k-1)} = \hat{P}^{(k-1)}$) imply that

$$F^{(k)} \text{ is } \left(\mu_{\mathrm{RAL}} \frac{|\mathcal{K}_k(\hat{O}^{(k-1)})|}{|\mathcal{K}_k(\hat{P}^{(k-1)})|}\right)\text{-regular w.r.t. } \hat{P}^{(k-1)}\,. \tag{4.32}$$

Clearly, $|\mathcal{K}_k(\hat{O}^{(k-1)})| \leq n^k$ and due to the choice of $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leq \varepsilon_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = \min_{2 \leq i < k} 1/a_i^{\mathscr{P}_{\mathrm{RL}}})$ in (4.23), the appropriate choice of $n_{\mathrm{RAL}} \geq t_{\mathrm{RL}} \times m_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = t^{-1}_{\mathrm{RL}})$ in (4.25), and (RL.$P1$), by Theorem 4.19, we infer

$$\left|\mathcal{K}_k(\hat{P}^{(k-1)})\right| \geq \frac{1}{2} \prod_{j=2}^{k-1} \left(\frac{1}{a_j^{\mathscr{P}_{\mathrm{RL}}}}\right)^{\binom{k}{j}} \times \left(\frac{n}{a_1^{\mathscr{P}_{\mathrm{RL}}}}\right)^k \geq \frac{n^k}{2t^{2k}_{\mathrm{RL}}}\,.$$

and the claim follows. □

We now continue with the proof of the existence of the partition $\mathscr{G}^{(k)}$ of $\binom{V}{k}$ which satisfies (RAL.$G1$)–(RAL.$G3$). For that we will mainly use Lemma 4.38 applied to the polyads of $\mathscr{P}_{\mathrm{RL}}$. However, we distinguish between two types of polyads and set

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} = \Big\{\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}} \colon\ H_i^{(k)} \text{ is } (\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))\text{-regular}$$
$$\text{w.r.t. } \hat{P}^{(k-1)} \text{ for every } i \in [s_{\mathrm{RL}}]\Big\}\,.$$

Case 1 ($\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$)**.** In this case let $K \in \mathcal{K}_k(\hat{P}^{(k-1)})$ and set $\boldsymbol{R} = \boldsymbol{P}(K) = \{\hat{P}^{(j)}(K)\}_{j=1}^{k-1}$ with $\hat{P}^{(k-1)}(K) = \hat{P}^{(k-1)}$ (see (4.1)). Let $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ be such that

$$F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \neq \varnothing\,, \tag{4.33}$$

and set

$$\widetilde{H}_i^{(k)} = H_i^{(k)} \cap F^{(k)} = H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \qquad \text{for} \quad i \in I(O^{(k)})\,. \tag{4.34}$$

We want to apply Lemma 4.38 with parameters $s = s_{4.38}$, $\nu = \nu_{4.38}$, $\varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$, and $d_i = 1/a_i^{\mathscr{P}_{\mathrm{RL}}}$ for $2 \le i < k$. Note that due to the definition of the functions δ_{aux}, ξ_{aux}, t_{aux}, and m_{aux} in view of (4.20), Lemma 4.38 yields constants $\delta_{4.38}$, $\xi_{4.38}$, $t_{4.38}$ and $m_{4.38}$ which satisfy

$$\delta_{4.38} = \delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}),$$
$$\xi_{4.38} = \xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}),$$
$$t_{4.38} = t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}),$$

and

$$m_{4.38} = m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}).$$

In order to apply Lemma 4.38 with the chosen parameters to $m = n/a_1^{\mathscr{P}_{\mathrm{RL}}}$, $\boldsymbol{R}$, $F^{(k)}$, and $\{\widetilde{H}_i^{(k)} \colon i \in I(O^{(k)})\}$ we have to verify

(L.4.38.a) $n/a_1^{\mathscr{P}_{\mathrm{RL}}} = m \ge m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ and $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides m,

(L.4.38.b) $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{d})$-regular $(m, k, k-1)$-complex for the density vector $\boldsymbol{d} = (1/a_2^{\mathscr{P}_{\mathrm{RL}}}, \dots, 1/a_{k-1}^{\mathscr{P}_{\mathrm{RL}}})$,

(L.4.38.c) $F^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ is $\xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $R^{(k-1)}$, and

(L.4.38.d) $\{\widetilde{H}_i^{(k)} \colon i \in I(O^{(k)})\}$ partitions $F^{(k)}$, we have $|I(O^{(k)})| \le s_{4.38}$, and $\widetilde{H}_i^{(k)}$ is $(\nu_{4.38}/12, *, (t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))^{2^k})$-regular w.r.t. $R^{(k-1)}$ for every $i \in I(O^{(k)})$.

The verification of (L.4.38.a)–(L.4.38.d) is straightforward, but somewhat technical. We give the details below.

Due to (RAL.a), (4.25), (4.24), (RL.b) and the monotonicity of the function m_{aux} we have

$$n \ge t_{\mathrm{RL}} \times m_{\mathrm{worst}} \ge a_1^{\mathscr{P}_{\mathrm{RL}}} \times m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}).$$

In order to verify (L.4.38.a) it is left to show that $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides $m = n/a_1^{\mathscr{P}_{\mathrm{RL}}}$. For that we note that due to the definition of t_{RAL} in (4.25) we have $t_{\mathrm{RAL}} = t_{\mathrm{RL}} + t_{\mathrm{worst}}$, which due to (RAL.$a$) yields $(t_{\mathrm{RL}} + t_{\mathrm{worst}})!$ divides n. Consequently, $(t_{\mathrm{RL}})!(t_{\mathrm{worst}})!$ divides n (to see this consider $\binom{t_{\mathrm{RL}}+t_{\mathrm{worst}}}{t_{\mathrm{worst}}}$). Hence, from $a_1^{\mathscr{P}_{\mathrm{RL}}} \le t_{\mathrm{RL}}$ (cf. (RL.b)) it follows that $n/a_1^{\mathscr{P}_{\mathrm{RL}}} = m$ is divisible by $(t_{\mathrm{worst}})!$. It now follows that $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides m since $t_{\mathrm{worst}} \ge t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ due to the monotonicity of the function t_{aux}.

Part (L.4.38.b) follows easily from (RL.b) and the choice of the function δ_{RL} in (4.23) ensuring that $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \le \delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$.

Next we verify (L.4.38.c). It follows from the definition of $F^{(k)}$ that $R^{(k-1)} = \hat{P}^{(k-1)}$ underlies $F^{(k)}$. The second assertion of (L.4.38.c) follows from

$$2t_{\mathrm{RL}}^{2^k}\mu_{\mathrm{RAL}} \le \xi_{\mathrm{worst}} \le \xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$$

(cf. (4.25) and (4.24)) and Claim 4.39.

Finally, it is left to verify (L.4.38.d). It follows from the definitions in (4.30) and (4.34) and the fact that $\mathscr{H}^{(k)}$ is a partition of $\binom{V}{k}$ (cf. (RAL.d)), that $\{\widetilde{H}_i^{(k)}\colon i \in I(O^{(k)})\}$ partitions $F^{(k)}$. Clearly, $|I(O^{(k)})| \leq s_{4.38}$. Moreover, from the assumption of this case ($R^{(k-1)} = \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$) we know that $\widetilde{H}_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $R^{(k-1)} = \hat{P}^{(k-1)}$ for every $i \in I(O^{(k)})$. Therefore, (L.4.38.d) follows from the choice of $\delta_{k,\mathrm{RL}} \leq \nu_{4.38}/12$ in (4.21) and $r_{\mathrm{RL}}(a^{\mathscr{P}_{\mathrm{RL}}}) = (t_{\mathrm{aux}}(a^{\mathscr{P}_{\mathrm{RL}}}))^{2^k}$ in (4.22).

Having verified (L.4.38.a)–(L.4.38.d), we can apply Lemma 4.38 and infer the existence of a partition $\{\widetilde{G}_i^{(k)}\colon i \in I(O^{(k)})\}$ of $F^{(k)}$ so that for every $i \in I(O^{(k)})$

(L.4.38.i) $\widetilde{G}_i^{(k)}$ is $\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\hat{P}^{(k-1)} = R^{(k-1)}$, and

(L.4.38.ii) $|\widetilde{G}_i^{(k)} \triangle \widetilde{H}_i^{(k)}| \leq \nu_{4.38}|\mathcal{K}_k(\hat{P}^{(k-1)})|$.

For $i \in I(O^{(k)})$ each $\widetilde{G}_i^{(k)}$ given above defines $G_i^{(k)}$ restricted to the polyad $\hat{P}^{(k-1)}$. Formally we set

$$G_i^{(k)}(\hat{P}^{(k-1)}) = \widetilde{G}_i^{(k)} \quad \text{for} \quad i \in I(O^{(k)})\,, \tag{4.35}$$

and repeat the procedure for every $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ satisfying (4.33). ♢

Case 2 ($\hat{P}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$)**.** Let $K \in \mathcal{K}_k(\hat{P}^{(k-1)})$ and set $\boldsymbol{P} = \boldsymbol{P}(K) = \{\hat{P}^{(j)}(K)\}_{j=1}^{k-1}$ with $\hat{P}^{(k-1)}(K) = \hat{P}^{(k-1)}$ (see (4.1)). Let $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ be such that

$$F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \neq \varnothing\,. \tag{4.36}$$

In this case fix some index $i_0 \in I(O^{(k)})$. We then define for $i \in I(O^{(k)})$

$$G_i^{(k)}(\hat{P}^{(k-1)}) = \begin{cases} F^{(k)} & \text{for} \quad i = i_0\,, \\ \varnothing & \text{for} \quad i \neq i_0 \in I(O^{(k)})\,. \end{cases} \tag{4.37}$$

For later reference we note that for every $i \in I(O^{(k)})$

$$G_i^{(k)}(\hat{P}^{(k-1)}) \subseteq O^{(k)} \tag{4.38}$$

and

$$G_i^{(k)}(\hat{P}^{(k-1)}) \text{ is } \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\text{-regular w.r.t. } \hat{P}^{(k-1)}\,. \tag{4.39}$$

Indeed, (4.38) is trivial for every $i \in I(O^{(k)})$ and (4.39) is trivial for $i \neq i_0$. In the case $i = i_0$ we have $G_i^{(k)}(\hat{P}^{(k-1)}) = F^{(k)} = O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ and (4.39) follows from Claim 4.39 and the choice of μ_{RAL} in (4.25) ensuring $2t_{\mathrm{RL}}^{2^k} \times \mu_{\mathrm{RAL}} \leq \varepsilon_{\mathrm{RAL}}(t_{\mathrm{RL}}, \ldots, t_{\mathrm{RL}}) \leq \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$.

Again we repeat this procedure for every $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ satisfying (4.36). ♢

We note that due to the both cases above the following statement holds:

($*$) For every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ and every $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ satisfying $O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \neq \varnothing$ we have that $\{G_i^{(k)}(\hat{P}^{(k-1)})\colon i \in I(O^{(k)})\}$ is a partition of $O^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$.

Now we define the partition $\mathscr{G}^{(k)}$ and verify (RAL.*G1*)–(RAL.*G3*). For that we set for $i \in [s_{\mathrm{RAL}}]$

$$G_i^{(k)} = \bigcup \Big\{ G_i^{(k)}(\hat{P}^{(k-1)}) \colon \; \hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)} \Big\} . \tag{4.40}$$

Since $\widetilde{\mathscr{O}}^{(k)}$ is a partition of $\binom{V}{k}$ we infer from $(*)$ that $\mathscr{G}^{(k)} = \{G_1^{(k)}, \dots, G_{s_{\mathrm{RAL}}}^{(k)}\}$ forms a partition of $\binom{V}{k}$.

We verify property (RAL.*G1*). From (L.4.38.*i*) (combined with (4.35)) and (4.39) we conclude that for all $i \in [s_{\mathrm{RAL}}]$ and all $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$ the defined $G_i^{(k)}$ is $\varepsilon_{\mathrm{RAL}}(a^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\hat{P}^{(k-1)}$. Consequently, the definition of $\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RL}}$ and $\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}} = \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}$ in (4.27) yields (RAL.*G1*).

In order to show (RAL.*G2*), let $i \in [s_{\mathrm{RAL}}]$ be fixed. It then follows from (L.4.38.*ii*) that

$$\sum \Big\{ |(G_i^{(k)} \triangle H_i^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)})| \colon \; \hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL},\mathscr{H}\text{-reg}}^{(k-1)} \Big\} \overset{(4.19)}{\leq} \frac{\nu_{\mathrm{RAL}} n^k}{2} \tag{4.41}$$

Moreover, from (RL.*H*) and Definition 4.13 we infer

$$\begin{aligned} \sum \Big\{ |(G_i^{(k)} \triangle H_i^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)})| \colon \; \hat{P}^{(k-1)} \notin \hat{\mathscr{P}}_{\mathrm{RL},\mathscr{H}\text{-reg}}^{(k-1)} \Big\} \\ \leq \sum \Big\{ |\mathcal{K}_k(\hat{P}^{(k-1)})| \colon \; \hat{P}^{(k-1)} \notin \hat{\mathscr{P}}_{\mathrm{RL},\mathscr{H}\text{-reg}}^{(k-1)} \Big\} \\ \leq s_{\mathrm{RL}} \delta_{k,\mathrm{RL}} n^k \overset{(4.21)}{\leq} \frac{1}{2} \nu_{\mathrm{RAL}} n^k \end{aligned} \tag{4.42}$$

In view of (4.27) the inequalities (4.41) and (4.42) then yield (RAL.*G2*).

Finally, we consider (RAL.*G3*). For that for each $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ we set

$$J(O^{(k)}) = \{ i \in [s_{\mathrm{RAL}}] \colon \; G_i^{(k)} \cap O^{(k)} \neq \varnothing \} .$$

Since (4.26) and $\mathscr{G}^{(k)}$ is a partition of $\binom{V}{k}$, the two assertions in (RAL.*G3*) are implied by the following two statements which we verify below

$$J\left(\binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)}) \right) \subseteq I\left(\binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)}) \right) , \tag{4.43}$$

and

$$J(O_1^{(k)}) \cap J(O_2^{(k)}) = \varnothing \quad \text{for all} \quad O_1^{(k)} \neq O_2^{(k)} \in \widetilde{\mathscr{O}}^{(k)} . \tag{4.44}$$

From $(*)$ we infer for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$ that if $G_i^{(k)}(\hat{P}^{(k-1)}) \cap O^{(k)} \neq \varnothing$ then $G_i^{(k)}(\hat{P}^{(k-1)}) \subseteq O^{(k)}$. Consequently, $(*)$ yields

$$J(O^{(k)}) \subseteq I(O^{(k)}) \tag{4.45}$$

for every $O^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$, which gives (4.43).

Moreover, since $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}$ and, therefore, $\mathscr{H}^{(k)} \prec \widetilde{\mathscr{O}}^{(k)}$ (see (4.29), we have $I(O_1^{(k)}) \cap I(O_2^{(k)}) = \varnothing$ for all distinct $O_1^{(k)}$ and $O_2^{(k)}$ from $\widetilde{\mathscr{O}}^{(k)}$. Hence, (4.44) holds as well, and consequently (RAL.*G3*) follows.

From the discussion above and (4.28) we infer that $\mathscr{P}_{\mathrm{RAL}}$ defined in (4.27) and $\mathscr{G}^{(k)}$ defined in (4.40) satisfy the conclusion of RAL(k), Lemma 4.36, i.e., (RAL.*P1*)–(RAL.*G3*). □

4.4.2 RL(k) implies Lemma 4.38

The proof of Lemma 4.38 is the heart of the implication RL(k) $\Longrightarrow$ RAL(k) and its idea resembles the main idea from the work of Nagle, Rödl, and Schacht in [NRS06a]. Before we give with the detailed proof below, we briefly discuss the main idea.

Recall that in Lemma 4.38 a $(\delta_{4.38}, \boldsymbol{d})$-regular $(m, k, k-1)$-complex $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ and a ξ-regular k-uniform hypergraph $F^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ are given. Moreover, we are given a partition $\mathscr{H}^{(k)} = \{H_i^{(k)} \colon i \in [s_{4.38}]\}$ of $F^{(k)}$, where every $H_i^{(k)}$ is $(\nu, *, t_{4.38}^{2^k})$-regular w.r.t. $R^{(k-1)}$. We will apply RL(k) to regularize every $H_i^{(k)} \in \mathscr{H}^{(k)}$ with some appropriately chosen δ_k less than the given ε. For this regularization we apply the variant of RL(k) discussed in Remark 4.35, which allows us to find a t_{RL}-bounded family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) = \{\mathscr{P}_{\mathrm{RL}}^{(j)}\}_{j=1}^{k-1}$ such that for each $j = 1, \dots, k-1$ and each $P^{(j)} \in \mathscr{P}_{\mathrm{RL}}^{(j)}$ either $P^{(j)} \subseteq R^{(j)}$ or $P^{(j)} \cap R^{(j)} = \varnothing$. Since each $H_i^{(k)} \subseteq F^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$, we will focus on the "interesting" part of the partition $\mathscr{P}_{\mathrm{RL}}$ and consider only those polyads $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$ which are subsets of $R^{(k-1)}$. For that we set

$$\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R}) = \left\{\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)} \colon \hat{P}^{(k-1)} \subseteq R^{(k-1)}\right\}.$$

From RL(k) we infer that for every "typical" $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$

(*i*) $H_i^{(k)}$ is $(\delta_k, d(H_i^{(k)}|\hat{P}^{(k-1)}), 1)$-regular w.r.t. $\hat{P}^{(k-1)}$ for every $i \in [s_{4.38}]$.

Moreover, we will prove (cf. Claim 4.41) that for every $i \in [s_{4.38}]$ the typical density $d(H_i^{(k)}|\hat{P}^{(k-1)})$ will be "near" to the density of $H_i^{(k)}$ in $R^{(k-1)}$, i.e.,

(*ii*) $|d(H_i^{(k)}|\hat{P}^{(k-1)}) - d(H_i^{(k)}|R^{(k-1)})| \le \nu/6$ for "most" $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$.

Property (*ii*) is the key observation in the proof of Lemma 4.38. Its proof is based on our choice of $t_{4.38} \ge t_{\mathrm{RL}}$ and $t_{\mathrm{RL}}^{2^k} \ge |\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})|$. The proof of (*ii*) then is simple. Assuming that there is a constant fraction of polyads in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$ which violate (*ii*) gives rise to a witness that is $(\nu/12, *, t_{4.38}^{2^k})$-irregular w.r.t. $R^{(k-1)}$. (The choice of $t_{4.38} \ge t_{\mathrm{RL}}$ allows us to "look" into a constant fraction of polyads in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$.)

Combining, (*i*) and (*ii*) with an appropriate use of the slicing lemma, Proposition 4.33, allows us to prove that for a typical $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$, $H_i^{(k)}$ needs to be altered only slightly (in less than $\nu/6$ proportion of the number of cliques in $\hat{P}^{(k-1)}$) to become

$(\varepsilon^2/4, d(H_i^{(k)}|R^{(k-1)}))$-regular w.r.t. $\hat{P}^{(k-1)}$. In other words, the resulting graph, which we denote by $G_i^{(k)}(\hat{P}^{(k-1)})$, maintains large degree of regularity (we will choose $\delta_k \ll \varepsilon$), while its density will be $\sim d(H_i^{(k)}|R^{(k-1)})$.

On the other hand in the rare case of an atypical polyad $\hat{P}^{(k-1)}$ for which (i) or (ii) does not hold for $H_i^{(k)}$ we use slicing lemma to replace $H_i^{(k)}$ by a randomly chosen (and therefore extremely regular) $G_i^{(k)}(\hat{P}^{(k-1)})$, with

$$d(G_i^{(k)}(\hat{P}^{(k-1)})|\hat{P}^{(k-1)}) \sim d(H_i^{(k)}|R^{(k-1)}) .$$

For each $i \in [s_{4.38}]$ we set

$$G_i^{(k)} = \bigcup G_i^{(k)}(\hat{P}^{(k-1)})$$

where the union is taken over all (typical and atypical) $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$. Since, $G_i^{(k)}$ obtained that way is $(\varepsilon^2/4, d(H_i^{(k)}|R^{(k-1)}))$-regular for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$, Proposition 4.30 then yields that $G_i^{(k)}$ is $(\varepsilon, d(H_i^{(k)}|R^{(k-1)})$-regular w.r.t. $R^{(k-1)}$. Since in the typical case we changed $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ only "slightly" to become $G_i^{(k)}(\hat{P}^{(k-1)})$ and since the atypical case, in which more drastic changes are needed, happens rarely, we will be able to prove that $|G_i^{(k)} \triangle H_i^{(k)}| \le \nu n^k$.

We now give the technical details of the proof of Lemma 4.38, sketched above.

Proof: RL(k) $\Longrightarrow$ Lemma 4.38. Let positive reals $s_{4.38}$, $\nu_{4.38}$, and $\varepsilon_{4.38}$ and a vector $\boldsymbol{d}_{4.38} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \le i < k$ be given. Lemma 4.38 is trivial for $\nu_{4.38} > 1$. Moreover, without loss of generality we may assume that

$$\varepsilon_{4.38} < \nu_{4.38} \le 1 . \tag{4.46}$$

We will apply RL(k). For that we set[2]

$$o_{\mathrm{RL}} = \max_{2 \le i < k} 1/d_i , \qquad s_{\mathrm{RL}} = s_{4.38} + 1 , \qquad \eta_{\mathrm{RL}} = 10^{-2} , \tag{4.47}$$

and

$$\delta_{k,\mathrm{RL}} = \min \left\{ \frac{\nu_{4.38} \prod_{h=2}^{k-1} d_h^{\binom{k}{h}}}{6 \times s_{4.38} \times k^k} , \frac{\varepsilon_{4.38}^2}{384 s_{4.38}}, \frac{\nu_{4.38}}{18} \right\} \tag{4.48}$$

and consider functions $r_{\mathrm{RL}} \colon \mathbb{N}^{k-1} \to \mathbb{N}$, and $\delta_{\mathrm{RL}} \colon \mathbb{N}^{k-1} \to (0,1]$ defined for every $\boldsymbol{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$ by

$$r_{\mathrm{RL}}(\boldsymbol{a}) = 1 \quad \text{and} \quad \delta_{\mathrm{RL}}(\boldsymbol{a}) = \varepsilon_{\mathrm{DCL}}\Big(h = k-1, \ell = k, \gamma = \tfrac{\nu_{4.38}}{48}, d_0 = \min_{2 \le i < k} a_i^{-1}\Big) , \tag{4.49}$$

[2] Since we later are only interested in partition classes $P^{(j)}$, which are sub-hypergraphs of the given $R^{(j)}$ (see, e.g., (4.55)), the constant η_{RL} is unessential for our proof and any positive constant value would do.

where $\varepsilon_{\mathrm{DCL}}(h,\ell,\gamma,d_0)$ is given by Theorem 4.19.

Having defined all input parameters of Lemma 4.34 in (4.47), (4.48), and (4.49), this lemma now yields positive constants μ_{RL}, t_{RL}, and n_{RL}. We then define $\delta_{4.38}$, $\xi_{4.38}$, $t_{4.38}$, and $m_{4.38}$ promised by Lemma 4.38. For that we set $t_{4.38}=t_{\mathrm{RL}}$,

$$\begin{aligned}\delta_{4.38}&=\min\left\{\tfrac{\mu_{\mathrm{RL}}}{3},\varepsilon_{\mathrm{DCL}}\Big(h=k-1,\ell=k,\gamma=\tfrac{1}{2},d_0=\min_{2\le i<k}d_i\Big)\right\},\\ \xi_{4.38}&=\frac{\varepsilon_{4.38}^2}{192t_{\mathrm{RL}}^{2^k}},\end{aligned}\tag{4.50}$$

and

$$\begin{aligned}m_{4.38}=\max\Bigg\{n_{\mathrm{RL}},\ &\exp(2t_{\mathrm{RL}}^{2^k}),\\ &m_{\mathrm{DCL}}\left(h=k-1,\ell=k,\gamma=\frac{1}{2},d_0=\min_{2\le i<k}d_i\right),\\ &t_{\mathrm{RL}}m_{\mathrm{DCL}}\left(k-1,k,\frac{\nu_{4.38}}{48},t_{\mathrm{RL}}^{-1}\right),\\ &t_{\mathrm{RL}}m_{\mathrm{SL}}\left(k,s_{4.38},1,\frac{\varepsilon_{4.38}^2}{96},\frac{\varepsilon_{4.38}^2}{16},\frac{\varepsilon_{4.38}^2}{192s_{4.38}}\right),\\ &t_{\mathrm{RL}}m_{\mathrm{SL}}\left(k,2,1,\delta_{k,\mathrm{RL}},\frac{\varepsilon_{4.38}^2}{192s_{4.38}},\frac{\varepsilon_{4.38}^2}{192s_{4.38}}\right),\\ &t_{\mathrm{RL}}m_{\mathrm{SL}}\left(k,s_{4.38},1,\frac{\varepsilon_{4.38}^2}{24},\frac{\varepsilon_{4.38}^2}{12},\frac{\varepsilon_{4.38}^2}{192s_{4.38}}\right)\Bigg\}.\end{aligned}\tag{4.51}$$

Having defined all the parameters of Lemma 4.38, now let m, $\boldsymbol{R}$, $F^{(k)}$, and $\mathscr{H}^{(k)}$ satisfying (a)–(d) of Lemma 4.38 for these parameters be given, i.e.,

(L.4.38.a) $m\ge m_{4.38}$ and $(t_{4.38})!$ divides m,

(L.4.38.b) $\boldsymbol{R}=\{R^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{4.38},\boldsymbol{d}_{4.38})$-regular $(m,k,k-1)$-complex with vertex set $V=V_1\cup\dots\cup V_k$,

(L.4.38.c) $F^{(k)}\subseteq\mathcal{K}_k(R^{(k-1)})$ is $\xi_{4.38}$-regular w.r.t. $R^{(k-1)}$, and

(L.4.38.d) the family $\mathscr{H}^{(k)}=\{H_1^{(k)},\dots,H_{s_{4.38}}^{(k)}\}$ is a partition of $F^{(k)}$ and every $H_i^{(k)}$ is $(\nu_{4.38}/12,*,t_{4.38}^{2^k})$-regular w.r.t. $R^{(k-1)}$ for $i\in[s_{4.38}]$.

We have to ensure the existence of a partition $\mathscr{G}^{(k)}=\{G_1^{(k)},\dots,G_{s_{4.38}}^{(k)}\}$ of $F^{(k)}$ so that for every $i\in[s_{4.38}]$

(L.4.38.i) $G_i^{(k)}$ is $\varepsilon_{4.38}$-regular w.r.t. $R^{(k-1)}$, and

(L.4.38.ii) $|G_i^{(k)}\triangle H_i^{(k)}|\le\nu_{4.38}|\mathcal{K}_k(R^{(k-1)})|$.

Before we start we note for later use that due to (L.4.38.b) and the choice of $\delta_{4.38} \le \varepsilon_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = \min_{2\le i<k} d_i)$ in (4.50) and $m \ge m_{4.38} \ge m_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = \min_{2\le i<k} d_i)$ in (4.51) we infer by DCL, Theorem 4.19, that

$$|\mathcal{K}_k(R^{(k-1)})| = \left(1 \pm \frac{1}{2}\right) \prod_{h=2}^{k-1} d_h^{\binom{k}{h}} \times m^k \,. \tag{4.52}$$

Our proof is based on the variant of RL(k), Lemma 4.34, discussed in Remark 4.35. More precisely we want to apply this variant of Lemma 4.34 with the constants and functions chosen in (4.47), (4.48), and (4.49) to V, $\boldsymbol{R}$, and $H_0^{(k)} \cup \{H_1^{(k)}, \dots, H_{s_{4.38}}^{(k)}\}$, where

$$H_0^{(k)} = \binom{V}{k} \setminus F^{(k)} = \binom{V}{k} \Big\backslash \bigcup_{i\in[s_{4.38}]} H_i^{(k)} \,. \tag{4.53}$$

We artificially add $H_0^{(k)}$ only to obtain a partition of $\binom{V}{k}$, to formally match the assumption (c) of RL(k) (see (RL.c) below). We have to verify the following assumptions of Lemma 4.34 (see also Remark 4.35).

(RL.a') $|V| = km \ge n_{\mathrm{RL}}$, $V = V_1 \cup \dots \cup V_k$ with $V_i = m$ and $t_{\mathrm{RL}}!$ divides m,

(RL.b') $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$ is a $(\mu_{\mathrm{RL}}/3, \boldsymbol{d}_{4.38})$-regular $(m,k,k-1)$-complex, where $\boldsymbol{d}_{4.38} = (d_2, \dots, d_{k-1})$, $1/d_i \in \mathbb{N}$ and $1/d_i \le o_{\mathrm{RL}}$ for $2 \le i < k$, and $R^{(1)} = V_1 \cup \dots \cup V_k$, and

(RL.c) $\{H_0^{(k)}, H_1^{(k)}, \dots, H_{s_{4.38}}^{(k)}\}$ is a partition of $\binom{V}{k}$ into s_{RL} parts.

We note that (RL.a') follows from (L.4.38.a) and the choice of $m_{4.38}$ in (4.51) and $t_{4.38}$ in (4.50). Moreover, (RL.b') is a consequence of the assumption on $\boldsymbol{d}_{4.38}$, and (L.4.38.b) combined with the choice of $\delta_{4.38}$ in (4.50) and o_{RL} in (4.47). Similarly, (RL.c) follows from (L.4.38.d) in conjunction with (4.53) and the choice of s_{RL} in (4.47).

Having verified that (RL.a'), (RL.b'), and (RL.c) hold, Lemma 4.34 then ensures the existence of a family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ on V which satisfies the following properties:

(RL.$P1$) $\mathscr{P}_{\mathrm{RL}}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-equitable and t_{RL}-bounded,

(RL.$P2'$) $\mathscr{P}^{(1)} \prec R^{(1)} = V_1 \cup \dots \cup V_k$ and for every $2 \le j < k$ and every $P^{(j)} \in \mathscr{P}^{(j)}$ we have either $P^{(j)} \subseteq R^{(j)}$ or $P^{(j)} \cap R^{(j)} = \varnothing$, and

(RL.H) $H_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathscr{P}_{\mathrm{RL}}$ for every $i \in [s_{\mathrm{RL}}]$.

Before we continue with the proof we make a few observations and develop some notation. To an arbitrary polyad $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$ consider its corresponding $(m/a_1^{\mathscr{P}_{\mathrm{RL}}}, k, k-1)$-complex $\boldsymbol{P} = \{\hat{P}^{(j)}\}_{j=1}^{k-1}$. (More precisely, recalling (4.1), $\boldsymbol{P} = \boldsymbol{P}(K) = \{\hat{P}^{(j)}(K)\}_{j=1}^{k-1}$ for any $K \in \mathcal{K}_k(\hat{P}^{(k-1)})$). It follows from (RL.$P1$) and part ($c$) of Definition 4.10 that the complex $\boldsymbol{P}$ is an $(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}),(1/a_2^{\mathscr{P}_{\mathrm{RL}}},\dots,1/a_{k-1}^{\mathscr{P}_{\mathrm{RL}}})$-regular $(m/a_1^{\mathscr{P}_{\mathrm{RL}}},k,k-1)$-complex. From the choice of the function δ_{RL} in (4.49) and of

$$m \geq m_{4.38} \geq t_{\mathrm{RL}} \times m_{\mathrm{DCL}}\left(h=k-1,\ell=k,\gamma=\frac{\nu_{4.38}}{48},d_0=t_{\mathrm{RL}}^{-1}\right)$$

in (4.51) we infer by Theorem 4.19 that

$$|\mathcal{K}_k(\hat{P}^{(k-1)})| = \left(1\pm\frac{\nu_{4.38}}{48}\right)\prod_{h=2}^{k-1}\left(\frac{1}{a_h^{\mathscr{P}_{\mathrm{RL}}}}\right)^{\binom{k}{h}} \times \left(\frac{m}{a_1^{\mathscr{P}_{\mathrm{RL}}}}\right)^{k}, \tag{4.54}$$

holds for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$.

Since each $H_i^{(k)} \subseteq F^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ for the rest of the proof we will focus to the "interesting" part of the partition $\mathscr{P}_{\mathrm{RL}}$ and consider only those polyads which are sub-hypergraphs of $R^{(k-1)}$. To this end we set

$$\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R}) = \left\{\hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}\colon\ \hat{P}^{(k-1)} \subseteq R^{(k-1)}\right\}. \tag{4.55}$$

Note that due to (RL.*P2′*) and the properties of $\mathscr{P}_{\mathrm{RL}}$ we have that

$$\left\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})\right\} \text{ partitions } \mathcal{K}_k(R^{(k-1)}). \tag{4.56}$$

To simplify the notation we set

$$d_{\mathscr{H},R}(i) = d(H_i^{(k)}|R^{(k-1)}).$$

The following claim, Claim 4.40 ensures the existence of a partition

$$\left\{G_i^{(k)}(\hat{P}^{(k-1)})\colon\ i\in[s_{4.38}]\right\}$$

of $F^{(k)}\cap\mathcal{K}_k(\hat{P}^{(k-1)})$ for every $\hat{P}^{(k-1)}$ in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})$ with the property that $G_i^{(k)}(\hat{P}^{(k-1)})$ is $(\varepsilon_{4.38}^2/4, d_{\mathscr{H},R}(i))$-regular w.r.t. $\hat{P}^{(k-1)}$ for each $i\in[s_{4.38}]$. This property will enable us to use Proposition 4.32 to infer property (L.4.38.*i*) for $G_i^{(k)}$ defined in the obvious way.

In order to verify (L.4.38.*ii*) we will need some additional information concerning the $\{G_i^{(k)}(\hat{P}^{(k-1)})\colon\ i\in[s_{4.38}]\}$. Here our analysis splits into two cases and we define[3]

$$\hat{\mathscr{P}}_{\mathrm{RL},\mathscr{H}\text{-reg}}^{(k-1)}(\boldsymbol{R})=\Big\{\hat{P}^{(k-1)}\in\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{R})\colon H_i^{(k)} \text{ is } (\delta_{k,\mathrm{RL}},*,r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))\text{-regular w.r.t. } \hat{P}^{(k-1)} \text{ for every } i\in[s_{4.38}]\Big\}. \tag{4.57}$$

Below we present two claims, from which we infer the existence of the partition

[3] Note that we exclude the artificially added hypergraph $H_0^{(k)}$ in the definition of $\hat{\mathscr{P}}_{\mathrm{RL},\mathscr{H}\text{-reg}}^{(k-1)}(\boldsymbol{R})$.

$\{G_i^{(k)}\colon i \in [s_{4.38}]\}$ with the desired properties (L.4.38.i) and (L.4.38.ii). We then give the proofs of the claims.

Claim 4.40. *For every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{RL}(\boldsymbol{R})$ there exist a partition*

$$\{G_i^{(k)}(\hat{P}^{(k-1)})\colon i \in [s_{4.38}]\}$$

of $F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ such that for every $i \in [s_{4.38}]$

$$G_i^{(k)}(\hat{P}^{(k-1)}) \text{ is } (\varepsilon^2_{4.38}/4, d_{\mathscr{H},R}(i))\text{-regular w.r.t. } \hat{P}^{(k-1)}\,. \tag{4.58}$$

Moreover, if $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{RL,\mathscr{H}\text{-}reg}(\boldsymbol{R})$, then the partition $\{G_i^{(k)}\colon i \in [s_{4.38}]\}$ has the additional property that for every $i \in [s_{4.38}]$

$$\begin{aligned}\Big|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))\Big|&\\ \le \left(|d_{\mathscr{H},R}(i) - d(H_i^{(k)}|\hat{P}^{(k-1)})| + \frac{\nu_{4.38}}{6}\right)&|\mathcal{K}_k(\hat{P}^{(k-1)})|\end{aligned} \tag{4.59}$$

In order to verify (L.4.38.ii) we need further control over the quantity considered in (4.59). The following claim ensures that “typically”

$$|d_{\mathscr{H},R}(i) - d(H_i^{(k)}|\hat{P}^{(k-1)})| \le \frac{\nu_{4.38}}{6}\,.$$

For that we define for every $i \in [s_{4.38}]$

$$\begin{aligned}&\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})\\ &\qquad= \left\{\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})\colon\ |d_{\mathscr{H},R}(i) - d(H_i^{(k)}|\hat{P}^{(k-1)})| > \frac{\nu_{4.38}}{6}\right\}.\end{aligned} \tag{4.60}$$

Claim 4.41. *For every $i \in [s_{4.38}]$*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{RL,BAD}(\boldsymbol{R}, H_i^{(k)})\Big\}\Big| \le \frac{\nu_{4.38}}{3}|\mathcal{K}_k(R^{(k-1)})|\,. \tag{4.61}$$

We finish the proof of Lemma 4.38 based on Claim 4.40 and Claim 4.41. We use Claim 4.40 and set $G_i^{(k)}$ for every $i \in [s_{4.38}]$ equal to

$$G_i^{(k)} = \bigcup\Big\{G_i^{(k)}(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})\Big\}\,. \tag{4.62}$$

From $F^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$ (cf. (L.4.38.c)) combined with (4.56) and Claim 4.40 we infer that $\mathscr{G}^{(k)} = \{G_i^{(k)}\colon i \in [s_{4.38}]\}$ defined in (4.62) is a partition of $F^{(k)}$.

Now, we have to verify (L.4.38.i) and (L.4.38.ii) for every fixed $i \in [s_{4.38}]$ and this choice of $\mathscr{G}^{(k)}$. So let $i \in [s_{4.38}]$ be fixed.

We start with (L.4.38.i). We infer from (4.56) that the two families $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})$ and

$\{G_i^{(k)}(\hat{P}^{(k-1)})\colon \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})\}$ satisfy Setup 4.31 for $j = k$, and $t = |\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})|$. Consequently, in view of (4.58) we can apply Proposition 4.32 with $r = 1$, $\delta = \varepsilon^2_{4.38}/4$, and $d = d_{\mathscr{H},R}(i)$, to infer

$$G_i^{(k)} \text{ is } (\varepsilon_{4.38}, d_{\mathscr{H},R}(i))\text{-regular w.r.t. } \bigcup\Big\{\hat{P}^{(k-1)}\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})\Big\} = R^{(k-1)}\,,$$

and, therefore, (L.4.38.i) holds.

We now focus on (L.4.38.ii) for a fixed $i \in [s_{4.38}]$. We will estimate $|G_i^{(k)} \triangle H_i^{(k)}|$ as the sum of the symmetric difference taken over all polyads $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})$. In this sum we distinguish among polyads in which some $H^{(k)} \in \mathscr{H}^{(k)}$ is "irregular", in which $H_i^{(k)}$ has "bad" (atypical) density and the remaining "typical" polyads in which $H_i^{(k)}$ has the correct density and every $H^{(k)} \in \mathscr{H}^{(k)}$ is regular. With this in mind we set

$$\begin{aligned}
\mathfrak{D}_{\mathrm{irreg}}(i) = \sum\Big\{\Big|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))\Big|\colon&\\
\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R})\Big\}&\\
\mathfrak{D}_{\mathrm{typ}}(i) = \sum\Big\{\Big|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))\Big|\colon&\\
\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})\Big\}&\\
\mathfrak{D}_{\mathrm{bad}}(i) = \sum\Big\{\Big|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))\Big|\colon&\\
\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})\Big\}&
\end{aligned}$$

and note that

$$|G_i^{(k)} \triangle H_i^{(k)}| \le \mathfrak{D}_{\mathrm{irreg}}(i) + \mathfrak{D}_{\mathrm{typ}}(i) + \mathfrak{D}_{\mathrm{bad}}(i)\,. \tag{4.63}$$

In the following we bound each of the terms of (4.63) separately. We start with $\mathfrak{D}_{\mathrm{irreg}}(i)$. Due to (RL.$H$) and the definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R})$ in (4.57) we have

$$\sum\Big\{\Big|\mathcal{K}_k(\hat{P}^{(k-1)})\Big|\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R})\Big\} \le s_{4.38} \times \delta_{k,\mathrm{RL}} k^k m^k\,.$$

Clearly, the left-hand side of the last inequality is an upper bound on $\mathfrak{D}_{\mathrm{irreg}}(i)$ and we infer

$$\mathfrak{D}_{\mathrm{irreg}}(i) \le s_{4.38}\delta_{k,\mathrm{RL}} k^k m^k \overset{(4.48)}{\le} \frac{\nu_{4.38}}{6} \prod_{h=2}^{k-1} d_h^{\binom{k}{h}} \times m^k \overset{(4.52)}{\le} \frac{\nu_{4.38}}{3} |\mathcal{K}_k(R^{(k-1)})|\,. \tag{4.64}$$

Next we consider $\mathfrak{D}_{\mathrm{typ}}(i)$. Owing to (4.60), for each $\hat{P}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})$ we have

$$|d_{\mathscr{H},R}(i) - d(H_i^{(k)}|\hat{P}^{(k-1)})| \le \frac{\nu_{4.38}}{6}\,,$$

and we infer from (4.59) that for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})$

$$\left|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))\right| \le \frac{\nu_{4.38}}{3} |\mathcal{K}_k(R^{(k-1)})| .$$

Consequently, it follows directly from the definition of $\mathfrak{D}_{\mathrm{typ}}(i)$ that

$$\begin{aligned}\mathfrak{D}_{\mathrm{typ}}(i) &\le \frac{\nu_{4.38}}{3} \sum \left\{ \left|\mathcal{K}_k(\hat{P}^{(k-1)})\right| \colon \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)}) \right\} \\ &\le \frac{\nu_{4.38}}{3} |\mathcal{K}_k(R^{(k-1)})| . \end{aligned} \tag{4.65}$$

Finally, we derive the same bound for $\mathfrak{D}_{\mathrm{bad}}(i)$ directly from the definition of $\mathfrak{D}_{\mathrm{bad}}(i)$ and (4.61)

$$\mathfrak{D}_{\mathrm{bad}}(i) \le \frac{\nu_{4.38}}{3} |\mathcal{K}_k(R^{(k-1)})| . \tag{4.66}$$

We now conclude (L.4.38.*ii*) from (4.64), (4.65), and (4.66), applied to (4.63). In order to complete the proof of Lemma 4.38 we still have to verify Claim 4.40 and Claim 4.41, which we will do below. □

Proof of Claim 4.40. Let $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})$ be fixed. First we recall (4.54). Below we will apply the slicing lemma, Proposition 4.33 to sub-hypergraphs of $\hat{P}^{(k-1)}$. For that, among others, we have to verify the assumption (i) of Proposition 4.33, i.e.,

$$|\mathcal{K}_k(\hat{P}^{(k-1)})| \ge \frac{m^k}{\ln m} . \tag{4.67}$$

This, however, follows from (4.54) and $m \ge m_{4.38} \ge \exp(2t_{\mathrm{RL}}^{2^k})$ (cf. (4.51)). Therefore, we don't have to verify this condition in future applications of the slicing lemma. We begin with the following consequence of the choice of $\xi_{4.38} \le \varepsilon_{4.38}^2/(192 t_{\mathrm{RL}}^{2^k})$ in (4.50), and Proposition 4.29;

$$F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \text{ is } (\varepsilon_{4.38}^2/96, d(F^{(k)}|R^{(k-1)}))\text{-regular w.r.t. } \hat{P}^{(k-1)} . \tag{4.68}$$

The proof of Claim 4.40 splits into two main cases.

Case 1 $(d(F^{(k)}|R^{(k-1)}) > \varepsilon_{4.38}^2/16)$**.** In this case we will treat "thin" hypergraphs $H_i^{(k)}$ w.r.t. $R^{(k-1)}$ somewhat differently. To this end we set

$$R_{\mathrm{THIN}} = \left\{ i \in [s_{4.38}] \colon d_{\mathscr{H},R}(i) < \frac{\varepsilon_{4.38}^2}{192 s_{4.38}} \right\} . \tag{4.69}$$

Due to the definition of R_{THIN} and the assumption of Case 1 we have

$$[s_{4.38}] \setminus R_{\mathrm{THIN}} \ne \varnothing . \tag{4.70}$$

We distinguish two sub-cases of Case 1 depending on whether $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R})$.

Case 1.1 ($\hat{P}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R})$)**.** In this particular case it suffices to prove the existence of a partition $\{G_i^{(k)}(\hat{P}^{(k-1)})\colon i \in [s_{4.38}]\}$ of $F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ which satisfies (4.58) only. For this we will simply appeal to the slicing lemma, Proposition 4.33, to decompose $F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ into hypergraphs with the appropriate densities (as required for (4.58)). More precisely, we apply Proposition 4.33, with

$$j = k\,, \quad s_0 = s_{4.38}\,, \quad r = 1\,, \quad \delta_0 = \frac{\varepsilon_{4.38}^2}{96}\,, \quad \varrho_0 = \frac{\varepsilon_{4.38}^2}{16}\,, \quad \text{and} \quad p_0 = \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}$$

to $\hat{P}^{(k-1)}$ and $F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ with

$$s = \big|[s_{4.38}] \setminus R_{\mathrm{THIN}}\big|\,, \quad \delta = \frac{\varepsilon_{4.38}^2}{96}\,, \quad \varrho = d(F^{(k)}|R^{(k-1)})\,,$$

and

$$p_i = \frac{d_{\mathscr{H},R}(i)}{d(F^{(k)}|R^{(k-1)})} \quad \text{for every} \quad i \in [s_{4.38}] \setminus R_{\mathrm{THIN}}\,.$$

Due to (4.51) we have

$$\frac{m}{a_1^{\mathscr{P}_{\mathrm{RL}}}} \geq m_{\mathrm{SL}}\left(k,\, s_{4.38},\, 1,\, \frac{\varepsilon_{4.38}^2}{96},\, \frac{\varepsilon_{4.38}^2}{16},\, \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}\right)$$

and the other conditions of Proposition 4.33 are immediate consequences of (4.67)–(4.69), and the assumption of Case 1.1. By Proposition 4.33 we obtain a family of hypergraphs $T_0^{(k)} \cup \{T_i^{(k)}\colon i \in [s_{4.38}] \setminus R_{\mathrm{THIN}}\}$ satisfying the following properties

$$T_0^{(k)} \cup \{T_i^{(k)}\colon i \in [s_{4.38}] \setminus R_{\mathrm{THIN}}\} \text{ partitions } F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})\,, \tag{4.71}$$

$$T_i^{(k)} \text{ is } (\varepsilon_{4.38}^2/32, d_{\mathscr{H},R}(i))\text{-regular w.r.t } \hat{P}^{(k-1)} \text{ for } i \in [s_{4.38}] \setminus R_{\mathrm{THIN}} \tag{4.72}$$

$$T_0^{(k)} \text{ is } (\varepsilon_{4.38}^2/32, d_{T_0^{(k)}})\text{-regular w.r.t. } \hat{P}^{(k-1)}\,, \tag{4.73}$$

where

$$d_{T_0^{(k)}} = d(F^{(k)}|R^{(k-1)}) - \sum \{d_{\mathscr{H},R}(i)\colon i \in [s_{4.38}] \setminus R_{\mathrm{THIN}}\} \overset{(4.69)}{\leq} \frac{\varepsilon_{4.38}^2}{192}\,. \tag{4.74}$$

Fix some $i_0 \in [s_{4.38}] \setminus R_{\mathrm{THIN}}$ (due to (4.70) such an i_0 exists). We then define the family $G_i^{(k)}(\hat{P}^{(k-1)})$ for $i \in [s_{4.38}]$ as follows

$$G_i^{(k)}(\hat{P}^{(k-1)}) = \begin{cases} \varnothing & \text{if} \quad i \in R_{\mathrm{THIN}} \\ T_i^{(k)} \cup T_0^{(k)} & \text{if} \quad i = i_0 \\ T_i^{(k)} & \text{otherwise.} \end{cases}$$

From (4.71) we infer that $\{G_i^{(k)}(\hat{P}^{(k-1)}) \in [s_{4.38}]\}$ defined that way forms a partition of

$F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ and it is left to verify (4.58) for every $i \in [s_{4.38}]$.

First, let $i \in R_{\mathrm{THIN}}$. Since by definition, $G_i^{(k)}(\hat{P}^{(k-1)}) = \varnothing$ we infer that $G_i^{(k)}(\hat{P}^{(k-1)})$ is $(\varepsilon', 0)$-regular w.r.t. $\hat{P}^{(k-1)}$ for all $\varepsilon' > 0$. Since $i \in R_{\mathrm{THIN}}$, $d_{\mathscr{H},R}(i) < \varepsilon_{4.38}^2/4$. Consequently, the $(\varepsilon', 0)$-regularity for every $\varepsilon' > 0$ yields that the hypergraph $G_i^{(k)}(\hat{P}^{(k-1)})$ is $(\varepsilon_{4.38}^2/4, d_{\mathscr{H},R}(i))$-regular (i.e., (4.58) holds for $i \in R_{\mathrm{THIN}}$).

If $i \in [s_{4.38}] \setminus R_{\mathrm{THIN}}$ and $i \neq i_0$, then (4.72) and $G_i^{(k)}(\hat{P}^{(k-1)}) = T_i^{(k)}$ immediately implies (4.58).

It is left to verify (4.58) for $i = i_0$. In that case Proposition 4.30 applied to $T_{i_0}^{(k)}$ and $T_0^{(k)}$ implies by (4.72) and (4.73) that $G_{i_0}^{(k)}(\hat{P}^{(k-1)})$ is $(\varepsilon_{4.38}^2/16)$-regular w.r.t. $\hat{P}^{(k-1)}$, with density between $d_{\mathscr{H},R}(i_0)$ and $d_{\mathscr{H},R}(i_0) + \varepsilon_{4.38}^2/192$ (cf. (4.74)). Consequently, $G_{i_0}^{(k)}(\hat{P}^{(k-1)})$ is $((\varepsilon_{4.38}^2/16 + \varepsilon_{4.38}^2/192), d_{\mathscr{H},R}(i_0))$-regular with respect to $\hat{P}^{(k-1)}$, which yields (4.58).

Having verified (4.58) for every $i \in [s_{4.38}]$, we conclude Case 1.1. ◊

Case 1.2 $(\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{R}))$**.** In this case we have to guarantee the existence of a partition of $F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ which satisfies both (4.58) and (4.59) of Claim 4.40. Due to (4.59) we have to be more careful in defining the desired partition. On the other hand, the assumption in this case says that $H_i^{(k)}$ is $\delta_{k,\mathrm{RL}}$-regular w.r.t. $\hat{P}^{(k-1)}$ for every $i \in [s_{4.38}]$. This allows us to apply the slicing lemma, to any $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$.

Below we give a short outline how we use this additional assumption. To simplify the notation we set for every $i \in [s_{4.38}]$

$$d_{\mathscr{H},\hat{P}}(i) = d(H_i^{(k)} | \hat{P}^{(k-1)}) .$$

We first consider the hypergraphs $H_i^{(k)}$ which are too "fat" in $\hat{P}^{(k-1)}$, i.e., we consider

$$I_{\mathrm{FAT}}(\hat{P}) = \left\{ i \in [s_{4.38}] \setminus R_{\mathrm{THIN}} \colon \ d_{\mathscr{H},\hat{P}}(i) > d_{\mathscr{H},R}(i) + \frac{\varepsilon_{4.38}^2}{192 s_{4.38}} \right\} . \tag{4.75}$$

We apply the slicing lemma to split each $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ for $i \in I_{\mathrm{FAT}}(\hat{P})$ into a "main" part $M_i^{(k)}$ of density $d_{\mathscr{H},R}(i)$ and a "leftover" $L_i^{(k)}$. The $M_i^{(k)}$ will be used to define $G_i^{(k)}(\hat{P}^{(k-1)})$. Furthermore, since each $L_i^{(k)}$ is regular, and since each $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ for $i \in R_{\mathrm{THIN}}$ is regular, as well, (by the assumption of the case), we will infer that their union $U^{(k)} = \bigcup_{i \in I_{\mathrm{FAT}}(\hat{P})} L_i^{(k)} \cup \bigcup_{i \in R_{\mathrm{THIN}}} (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ is regular with density "very close" to

$$\Delta_{\mathrm{SLIM}}(\hat{P}) = \sum \left\{ d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i) \colon \ i \in I_{\mathrm{SLIM}}(\hat{P}) \right\} , \tag{4.76}$$

where

$$I_{\mathrm{SLIM}}(\hat{P}) = \left\{ i \in [s_{4.38}] \setminus R_{\mathrm{THIN}} \colon \ d_{\mathscr{H},\hat{P}}(i) < d_{\mathscr{H},R}(i) - \frac{\varepsilon_{4.38}^2}{192 s_{4.38}} \right\} . \tag{4.77}$$

We then apply the slicing lemma again, this time to $U^{(k)}$, to split it into regular pieces of densities $\{d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i)\colon\ i \in I_{\mathrm{SLIM}}(\hat{P})\}$. For $i \in I_{\mathrm{SLIM}}(\hat{P})$ uniting $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ with the appropriate slice from $U^{(k)}$ then gives rise to the desired partition. We now implement the technical details of this plan.

Let $I_{\mathrm{FAT}}(\hat{P})$ and $I_{\mathrm{SLIM}}(\hat{P})$ be defined as in (4.75) and (4.77). We set

$$I_{\mathrm{OK}}(\hat{P}) = \left\{ i \in [s_{4.38}] \setminus R_{\mathrm{THIN}}\colon\ d_{\mathscr{H},\hat{P}}(i) = d_{\mathscr{H},R}(i) \pm \frac{\varepsilon_{4.38}^2}{192 s_{4.38}} \right\} \tag{4.78}$$

and note that $[s_{4.38}]$ is the disjoint union of $I_{\mathrm{FAT}}(\hat{P})$, $I_{\mathrm{OK}}(\hat{P})$, $I_{\mathrm{SLIM}}(\hat{P})$, and R_{THIN}. We will later need the following observation

$$\begin{aligned}
&\left|\left(\sum_{i \in I_{\mathrm{FAT}}(\hat{P})} (d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)) + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\hat{P}}(i) \right) - \sum_{i \in I_{\mathrm{SLIM}}(\hat{P})} (d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i)) \right| \\
&\qquad = \Big| \sum \left\{ d_{\mathscr{H},\hat{P}}(i)\colon\ i \in I_{\mathrm{FAT}}(\hat{P}) \cup R_{\mathrm{THIN}} \cup I_{\mathrm{SLIM}}(\hat{P}) \right\} \\
&\qquad\qquad\qquad - \sum \left\{ d_{\mathscr{H},R}(i)\colon\ i \in I_{\mathrm{FAT}}(\hat{P}) \cup I_{\mathrm{SLIM}}(\hat{P}) \right\} \Big| \\
&\qquad = \Big| \sum \left\{ d_{\mathscr{H},\hat{P}}(i)\colon\ i \in [s_{4.38}] \setminus I_{\mathrm{OK}}(\hat{P}) \right\} \\
&\qquad\qquad\qquad - \sum \left\{ d_{\mathscr{H},R}(i)\colon\ i \in [s_{4.38}] \setminus (I_{\mathrm{OK}}(\hat{P}) \cup R_{\mathrm{THIN}}) \right\} \Big| \\
&\qquad = \left| d(F^{(k)} | \hat{P}^{(k-1)}) - d(F^{(k)} | R^{(k-1)}) \right| \\
&\qquad\qquad\qquad + \sum_{i \in I_{\mathrm{OK}}(\hat{P})} \left| d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i) \right| + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},R}(i)\,.
\end{aligned}$$

Thus in view of (4.76) and (4.68), (4.69), and (4.78) we derive the following bound on the left-hand side from above

$$\begin{aligned}
&\left|\left(\sum_{i \in I_{\mathrm{FAT}}(\hat{P})} \left(d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)\right) + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\hat{P}}(i) \right) - \Delta_{\mathrm{SLIM}}(\hat{P}) \right| \\
&\qquad\qquad \leq \frac{\varepsilon_{4.38}^2}{96} + \frac{\varepsilon_{4.38}^2}{192} + \frac{\varepsilon_{4.38}^2}{192} = \frac{\varepsilon_{4.38}^2}{48}\,.
\end{aligned} \tag{4.79}$$

Case 1.2 splits into two sub-cases depending on the size of $\Delta_{\mathrm{SLIM}}(\hat{P})$.

Case 1.2.1 ($\Delta_{\mathrm{SLIM}}(\hat{P}) > \varepsilon_{4.38}^2/12$)**.** For every $i \in I_{\mathrm{FAT}}(\hat{P})$ we have, due to the assumption of Case 1.2, that $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}}, d_{\mathscr{H},\hat{P}}(i))$-regular w.r.t. $\hat{P}^{(k-1)}$. We apply the slicing lemma, Proposition 4.33, with

$$j = k\,, \quad s_0 = 2\,, \quad r = 1\,, \quad \delta_0 = \delta_{k,\mathrm{RL}}\,, \quad \varrho_0 = \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}\,, \quad \text{and} \quad p_0 = \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}$$

to $\hat{P}^{(k-1)}$ and $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ with

$$s = 2\,, \quad \delta = \delta_{k,\mathrm{RL}}\,, \quad \varrho = d_{\mathscr{H},\hat{P}}(i)\,,$$

and

$$p_1 = \frac{d_{\mathscr{H},R}(i)}{d_{\mathscr{H},\hat{P}}(i)} \qquad \text{and} \qquad p_2 = \frac{d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)}{d_{\mathscr{H},\hat{P}}(i)}\,.$$

For this choice of parameters the assumptions of Proposition 4.33 are satisfied. Indeed we have (4.67),

$$\frac{m}{a_1^{\mathscr{P}_{\mathrm{RL}}}} \geq m_{\mathrm{SL}}\left(k, 2, 1, \delta_{k,\mathrm{RL}}, \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}, \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}\right)$$

by (4.51), $\varrho \geq \varrho_0$ since $i \in I_{\mathrm{FAT}}(\hat{P})$ and (4.75), $\varrho_0 \geq 2\delta = \delta_{k,\mathrm{RL}}$ (cf. (4.48)), $p_1 \geq p_0$ since $i \notin R_{\mathrm{THIN}}$ by definition of $I_{\mathrm{FAT}}(\hat{P})$, and $p_2 \geq p_0$ since $i \in I_{\mathrm{FAT}}(\hat{P})$.

Since $p_1 + p_2 = 1$, Proposition 4.33 yields a partition $M_i^{(k)} \cup L_i^{(k)}$ of $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ for every $i \in I_{\mathrm{FAT}}(\hat{P})$, where

$$M_i^{(k)} \text{ is } (3\delta_{k,\mathrm{RL}}, d_{\mathscr{H},R}(i))\text{-regular w.r.t. } \hat{P}^{(k-1)} \text{ and} \tag{4.80}$$

$$L_i^{(k)} \text{ is } (3\delta_{k,\mathrm{RL}}, (d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)))\text{-regular w.r.t. } \hat{P}^{(k-1)}\,. \tag{4.81}$$

We now collect all "leftovers" and redistribute them among the hypergraphs $H_i^{(k)}$ which are too "slim" in $\hat{P}^{(k-1)}$. For that we set

$$U^{(k)} = \bigcup_{i \in I_{\mathrm{FAT}}(\hat{P})} L_i^{(k)} \cup \bigcup_{i \in R_{\mathrm{THIN}}} \left(H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})\right).$$

From (4.81) and the assumption of Case 1.2 we infer with Proposition 4.30 that $U^{(k)}$ is $(3 s_{4.38} \delta_{k,\mathrm{RL}})$-regular w.r.t. $\hat{P}^{(k-1)}$. Moreover, by the choice of $\delta_{k,\mathrm{RL}}$ in (4.48) we have $3 s_{4.38} \delta_{k,\mathrm{RL}} \leq \varepsilon_{4.38}^2/48$ and by (4.79) it follows that

$$U^{(k)} \text{ is } (\varepsilon_{4.38}^2/24, \Delta_{\mathrm{SLIM}}(\hat{P}))\text{-regular w.r.t. } \hat{P}^{(k-1)}\,. \tag{4.82}$$

We then apply the slicing lemma again, this time with

$$j = k\,, \quad s_0 = s_{4.38}\,, \quad r = 1\,, \quad \delta_0 = \frac{\varepsilon_{4.38}^2}{24}\,, \quad \varrho_0 = \frac{\varepsilon_{4.38}^2}{12}\,, \quad \text{and} \quad p_0 = \frac{\varepsilon_{4.38}^2}{192 s_{4.38}}\,,$$

to $\hat{P}^{(k-1)}$ and $U^{(k)}$ with

$$s = |I_{\mathrm{SLIM}}(\hat{P})|\,, \quad \delta = \frac{\varepsilon_{4.38}^2}{24}\,, \quad \varrho = \Delta_{\mathrm{SLIM}}(\hat{P})\,,$$

and

$$p_i = \frac{d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i)}{\Delta_{\mathrm{SLIM}}(\hat{P})} \quad \text{for every} \quad i \in I_{\mathrm{SLIM}}(\hat{P})\,.$$

Here the assumptions of Proposition 4.33 are consequences of (4.67) (showing (i) of Proposition 4.33), (4.82) (showing that $U^{(k)}$ is sufficiently regular), (4.51) (which yields $m/a_1^{\mathscr{P}_{\mathrm{RL}}}$ is sufficiently large), the assumption of Case 1.2.1 (which yields $\varrho \geq \varrho_0$), and the definition of $I_{\mathrm{SLIM}}(\hat{P})$ in (4.77) combined with $\Delta_{\mathrm{SLIM}}(\hat{P}) \leq 1$ (which yields $p_i \geq p_0$).

Also, note that $\sum_{i \in I_{\mathrm{SLIM}}(\hat{P})} p_i = 1$ and, consequently, Proposition 4.33 yields a partition $\{T_i^{(k)} \colon i \in I_{\mathrm{SLIM}}(\hat{P})\}$ of $U^{(k)}$, which by (4.82) has density "close" to $\Delta_{\mathrm{SLIM}}(\hat{P})$, so that

$$T_i^{(k)} \text{ is } (\varepsilon_{4.38}^2/8, (d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i)))\text{-regular w.r.t. } \hat{P}^{(k-1)} . \tag{4.83}$$

Finally, we are ready to define the family $\{G_i^{(k)}(\hat{P}^{(k-1)}) \colon i \in [s_{4.38}]\}$. Set

$$G_i^{(k)}(\hat{P}^{(k-1)}) = \begin{cases} \varnothing & \text{if} \quad i \in R_{\mathrm{THIN}} \\ M_i^{(k)} & \text{if} \quad i \in I_{\mathrm{FAT}}(\hat{P}) \\ H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) & \text{if} \quad i \in I_{\mathrm{OK}}(\hat{P}) \\ (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})) \cup T_i^{(k)} & \text{if} \quad i \in I_{\mathrm{SLIM}}(\hat{P}) . \end{cases}$$

It is obvious that $\{G_i^{(k)}(\hat{P}^{(k-1)}) \colon i \in [s_{4.38}]\}$ defined this way is a partition of

$$F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) .$$

We still have to verify (4.58) and (4.59).

We start with showing (4.58). First let $i \in R_{\mathrm{THIN}}$. By definition of $G_i^{(k)}(\hat{P}^{(k-1)})$ it is $(\varepsilon', 0)$-regular for every $\varepsilon' > 0$ and, hence, it is $((\varepsilon' + d_{\mathscr{H},R}(i)), d_{\mathscr{H},R}(i))$-regular. Therefore, (4.58) follows from

$$d_{\mathscr{H},R}(i) \leq \varepsilon_{4.38}^2/(192 s_{4.38}) < \varepsilon_{4.38}^2/4$$

(cf. (4.69)).

If $i \in I_{\mathrm{FAT}}(\hat{P})$, then (4.58) follows from (4.80) and $3\delta_{k,\mathrm{RL}} < \varepsilon_{4.38}^2/4$ (cf. (4.48)).

Now let $i \in I_{\mathrm{OK}}(\hat{P})$. Then

$$G_i^{(k)}(\hat{P}^{(k-1)}) = H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$$

is $(\delta_{k,\mathrm{RL}}, d_{\mathscr{H},\hat{P}}(i))$-regular due to the assumption of Case 1.2. Moreover, due to (4.78)

$$|d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)| \leq \varepsilon_{4.38}^2/(192 s_{4.38})$$

and, therefore, $G_i^{(k)}(\hat{P}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}} + \varepsilon_{4.38}^2/(192 s_{4.38}), d_{\mathscr{H},R}(i))$-regular. Now (4.58) follows, since $\delta_{k,\mathrm{RL}} + \varepsilon_{4.38}^2/(192 s_{4.38}) \leq \varepsilon_{4.38}^2/4$ (cf. (4.48)).

Finally, let $i \in I_{\mathrm{SLIM}}(\hat{P})$. Proposition 4.30 applied to $H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$ and T_i implies that $G_i^{(k)}(\hat{P}^{(k-1)})$ is $((\delta_{k,\mathrm{RL}} + \varepsilon_{4.38}^2/8), d_{\mathscr{H},R}(i))$-regular (cf. assumption of Case 1.2 and (4.83)). Hence, (4.58) follows since $\delta_{k,\mathrm{RL}} + \varepsilon_{4.38}^2/8 \leq \varepsilon_{4.38}^2/4$ (cf. (4.48)).

It is left to verify (4.59) for $i \in [s_{4.38}]$ to conclude this case, Case 1.2.1. Again our argument is different for each partition class R_{THIN}, $I_{\mathrm{FAT}}(\hat{P})$, $I_{\mathrm{OK}}(\hat{P})$, and $I_{\mathrm{SLIM}}(\hat{P})$ of the set $[s_{4.38}]$.

For $i \in R_{\mathrm{THIN}}$, due to notational reasons it will be easier to verify (4.59) in terms of the corresponding density

$$d\Big(G_i^{(k)}(\hat{P}^{(k-1)}) \triangle H_i^{(k)} \Big| \hat{P}^{(k-1)}\Big) = \frac{|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))|}{|\mathcal{K}_k(\hat{P}^{(k-1)})|}.$$

If $i \in R_{\mathrm{THIN}}$, then $d_{\mathscr{H},R}(i) \le \varepsilon_{4.38}^2/(192 s_{4.38})$ and, consequently,

$$\begin{aligned} d\Big(G_i^{(k)}(\hat{P}^{(k-1)}) \triangle H_i^{(k)} \Big| \hat{P}^{(k-1)}\Big) &= d_{\mathscr{H},\hat{P}}(i) \\ &\le |d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i)| + \varepsilon_{4.38}^2/(192 s_{4.38}). \end{aligned}$$

Therefore, (4.59) follows for $i \in R_{\mathrm{THIN}}$ from $\varepsilon_{4.38}^2/(192 s_{4.38}) \le \nu_{4.38}/6$.

If $i \in I_{\mathrm{FAT}}(\hat{P})$, then by definition of $G_i^{(k)}(\hat{P}^{(k-1)}) = M_i^{(k)}$ we have

$$|G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))| = |L_i^{(k)}|.$$

Due to (4.81) we have

$$|L_i^{(k)}| \le (d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i) + 3\delta_{k,\mathrm{RL}})|\mathcal{K}_k(\hat{P}^{(k-1)})|,$$

which combined with the choice of $\delta_{k,\mathrm{RL}} \le \nu_{4.38}/18$ (cf. (4.48)) yields (4.59) for $i \in I_{\mathrm{FAT}}(\hat{P})$.

If $i \in I_{\mathrm{OK}}(\hat{P})$, then (4.59) is a consequence of the definition $G_i^{(k)}(\hat{P}^{(k-1)}) = H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$, which yields that the left-hand side in (4.59) is 0.

Finally, we consider the case $i \in I_{\mathrm{SLIM}}(\hat{P})$. It follows from the definition of the hypergraph $G_i^{(k)}(\hat{P}^{(k-1)})$ and (4.83) that

$$\begin{aligned} |G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}))| &= |T_i^{(k)}| \\ &\le \left(d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i) + \frac{\varepsilon_{4.38}^2}{8}\right)|\mathcal{K}_k(\hat{P}^{(k-1)})|. \end{aligned}$$

Consequently, (4.59) for $i \in I_{\mathrm{SLIM}}(\hat{P})$ follows from (4.46).

Having verified (4.58) and (4.59) for every $i \in [s_{4.38}]$ we conclude the proof of Claim 4.40 in Case 1.2.1. In order to finish Case 1.2, we have to consider the complementing and rather trivial sub-case when $\Delta_{\mathrm{SLIM}}(\hat{P})$ is small. ◊

Case 1.2.2 ($\Delta_{\mathrm{SLIM}}(\hat{P}) \le \varepsilon_{4.38}^2/12$)**.** In this case we set for every $i \in [s_{4.38}]$

$$G_i^{(k)}(\hat{P}^{(k-1)}) = H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}).$$

Therefore, (4.59) of Claim 4.40 holds trivially, and we only focus on (4.58). For that we note, that due to the assumption of Case 1.2 we have $G_i^{(k)}(\hat{P}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}}, d_{\mathscr{H},\hat{P}}(i))$-regular w.r.t. $\hat{P}^{(k-1)}$ and consequently for every $i \in [s_{4.38}]$

$$G_i^{(k)}(\hat{P}^{(k-1)}) \text{ is } \Big(\delta_{k,\mathrm{RL}} + |d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)|, d_{\mathscr{H},R}(i)\Big)\text{-regular w.r.t. } \hat{P}^{(k-1)}\,. \tag{4.84}$$

In what follows we show that

$$\Big|d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i)\Big| \le \frac{\varepsilon_{4.38}^2}{6} \quad \text{for every } i \in [s_{4.38}]\,, \tag{4.85}$$

which combined with (4.84) and $\delta_{k,\mathrm{RL}} + \varepsilon_{4.38}^2/6 \le \varepsilon_{4.38}^2/4$ (cf. (4.48)), yields (4.58) for every $i \in [s_{4.38}]$.

First we consider $i \in R_{\mathrm{THIN}}$. Due to (4.79) and the assumption of Case 1.2.2 we have

$$\sum_{i \in I_{\mathrm{FAT}}(\hat{P})} \Big(d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i)\Big) + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\hat{P}}(i) \le \left(\frac{1}{12} + \frac{1}{48}\right)\varepsilon_{4.38}^2 < \frac{\varepsilon_{4.38}^2}{6}\,, \tag{4.86}$$

where all terms on the left-hand side are positive (cf. (4.75)). Therefore, $d_{\mathscr{H},\hat{P}}(i) \le \varepsilon_{4.38}^2/6$ for every $i \in R_{\mathrm{THIN}}$. Since $d_{\mathscr{H},R}(i) \le \varepsilon_{4.38}^2/(192 s_{4.38})$ for every $i \in R_{\mathrm{THIN}}$, (4.85) holds for every $i \in R_{\mathrm{THIN}}$.

If $i \in I_{\mathrm{FAT}}(\hat{P})$, then (4.86) yields

$$0 \le d_{\mathscr{H},\hat{P}}(i) - d_{\mathscr{H},R}(i) \le \frac{\varepsilon_{4.38}^2}{6}$$

and consequently (4.85) holds for those i.

For $i \in I_{\mathrm{OK}}(\hat{P})$, (4.85) follows from the definition of $I_{\mathrm{OK}}(\hat{P})$ in (4.78).

Finally, we consider $i \in I_{\mathrm{SLIM}}(\hat{P})$. From the assumption of this case, Case 1.2.2, and the definition of $\Delta_{\mathrm{SLIM}}(\hat{P})$ in (4.76) we infer

$$0 \le d_{\mathscr{H},R}(i) - d_{\mathscr{H},\hat{P}}(i) \le \frac{\varepsilon_{4.38}^2}{12}\,,$$

which clearly implies (4.85) for $i \in I_{\mathrm{SLIM}}(\hat{P})$.

This concludes Case 1.2.2 the last sub-case of Case 1. ◊

Case 2 ($d(F^{(k)}|R^{(k-1)}) \le \varepsilon_{4.38}^2/16$)**.** In this case we set

$$G_1^{(k)}(\hat{P}^{(k-1)}) = F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$$

and

$$G_2^{(k)}(\hat{P}^{(k-1)}) = \dots = G_{s_{4.38}}^{(k)}(\hat{P}^{(k-1)}) = \varnothing\,.$$

Again we have to show (4.58) and (4.59) of Claim 4.40. We start with (4.58). Note that $G_i^{(k)}(\hat{P}^{(k-1)})$ is $(\varepsilon_{4.38}^2/96)$-regular w.r.t. $\hat{P}^{(k-1)}$ for every $i \in [s_{4.38}]$. (This is trivial for $i \ge 2$ and follows from (4.68) for $i = 1$.) In order to show that $G_i^{(k)}(\hat{P}^{(k-1)})$ is also $(\varepsilon_{4.38}^2/4, d_{\mathscr{H},R}(i))$-regular recall the assumption $d(F^{(k)}|R^{(k-1)}) \leq \varepsilon_{4.38}^2/16$, which implies that $d_{\mathscr{H},R}(i) \leq \varepsilon_{4.38}^2/16$ for every $i \in [s_{4.38}]$. Consequently, $G_i^{(k)}(\hat{P}^{(k-1)})$ is $((\varepsilon_{4.38}^2/96 + \varepsilon_{4.38}^2/16), d_{\mathscr{H},R}(i))$-regular for every $i \in [s_{4.38}]$ and (4.58) follows.

In order to infer (4.59) we observe that for $i \in [s_{4.38}]$

$$G_i^{(k)}(\hat{P}^{(k-1)}) \triangle (H_i^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})\,.$$

Moreover, due to the assumption $d(F^{(k)}|R^{(k-1)}) \leq \varepsilon_{4.38}^2/16$ and (4.68) we have

$$|F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})| \leq (\varepsilon_{4.38}^2/16 + \varepsilon_{4.38}^2/96)|\mathcal{K}_k(\hat{P}^{(k-1)})|\,.$$

Property (4.59) then follows from $\varepsilon_{4.38}^2/16 + \varepsilon_{4.38}^2/96 \leq \nu_{4.38}/6$ (see (4.46)). $\Diamond$

In all cases we ensured the existence of a partition of $F^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)})$, which satisfies the conclusions of Claim 4.40. This concludes the proof of Claim 4.40. $\square$

Proof of Claim 4.41. We assume the contrary, i.e.,

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})\Big\}\Big| > \frac{\nu_{4.38}}{3}|\mathcal{K}_k(R^{(k-1)})|\,.$$

Without loss of generality we may assume that $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)}) \subseteq \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)})$ defined by

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)}) = \Big\{\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{R})\colon\ d(H_i^{(k)}|\hat{P}^{(k-1)}) > d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{6}\Big\}$$

satisfies

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)})\Big\}\Big| \geq \frac{\nu_{4.38}}{6}|\mathcal{K}_k(R^{(k-1)})|\,. \tag{4.87}$$

The complementing case concerning

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,SLIM}}(\boldsymbol{R}, H_i^{(k)}) = \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{R}, H_i^{(k)}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)})$$

instead of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)})$ is very similar. In what follows we will show that (4.87) contradicts the $(\nu_{4.38}/12, *, t_{4.38}^{2^k})$-regularity of $H_i^{(k)}$ w.r.t. $R^{(k-1)}$ (see (L.4.38.d)). Since

$$|\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)})| \leq |\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}| = \prod_{h=1}^{k-1} (a_h^{\mathscr{P}_{\mathrm{RL}}})^{\binom{k}{h}} \leq t_{\mathrm{RL}}^{2^k} \overset{(4.50)}{=} t_{4.38}^{2^k}$$

this contradiction follows once we establish the following inequality

$$\frac{\Big|H_i^{(k)} \cap \bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)})\Big\}\Big|}{\Big|\bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H_i^{(k)})\Big\}\Big|} \geq d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{12}\,. \tag{4.88}$$

By definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H^{(k)}_i)$ we have $d(H^{(k)}_i|\hat{P}^{(k-1)}) \geq d_{\mathscr{H},R}(i) + \nu_{4.38}/6$ for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H^{(k)}_i)$ and, since

$$\mathcal{K}_k(\hat{P}^{(k-1)}_1) \cap \mathcal{K}_k(\hat{P}^{(k-1)}_2) = \varnothing$$

for all distinct $\hat{P}^{(k-1)}_1$, $\hat{P}^{(k-1)}_2 \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H^{(k)}_i) \subseteq \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ (cf. 4.56) it suffices to verify

$$\left(d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{6}\right) \frac{\min\left\{|\mathcal{K}_k(\hat{P}^{(k-1)})|\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H^{(k)}_i)\right\}}{\max\left\{|\mathcal{K}_k(\hat{P}^{(k-1)})|\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{R}, H^{(k)}_i)\right\}} \geq d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{12} \tag{4.89}$$

to infer (4.88). In view of (4.54) we derive the following upper bound on the right-hand side of (4.89)

$$\begin{aligned}\left(d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{6}\right) \frac{1 - \nu_{4.38}/48}{1 + \nu_{4.38}/48} &\geq \left(d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{6}\right)\left(1 - \frac{\nu_{4.38}}{24}\right) \\ &\geq d_{\mathscr{H},R}(i) + \frac{\nu_{4.38}}{12}\,,\end{aligned}$$

which concludes the proof of Claim 4.41. □

4.5 Proof of: RAL(k) $\Longrightarrow$ RL($k+1$)

In what follows we deduce RL($k+1$) (Lemma 4.34) from RAL(k) (Lemma 4.36). The proof presented here resembles the main ideas from [FR02, RS04, Sze78] combined with some techniques from [NRS06a]. In the next section we recall the concept of an *index* of a partition (cf. Definition 4.43) and derive some facts about it. We then give the proof of RL($k+1$) in Section 4.5.2.

4.5.1 The index of a partition

The following propositions center around the notion of an *index*. Throughout this section we will work under the following setup.

Setup 4.42. Let $\mathscr{R}^{(1)}_0$ be a fixed partition of some vertex set V and $\mathscr{H}^{(k+1)}$ be a partition of $\binom{V}{k}$. Moreover, let $\mathscr{X}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$, i.e., for every $X^{(k)} \in \mathscr{X}^{(k)}$ we have $X^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ or $X^{(k)} \cap \mathrm{Cross}_k(\mathscr{R}^{(1)}_0) = \varnothing$. Let $U(\mathscr{X}^{(k)}) = \bigcup\{X^{(k)}\colon\ X^{(k)} \in \mathscr{X}^{(k)}\} \supseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ be the set of k-tuples partitioned by $\mathscr{X}^{(k)}$.

For any $K \in U(\mathscr{X}^{(k)})$ let $X^{(k)}(K)$ be that partition class of $\mathscr{X}^{(k)}$ which contains K, i.e.,

$$X^{(k)}(K) = X^{(k)} \in \mathscr{X}^{(k)} \quad \text{so that} \quad K \in X^{(k)}\,.$$

Moreover, for every $(k+1)$-tuple $K' \in \binom{V}{k+1}$ satisfying $\binom{K'}{k} \subseteq U(\mathscr{X}^{(k)})$ we set

$$\hat{X}^{(k)}(K') = \bigcup \left\{ X^{(k)}(K) \colon K \in \binom{K'}{k} \right\}$$

and

$$\hat{\mathscr{X}}^{(k)} = \left\{ \hat{X}^{(k)}(K') \colon K' \in \binom{V}{k+1} \text{ s.t. } \binom{K'}{k} \subseteq U(\mathscr{X}^{(k)}) \right\}.$$

Note that every $K' \in \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})$ satisfies $\binom{K'}{k} \subseteq U(\mathscr{X}^{(k)})$, since $\mathscr{X}^{(k)}$ refines $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. □

We then define the index of a partition $\mathscr{X}^{(k)}$ (satisfying the above setup) with respect to $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ as follows.

Definition 4.43 (Index). *For V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{X}^{(k)}$ as in Setup 4.42. We set the index of $\mathscr{X}^{(k)}$ w.r.t. $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ equal to*

$$\begin{aligned} ind(\mathscr{X}^{(k)}) &= \frac{1}{|V|^{k+1}} \sum_{H^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{K' \in \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})} d^2\big(H^{(k+1)} | \hat{X}^{(k)}(K')\big) \\ &= \frac{1}{|V|^{k+1}} \sum_{H^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\substack{\hat{X}^{(k)} \in \hat{\mathscr{X}}^{(k)} \\ \hat{X}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})}} d^2\big(H^{(k+1)} | \hat{X}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{X}^{(k)})| \,. \end{aligned}$$

The next observation follows straight from the definition of the index.

Fact 4.44. *For all V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{X}^{(k)}$ as in Setup 4.42, $ind(\mathscr{X}^{(k)})$ is bounded between 0 and 1.* □

We now derive a few more propositions related to the index , which allow a simpler presentation of the the proof of $\mathrm{RL}(k+1)$.

Proposition 4.45. *Let V, $\mathscr{R}_0^{(1)}$, and $\mathscr{H}^{(k+1)}$ be given as in Setup 4.42. Suppose $\mathscr{X}^{(k)} = \{X_1^{(k)}, \dots, X_s^{(k)}\}$ and $\mathscr{Y}^{(k)} = \{Y_1^{(k)}, \dots, Y_s^{(k)}\}$ are partitions which refine $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. Moreover, let ν be a given positive real. If for every $\ell \in [s]$ we have*

(i) $|X_\ell^{(k)} \triangle Y_\ell^{(k)}| \le \nu |V|^k$ *and*

(ii) *if* $X_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ *then* $Y_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$,

then

$$ind(\mathscr{Y}^{(k)}) \ge ind(\mathscr{X}^{(k)}) - 3(k+1)s^{k+1}|\mathscr{H}^{(k+1)}|\nu \,. \tag{4.90}$$

Proof. For every $(k+1)$-tuple $I \in \binom{[s]}{k+1}$ we set

$$\hat{X}_I^{(k)} = \bigcup_{i \in I} X_i^{(k)} \quad \text{and} \quad \hat{Y}_I^{(k)} = \bigcup_{i \in I} Y_i^{(k)} .$$

From (i) we infer that for every $I \in \binom{[s]}{k+1}$ we have

$$\Big| |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| \Big| \leq |\mathcal{K}_{k+1}(\hat{X}_I^{(k)}) \triangle \mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| \leq \nu(k+1)|V|^{k+1} . \quad (4.91)$$

Suppose the partition classes of $\mathscr{H}^{(k+1)}$ are labeled $H_1^{(k+1)}, \ldots, H_h^{(k+1)}$. For a more concise notation we set for every $I \in \binom{[s]}{k+1}$ and $\zeta \in [h]$

$$d(\zeta | \hat{X}_I^{(k)}) = d(H_\zeta^{(k+1)} | \hat{X}_I^{(k)})$$

and

$$d(\zeta | \hat{Y}_I^{(k)}) = d(H_\zeta^{(k+1)} | \hat{Y}_I^{(k)}) .$$

The triangle inequality and (4.91) gives for every $I \in \binom{[s]}{k+1}$ and $\zeta \in [h]$

$$\begin{aligned}
\Big| |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| d^2(\zeta|\hat{X}_I^{(k)}) &- |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| d^2(\zeta|\hat{Y}_I^{(k)}) \Big| \\
&\leq \Big| |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| d(\zeta|\hat{X}_I^{(k)}) - |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| d(\zeta|\hat{Y}_I^{(k)}) \Big| d(\zeta|\hat{X}_I^{(k)}) \\
&\qquad + \Big| |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| \Big| d(\zeta|\hat{Y}_I^{(k)}) d(\zeta|\hat{X}_I^{(k)}) \\
&\qquad + \Big| |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| d(\zeta|\hat{X}_I^{(k)}) - |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| d(\zeta|\hat{Y}_I^{(k)}) \Big| d(\zeta|\hat{Y}_I^{(k)}) \\
&\leq \Big| |H_\zeta^{(k+1)} \cap \mathcal{K}_{k+1}(\hat{X}_I^{(k)})| - |H_\zeta^{(k+1)} \cap \mathcal{K}_{k+1}(\hat{X}_I^{(k)})| \Big| \\
&\qquad + \Big| |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| \Big| \\
&\qquad + \Big| |H_\zeta^{(k+1)} \cap \mathcal{K}_{k+1}(\hat{X}_I^{(k)})| - |H_\zeta^{(k+1)} \cap \mathcal{K}_{k+1}(\hat{X}_I^{(k)})| \Big| \\
&\leq 3 \Big| |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| \Big| \\
&\leq 3\nu(k+1)|V|^{k+1} . \qquad (4.92)
\end{aligned}$$

Now let $\hat{\mathscr{X}}^{(k)}$ and $\hat{\mathscr{Y}}^{(k)}$ be defined as in Setup 4.42. Clearly, for every $\hat{X}^{(k)} \in \hat{\mathscr{X}}^{(k)}$ there exist a unique $I \in \binom{[s]}{k+1}$ so that $\hat{X}^{(k)} = \hat{X}_I^{(k)}$, while the converse fails to be true in general. We define

$$S(\mathscr{X}^{(k)}) = \left\{ I \in \binom{[s]}{k+1} : \ \hat{X}_I^{(k)} \in \hat{\mathscr{X}}^{(k)} \text{ and } \hat{X}_I^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)}) \right\} .$$

Then we have

$$\operatorname{ind}(\mathscr{X}^{(k)}) = \frac{1}{|V|^{k+1}} \sum_{\zeta\in[h]} \sum_{I\in S(\mathscr{X}^{(k)})} |\mathcal{K}_{k+1}(\hat{X}_I^{(k)})| d^2(\zeta|\hat{X}_I^{(k)})$$

and applying (4.92) for every $\zeta \in [h]$ and $I \in S(\mathscr{X}^{(k)})$ yields

$$\begin{aligned}&\operatorname{ind}(\mathscr{X}^{(k)})\\ &\quad\leq \frac{1}{|V|^{k+1}} \sum_{\zeta\in[h]} \sum_{I\in S(\mathscr{X}^{(k)})} |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| d^2(\zeta|\hat{Y}_I^{(k)}) + 3\nu(k+1)h|S(\mathscr{X}^{(k)})| \end{aligned} \tag{4.93}$$

Due to assumption (ii) we have that $\hat{Y}_I^{(k)} \subseteq \operatorname{Cross}_k(\mathscr{R}_0^{(1)})$ for every $I \in S(\mathscr{X}^{(k)})$. Consequently, $\hat{Y}_I^{(k)}$ is either in $\hat{\mathscr{Y}}^{(k)}$ or $\mathcal{K}_{k+1}(\hat{Y}_I^{(k)}) = \varnothing$ for every $I \in S(\mathscr{X}^{(k)})$ and, hence,

$$\operatorname{ind}(\mathscr{Y}^{(k)}) \geq \frac{1}{|V|^{k+1}} \sum_{\zeta\in[h]} \sum \Big\{ |\mathcal{K}_{k+1}(\hat{Y}_I^{(k)})| d^2(\zeta|\hat{Y}_I^{(k)}) \colon I \in S(\mathscr{X}^{(k)}) \Big\}.$$

Therefore, the last inequality combined with (4.93) implies

$$\operatorname{ind}(\mathscr{X}^{(k)}) \leq \operatorname{ind}(\mathscr{Y}^{(k)}) + 3\nu(k+1)hs^{k+1},$$

which concludes the proof of Proposition 4.45. □

The following proposition is a simple consequence of Jensen's inequality.

Proposition 4.46. *Suppose $\hat{Y}^{(k)}$ is an $(m, k+1, k)$-hypergraph and $\{\hat{Z}_1^{(k)}, \dots, \hat{Z}_z^{(k)}\}$ is is a family of $(m, k+1, k)$-hypergraphs such that the family $\{\mathcal{K}_{k+1}(\hat{Z}_i^{(k)}) \colon i \in [z]\}$ partitions $\mathcal{K}_{k+1}(\hat{Y}^{(k)})$, then*

$$d^2(H^{(k+1)}|\hat{Y}^{(k)})|\mathcal{K}_{k+1}(\hat{Y}^{(k)})| \leq \sum_{i\in[z]} d^2(H^{(k+1)}|\hat{Z}_i^{(k)})|\mathcal{K}_{k+1}(\hat{Z}_i^{(k)})| \tag{4.94}$$

for every hypergraph $H^{(k+1)} \subseteq \mathcal{K}_{k+1}(\hat{Y}^{(k)})$.

Proof. For all $K' \in \mathcal{K}_{k+1}(\hat{Y}^{(k)})$ let $\hat{Z}^{(k)}(K')$ be the unique member from $\{\hat{Z}_1^{(k)}, \dots, \hat{Z}_z^{(k)}\}$ with the property $K' \in \mathcal{K}_{k+1}(\hat{Z}^{(k)}(K'))$. Then we have

$$\begin{aligned} d(H^{(k+1)}|\hat{Y}^{(k)}) &= \frac{\sum_{i\in[z]} d(H^{(k+1)}|\hat{Z}_i^{(k)})|\mathcal{K}_{k+1}(\hat{Z}_i^{(k)})|}{|\mathcal{K}_{k+1}(\hat{Y}^{(k)})|}\\ &= \frac{\sum_{K'\in\mathcal{K}_{k+1}(\hat{Y}^{(k)})} d(H^{(k+1)}|\hat{Z}^{(k)}(K'))}{|\mathcal{K}_{k+1}(\hat{Y}^{(k)})|}, \end{aligned}$$

and Jensen's inequality yields (4.94), since

$$d^2(H^{(k+1)}|\hat{Y}^{(k)})|\mathcal{K}_{k+1}(\hat{Y}^{(k)})| = \frac{\left(\sum_{K'\in\mathcal{K}_{k+1}(\hat{Y}^{(k)})} d(H^{(k+1)}|\hat{Z}^{(k)}(K'))\right)^2}{|\mathcal{K}_{k+1}(\hat{Y}^{(k)})|} \leq \sum_{K'\in\mathcal{K}_{k+1}(\hat{Y}^{(k)})} d^2(H^{(k+1)}|\hat{Z}^{(k)}(K')) = \sum_{i\in[z]} d^2(H^{(k+1)}|\hat{Z}_i^{(k)})|\mathcal{K}_{k+1}(\hat{Z}_i^{(k)})|\,.$$

□

The following proposition is a corollary of Proposition 4.46 and asserts that the refinement of a family of partitions has the same or bigger index.

Proposition 4.47. *Let V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{Y}^{(k)}$ be given as in Setup 4.42 and let $\mathscr{Z}^{(k)}$ be a partition refining $Cross_k(\mathscr{R}_0^{(1)})$. If $\mathscr{Z}^{(k)} \prec \mathscr{Y}^{(k)}$, then $ind(\mathscr{Y}^{(k)}) \leq ind(\mathscr{Z}^{(k)})$.*

Proof. We observe that for every $\hat{Y}^{(k)} \in \hat{\mathscr{Y}}^{(k)}$ satisfying $\hat{Y}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ the family

$$\{\mathcal{K}_{k+1}(\hat{Z}^{(k)}):\ \hat{Z}^{(k)} \in \hat{\mathscr{Z}}^{(k)} \text{ and } \hat{Z}^{(k)} \subseteq \hat{Y}^{(k)}\}$$

partitions $\mathcal{K}_{k+1}(\hat{Y}^{(k)})$. Consequently, we can apply Proposition 4.46 to every $H^{(k+1)} \in \mathscr{H}^{(k+1)}$ and $\hat{Y}^{(k)} \in \hat{\mathscr{Y}}^{(k)}$, which yields the proposition. □

In the proof of RL($k+1$) we will also deal with partitions which "almost" refine each other (see Definition 4.48 below) and we need approximations of their index (Proposition 4.49).

Definition 4.48. *Given V, $\mathscr{R}_0^{(1)}$, and $\mathscr{Z}^{(k)}$ as in Setup 4.42. Moreover, let $\beta \geq 0$ and let $\mathscr{T}^{(k)}$ be a partition refining $Cross_k(\mathscr{R}_0^{(1)})$. We say the partition $\mathscr{T}^{(k)}$ is a β-*refinement *of $\mathscr{Z}^{(k)}$ if*

$$\sum\left\{|T^{(k)}|:\ T^{(k)} \in \mathscr{T}^{(k)},\ T^{(k)} \not\subseteq Z^{(k)} \text{ for every } Z^{(k)} \in \mathscr{Z}^{(k)}\right\} \leq \beta|V|^k\,.$$

The following proposition extends Proposition 4.47. A very similar statement appeared in [FR02, Lemma 3.6].

Proposition 4.49. *Let V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{Z}^{(k)}$ be given as in Setup 4.42, let $\mathscr{T}^{(k)}$ be a β-refinement of $\mathscr{Z}^{(k)}$ for some $\beta \geq 0$. Then*

$$ind(\mathscr{T}^{(k)}) \geq ind(\mathscr{Z}^{(k)}) - \beta\,.$$

Proof. We first define an auxiliary partition $\mathscr{S}^{(k)}$ which is a refinement of $\mathscr{T}^{(k)}$ and $\mathscr{Z}^{(k)}$. For that set

$$\mathscr{S}^{(k)} = \left\{T^{(k)} \cap Z^{(k)}:\ T^{(k)} \in \mathscr{T}^{(k)} \quad\text{and}\quad Z^{(k)} \in \mathscr{Z}^{(k)}\right\}.$$

Due to Proposition 4.47 we have

$$\operatorname{ind}(\mathscr{Z}^{(k)}) \leq \operatorname{ind}(\mathscr{S}^{(k)}) \,. \tag{4.95}$$

Let $\hat{\mathscr{T}}_0^{(k)}$ be the family of those polyads $\hat{T}^{(k)} \in \hat{\mathscr{T}}^{(k)}$ which are sub-hypergraphs of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ and for which there exists a $T^{(k)} \in \mathscr{T}^{(k)}$ such that

$$T^{(k)} \subseteq \hat{T}^{(k)} \qquad \text{and} \qquad T^{(k)} \nsubseteq Z^{(k)} \text{ for all } Z^{(k)} \in \mathscr{Z}^{(k)} \,.$$

Since $\mathscr{H}^{(k+1)}$ is a partition of $\binom{V}{k+1}$ and $\mathscr{T}^{(k)}$ is a β-refinement of $\mathscr{Z}^{(k)}$ we have

$$\sum_{H^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\hat{T}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \sum_{\substack{\hat{S}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{S}^{(k)} \subseteq \hat{T}^{(k)}}} d(H^{(k+1)} | \hat{S}^{(k)}) |\mathcal{K}_{k+1}(\hat{S}^{(k)})| \leq \beta |V|^{k+1} \,. \tag{4.96}$$

Note that for every $\hat{T}^{(k)} \notin \hat{\mathscr{T}}_0^{(k)}$ there exist some $\hat{S}^{(k)} \in \hat{\mathscr{S}}^{(k)}$ such that $\hat{S}^{(k)} = \hat{T}^{(k)}$. Consequently,

$$\begin{aligned}
&\operatorname{ind}(\mathscr{S}^{(k)}) - \operatorname{ind}(\mathscr{T}^{(k)}) \\
&\quad = \frac{1}{|V|^{k+1}} \sum_{H^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\hat{T}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \Bigg(\sum_{\substack{\hat{S}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{S}^{(k)} \subseteq \hat{T}^{(k)}}} d^2(H^{(k+1)} | \hat{S}^{(k)}) |\mathcal{K}_{k+1}(\hat{S}^{(k)})| \\
&\qquad\qquad - d^2(H^{(k+1)} | \hat{T}^{(k)}) |\mathcal{K}_{k+1}(\hat{T}^{(k)})| \Bigg) \\
&\quad \leq \frac{1}{|V|^{k+1}} \sum_{H^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\hat{T}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \sum_{\substack{\hat{S}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{S}^{(k)} \subseteq \hat{T}^{(k)}}} d(H^{(k+1)} | \hat{S}^{(k)}) |\mathcal{K}_{k+1}(\hat{S}^{(k)})| \\
&\quad \overset{(4.96)}{\leq} \beta \,,
\end{aligned}$$

and the proposition follows from (4.95). □

The last proposition in this section concerns the index of a family of partitions $\mathscr{R}$ failing to satisfy (H) of $\mathrm{RL}(k+1)$. It can be shown that a certain refinement of $\mathscr{R}$ has an index of at least the index of $\mathscr{R}$ plus some positive constant depending on δ_{k+1}. This observation is the crucial idea in the proof of $\mathrm{RL}(k+1)$. Since, it roughly shows (together with Fact 4.44) that there are only finitely many refinements which violate (H). The same idea was already used in [FR02, RS04, Sze78].

Proposition 4.50. *Let V, $\mathscr{R}_0^{(1)}$, and $\mathscr{H}^{(k+1)}$ be given as in Setup 4.42 and let $\mathscr{R}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. Moreover, let δ be a positive real and $r \geq 1$ be an integer. If*

$$|\,\mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})\,| \geq \left(1 - \frac{\delta}{2}\right) \binom{|V|}{k+1} \tag{4.97}$$

and if there is some $H^{(k+1)}_{irr} \in \mathscr{H}^{(k+1)}$ which is $(\delta, *, r)$-irregular[4] *w.r.t. $\mathscr{R}^{(k)}$, then there exists a partition $\mathscr{X}^{(k)}$ of $\binom{V}{k+1}$ satisfying*

(*i*) $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}$,

(*ii*) $|\mathscr{X}^{(k)}| \leq |\mathscr{R}^{(k)}| \times 2^{r \times |\hat{\mathscr{R}}^{(k)}|}$, *and*

(*iii*) $ind(\mathscr{X}^{(k)}) \geq ind(\mathscr{R}^{(k)}) + \delta^4/2$.

In the proof of Proposition 4.50 we will use the defect form of the Cauchy–Schwarz inequality, which we state first (see, e.g., [Sze78] and Lemma 2.4 for similar statements).

Proposition 4.51 (Defect Cauchy–Schwarz inequality)**.** *Suppose $\varnothing \neq J \subsetneq I$ are some index sets and $d_i \geq 0$ is some non-negative real number for every $i \in I$. If*

$$\frac{1}{|J|}\sum_{j \in J} d_j = \frac{1}{|I|}\sum_{i \in I} d_i + \alpha \tag{4.98}$$

for some (not necessarily non-negative) real α and if $|\alpha| \geq \delta$ and $|J| \geq \delta |I|$ for some $\delta \geq 0$, then

$$\sum_{i \in I} d_i^2 \geq \frac{1}{|I|}\left(\sum_{i \in I} d_i\right)^2 + \delta^3 |I| \,.$$

□

Proof of Proposition 4.50. Let $\hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ be the set of those polyads $\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ such that

$$H^{(k+1)}_{\mathrm{irr}} \text{ is } (\delta, *, r)\text{-irregular w.r.t. } \hat{R}^{(k)} \tag{4.99}$$

and

$$\hat{R}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0). \tag{4.100}$$

From the definition $(\delta, *, r)$-regularity w.r.t. $\mathscr{R}^{(k)}$ (see footnote 4) and (4.97) we infer that

$$\sum \left\{ |\mathcal{K}_k(\hat{R}^{(k)})| \colon \hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0} \right\} \geq \frac{\delta}{2} |V|^{k+1} \,. \tag{4.101}$$

For each $\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ there exist a witness of irregularity, i.e., there exists

$$\hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)}) = \{\hat{Q}^{(k)}_1, \ldots, \hat{Q}^{(k)}_r\}$$

[4] Strictly speaking in Definition 4.14 we only defined the regularity with respect to a family of partitions while here we only have a partition $\mathscr{R}^{(k)}$ of k-tuples. However, we can easily alter the definition based on $\hat{\mathscr{R}}^{(k)}$ meaning that $H^{(k+1)}$ is $(\delta, *, r)$-regular w.r.t. $\mathscr{R}^{(k)}$ if $\big|\bigcup\big\{\mathcal{K}_k(\hat{R}^{(k)}) \colon \hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ and $H^{(k+1)}$ is not $(\delta, *, r)$-regular w.r.t. $\hat{R}^{(k)}\big\}\big| \leq \delta |V|^{k+1}$.

such that $\hat{Q}_i^{(k)} \subseteq \hat{R}^{(k)}$ for every $i \in [r]$ and

$$\Big|\bigcup_{i\in[r]} \mathcal{K}_k(\hat{Q}_i^{(k)})\Big| \geq \delta\Big|\mathcal{K}_{k+1}(\hat{R}^{(k)})\Big| > 0\,, \tag{4.102}$$

$$d\big(H_{\mathrm{irr}}^{(k+1)}|\hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)})\big) = d\big(H_{\mathrm{irr}}^{(k+1)}|\hat{R}^{(k)})\big| + \alpha_{\hat{R}^{(k)}} \tag{4.103}$$

for some $\alpha_{\hat{R}^{(k)}}$ with $|\alpha_{\hat{R}^{(k)}}| > \delta$, where

$$d\big(H^{(k)}|\hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)})\big) = \frac{|H_{\mathrm{irr}}^{(k+1)} \cap \bigcup_{i\in[r]} \mathcal{K}_{k+1}(\hat{Q}^{(k)})|}{|\bigcup_{i\in[r]} \mathcal{K}_{k+1}(\hat{Q}^{(k)})|}\,.$$

Moreover, for every hypergraph $R^{(k)} \in \mathscr{R}^{(k)}$ we define the family $\boldsymbol{W}^{(k)}(R^{(k)})$ of those sub-hypergraphs of $R^{(k)}$ which are contained in some witness $\hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)})$ and for which $R^{(k)} \subseteq \hat{R}^{(k)}$. More precisely we set

$$\boldsymbol{W}^{(k)}(R^{(k)}) = \big\{R^{(k)} \cap \hat{Q}^{(k)} \colon \hat{R}^{(k)} \in \hat{\mathscr{R}}_{\mathrm{irr},0}^{(k)} \text{ with } R^{(k)} \subseteq \hat{R}^{(k)} \text{ and } \hat{Q}^{(k)} \in \hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)})\big\}\,.$$

We observe that $\boldsymbol{W}^{(k)}(R^{(k)})$ might be empty (e.g., if $R^{(k)} \nsubseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$), that the hypergraphs in $\boldsymbol{W}^{(k)}(R^{(k)})$ are not necessarily disjoint, and that for every $R^{(k)} \in \mathscr{R}^{(k)}$ we have the following trivial upper bound $w_{R^{(k)}}$ on the number of hypergraphs in $\boldsymbol{W}^{(k)}(R^{(k)})$

$$w_{R^{(k)}} = |\boldsymbol{W}^{(k)}(R^{(k)})| \leq r \times |\hat{\mathscr{R}}^{(k)}|\,. \tag{4.104}$$

We now define the promised refinement $\mathscr{X}^{(k)}$ of $\mathscr{R}^{(k)}$. We construct $\mathscr{X}^{(k)}$ for each $R^{(k)} \in \mathscr{R}^{(k)}$ separately. This partition of $R^{(k)}$ will be called $\mathscr{X}^{(k)}(R^{(k)})$ and is given by the atoms arising from the intersection of the hypergraphs in $\boldsymbol{W}^{(k)}(R^{(k)})$ (i.e., the regions of the Venn diagram of the family $\boldsymbol{W}^{(k)}(R^{(k)})$). More precisely, if $\boldsymbol{W}^{(k)}(R^{(k)}) \neq \varnothing$ let

$$\boldsymbol{W}^{(k)}(R^{(k)}) = \{W_i^{(k)} \colon\ i \in [w_{R^{(k)}}]\}$$

be some enumeration of the elements of $\boldsymbol{W}^{(k)}(R^{(k)})$ and set

$$\mathscr{X}^{(k)}(R^{(k)}) = \Big\{\bigcap_{i\in I} W_i^{(k)} \cap \bigcap_{i\in I^c} (R^{(k)} \setminus W_i^{(k)}) \colon\ \{I, I^c\} \text{ partitions } [w_{R^{(k)}}]\Big\}\,.$$

If $\boldsymbol{W}^{(k)}(R^{(k)}) = \varnothing$, then we set

$$\mathscr{X}^{(k)}(R^{(k)}) = \{R^{(k)}\}\,.$$

Collecting "contributions" for every $R^{(k)} \in \mathscr{R}^{(k)}$ in that way defines $\mathscr{X}^{(k)}$

$$\mathscr{X}^{(k)} = \bigcup\Big\{\mathscr{X}^{(k)}(R^{(k)}) \colon\ R^{(k)} \in \mathscr{R}^{(k)}\Big\}\,.$$

Owing to the construction above, the partition $\mathscr{X}^{(k)}$ clearly refines $\mathscr{R}^{(k)}$, i.e., it satisfies (i) of Proposition 4.50. Moreover, (4.104) and the construction yields (ii) of the proposition.

It is left to verify (iii). For that we first fix some $\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ and consider the witness of irregularity $\hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)}) = \{\hat{Q}^{(k)}_1, \dots, \hat{Q}^{(k)}_r\}$. Since, $\mathscr{X}^{(k)}$ refines $\mathscr{R}^{(k)}$ it satisfies the assumptions of Setup 4.42 with V, $\mathscr{R}^{(1)}_0$, and $\mathscr{H}^{(k+1)}$. In particular, the family of polyads $\hat{\mathscr{X}}^{(k)}$ is well defined and for every $K' \in \mathcal{K}_{k+1}(\hat{R}^{(k)})$ there exist a $\hat{X}^{(k)}(K') \in \hat{\mathscr{X}}^{(k)}$ so that $K' \in \mathcal{K}_{k+1}(\hat{X}^{(k)})$. We are heading towards an application of Proposition 4.51 with

$$I = \mathcal{K}_{k+1}(\hat{R}^{(k)}))\,, \quad J = \bigcup_{i=1}^{r} \mathcal{K}_{k+1}(\hat{Q}^{(k)}_i)\,, \quad \text{and} \quad d_{K'} = d(H^{(k+1)}_{\mathrm{irr}} | \hat{X}^{(k)}(K')) \tag{4.105}$$

for every $K' \in I$ and verify (4.98) below for $\alpha_{\hat{R}^{(k)}}$ and the choice above

$$\begin{aligned} \frac{1}{|J|} \sum_{K' \in J} d_{K'} &\overset{(4.105)}{=} d(H^{(k+1)}_{\mathrm{irr}} | \hat{\boldsymbol{Q}}^{(k)}(\hat{R}^{(k)})) \\ &\overset{(4.103)}{=} d(H^{(k+1)}_{\mathrm{irr}} | \hat{R}^{(k)}) + \alpha_{\hat{R}^{(k)}} \\ &\overset{(4.105)}{=} \frac{1}{|I|} \sum_{K' \in I} d_{K'} + \alpha_{\hat{R}^{(k)}}\,. \end{aligned}$$

Since, $|\alpha_{\hat{R}^{(k)}}| \geq \delta$ (cf. (4.103)) and

$$|J| \overset{(4.105)}{=} \Big| \bigcup_{i=1}^{r} \mathcal{K}_{k+1}(\hat{Q}^{(k)}_i) \Big| \overset{(4.102)}{\geq} \delta \Big| \mathcal{K}_{k+1}(\hat{R}^{(k)}) \Big| \overset{(4.105)}{=} \delta |I|\,,$$

Proposition 4.51 yields

$$\sum_{K' \in I} d^2_{K'} \geq \frac{1}{|I|} \Big(\sum_{K' \in I} d_{K'} \Big)^2 + \delta^3 |I|\,. \tag{4.106}$$

In view of (4.105) and since

$$\begin{aligned} \sum_{K' \in \mathcal{K}_{k+1}(\hat{R}^{(k)})} d(H^{(k+1)}_{\mathrm{irr}} | \hat{X}^{(k)}(K')) &= \sum_{\substack{\hat{X}^{(k)} \in \hat{\mathscr{X}}^{(k)} \\ \hat{X}^{(k)} \subseteq \hat{R}^{(k)}}} d(H^{(k+1)}_{\mathrm{irr}} | \hat{X}^{(k)}) |\mathcal{K}_{k+1}(\hat{X}^{(k)})| \\ &= d(H^{(k+1)}_{\mathrm{irr}} | \hat{R}^{(k)}) |\mathcal{K}_{k+1}(R^{(k)})| \end{aligned}$$

we can reformulate inequality (4.106) to

$$\sum_{K' \in \mathcal{K}_{k+1}(\hat{R}^{(k)})} d^2(H^{(k+1)}_{\mathrm{irr}} | \hat{X}^{(k)}(K')) \geq \sum_{K' \in \mathcal{K}_{k+1}(\hat{R}^{(k)})} \Big(d^2(H^{(k+1)}_{\mathrm{irr}} | \hat{R}^{(k)}) + \delta^3 \Big)\,. \tag{4.107}$$

Note that (4.107) holds for every irregular polyad $\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$. Summing over all such polyads inequality (4.107) together with (4.101) yields

$$\sum_{\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}} \sum_{K' \in \mathcal{K}_{k+1}(\hat{R}^{(k)})} d^2(H^{(k+1)}_{\mathrm{irr}} | \hat{X}^{(k)}(K'))$$
$$\geq \sum_{\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}} \sum_{K' \in \mathcal{K}_{k+1}(\hat{R}^{(k)})} d^2(H^{(k+1)}_{\mathrm{irr}} | \hat{R}^{(k)}) + \frac{\delta^4}{2} |V|^{k+1} .$$

Since $\mathscr{X}^{(k)}$ refines $\mathscr{R}^{(k)}$, we can apply Proposition 4.46 to every $\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)} \setminus \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ which is contained in $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ and we infer

$$\sum_{K' \in \mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)} d^2(H^{(k+1)}_{\mathrm{irr}} | \hat{X}^{(k)}(K')) \geq \sum_{K' \in \mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)} d^2(H^{(k+1)}_{\mathrm{irr}} | \hat{R}(K')) + \frac{\delta^4}{2} |V|^{k+1} .$$

Finally, part (*iii*) of Proposition 4.50 follows from the last inequality and Proposition 4.46 applied to every $H^{(k+1)} \in \mathscr{H}^{(k+1)}$, $H^{(k+1)} \neq H^{(k+1)}_{\mathrm{irr}}$ and every $\hat{R}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ contained in $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$. □

4.5.2 Proof of RL$(k+1)$

In what follows we give a proof of RL$(k+1)$ based on RAL(k), or more precisely, based on Lemma 4.36. In the next section, Section 4.5.2, we define all constants involved in the proof of this implication. In Section 4.5.2 we state the so called *index pumping lemma* and deduce RL$(k+1)$ from it. We then prove the index pumping lemma in Section 4.5.3.

Constants

We first recall the quantification of RL$(k+1)$, Lemma 4.34 for $k+1$

$$\forall\, o_{\mathrm{RL}},\, s_{\mathrm{RL}},\, \eta_{\mathrm{RL}},\, \delta_{k+1,\mathrm{RL}},\, r_{\mathrm{RL}} \colon \mathbb{N}^k \to \mathbb{N},\, \delta_{\mathrm{RL}} \colon \mathbb{N}^k \to (0,1] \quad \exists\, \mu_{\mathrm{RL}} > 0,\, t_{\mathrm{RL}},\, n_{\mathrm{RL}} .$$

So let positive integers o_{RL} and s_{RL}, positive reals η_{RL} and $\delta_{k+1,\mathrm{RL}}$, and positive functions r_{RL} and δ_{RL} be given. Without loss of generality we assume that

$$\eta_{\mathrm{RL}} \leq \delta_{k+1,\mathrm{RL}}/2 \text{ and } r_{\mathrm{RL}} \text{ and } \delta_{\mathrm{RL}} \text{ are monotone in every variable.} \tag{4.108}$$

For the definition of the promised constants μ_{RL}, t_{RL}, and n_{RL} we need auxiliary sequences of constants t_i, o_i, s_i, η_i, and ν_i and a sequence of functions $\varepsilon_i \colon \mathbb{N}^{k-1} \to (0,1]$ for $i \geq 0$. First we define t_0

$$t_0 = \min\left\{ t \geq \left\lceil \tfrac{(k+1)^{k+1}}{2\eta_{\mathrm{RL}}} \right\rceil \colon\ (o_{\mathrm{RL}})! \text{ divides } t \right\} > o_{\mathrm{RL}} . \tag{4.109}$$

Without loss of generality we may assume that the given function δ_{RL} is bounded for every $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ by

$$\delta_{\mathrm{RL}}(\boldsymbol{a}) \le \frac{\delta^4_{k+1,\mathrm{RL}}}{24t_0} < \frac{2}{t_0} \quad \text{and} \quad \delta_{\mathrm{RL}}(\boldsymbol{a}) \le \frac{3}{2a_k}\,. \tag{4.110}$$

For convenience we define the following integer-valued function $f\colon \mathbb{N} \to \mathbb{N}$

$$f(s) = \min\left\{x \in \mathbb{N}\colon\ x \ge \frac{24t_0 s}{\delta^4_{k+1,\mathrm{RL}}} \text{ and } (t_0)! \text{ divides } x\right\}. \tag{4.111}$$

We then define o_i, s_i, η_i, and ν_i in terms of t_i, $\delta_{k+1,\mathrm{RL}}$, η_{RL}, and $r_{\mathrm{RL}}(t_i, \ldots, t_i)$

$$o_i = t_0\,, \quad s_i = t_i^{2^k} 2^{r_{\mathrm{RL}}(t_i,\ldots,t_i)t_i^{2^{k+1}}}\,, \quad \eta_i = \eta_{\mathrm{RL}}\,, \quad \text{and} \quad \nu_i = \frac{\delta^4_{k+1,\mathrm{RL}}}{12(k+1)s_{\mathrm{RL}}s_i^{k+1}}\,. \tag{4.112}$$

Moreover, for $i \ge 0$ we define the function $\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1]$ defined for every vector $\boldsymbol{a} = (a_1, \ldots, a_{k-1}) \in \mathbb{N}^{k-1}$ as

$$\varepsilon_i(\boldsymbol{a}) = \min\left\{\frac{\delta_{\mathrm{RL}}(a_1, \ldots, a_{k-1}, f(s_i))}{18s_i}\,,\right.$$
$$\left.\varepsilon_{\mathrm{DCL}}\left(k-1, k, \frac{1}{2}, \min_{2\le j\le k-1}\frac{1}{a_j}\right), \frac{1}{2f(s_i)}, \frac{\delta^4_{k+1,\mathrm{RL}}}{72s_i t_0}\right\}, \tag{4.113}$$

where $\varepsilon_{\mathrm{DCL}}$ is given by Theorem 4.19. With out loss of generality we assume that ε_i is monotone in every variable.

We then define t_{i+1} using $t_{\mathrm{RAL}}(o, s, \eta, \nu, \varepsilon(\cdot, \ldots, \cdot))$ given by Lemma 4.36 and set

$$t_{i+1} = \max\left\{t_i\,, t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \ldots, \cdot))\,, f(s_i)\right\} \overset{(4.111)}{\ge} s_i\,. \tag{4.114}$$

This concludes the definition of the sequences t_i, o_i, s_i, η_i, ν_i and $\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1]$ for $i \ge 0$. We note that the sequence t_i is monotone by definition. In a similar way we define the monotone sequences μ_i for $i \ge 1$ by setting $\mu_1 = \delta_{\mathrm{RL}}(t_0, \ldots, t_0)$ and

$$\mu_{i+1} = \min\left\{\mu_i\,, \mu_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \ldots, \cdot))\,, \frac{\delta_{\mathrm{RL}}(\overbrace{t_{i+1}, \ldots, t_{i+1}}^{(k-1)\text{-times}}, f(s_i))}{12t_{i+1}^{2^k}}\right\} \tag{4.115}$$

and we define n_i by setting $n_1 = 1$ and

$$\begin{aligned} n_{i+1} = \max\Big\{ & n_i\,, n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot))\,, t_{i+1} m_{\mathrm{DCL}}(k-1, k, \tfrac{1}{2}, \tfrac{1}{t_{i+1}})\,, \\ & t_{i+1} m_{\mathrm{SL}}\Big(k, f(s_i), 1, \varepsilon_i(t_{i+1}, \dots, t_{i+1}), \tfrac{1}{f(s_i)}, \tfrac{1}{f(s_i)}\Big)\,, \\ & t_{i+1} m_{\mathrm{SL}}\Big(k, f(s_i), 1, \tfrac{1}{3}\delta_{\mathrm{RL}}(t_{i+1}, \dots, t_{i+1}, f(s_i)), \\ & \qquad \tfrac{1}{2}\delta_{\mathrm{RL}}(t_{i+1}, \dots, t_{i+1}, f(s_i)), \tfrac{1}{f(s_i)}\Big)\Big\}\,. \end{aligned} \tag{4.116}$$

We also define auxiliary constants

$$\begin{aligned} \mu^* &= \min\Big\{ \min_{2 \le j \le k} \{\varepsilon_{\mathrm{DCL}}(j, j+1, \tfrac{1}{2}, \tfrac{1}{t_0})\}\,,\ \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil} \Big\}\,, \\ n^* &= \max_{2 \le j \le k} \max\Big\{ t_0 m_{\mathrm{DCL}}(j, j+1, \tfrac{1}{2}, \tfrac{1}{t_0})\,, t_0 m_{\mathrm{SL}}(j+1, o_{\mathrm{RL}}, 1, \tfrac{\mu^*}{3}, 1, \tfrac{1}{o_{\mathrm{RL}}}) \Big\}\,. \end{aligned} \tag{4.117}$$

Finally, we fix the constants μ_{RL}, t_{RL}, and n_{RL} promised by Lemma 4.34 in the following way

$$\mu_{\mathrm{RL}} = \mu^*/(2t_0^{2^k})\,, \quad t_{\mathrm{RL}} = t_{\lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil}\,, \quad \text{and} \quad n_{\mathrm{RL}} = \max\{n^*, n_{\lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil}\}\,. \tag{4.118}$$

For the rest of this section let all constants and functions be fixed as stated in (4.109)–(4.118).

The index pumping lemma

Now let a set V, a family of partitions $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ and a family of $(k+1)$-uniform hypergraphs $\mathscr{H}^{(k+1)}$ satisfying the assumptions (a)–(c) of RL$(k+1)$ be given, i.e.,

(RL.a) $|V| = n \ge n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divided n,

(RL.b) $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and o_{RL}-bounded family of partitions on V, and

(RL.c) $\mathscr{H}^{(k+1)} = \{H_1^{(k+1)}, \dots, H_{s_{\mathrm{RL}}}^{(k+1)}\}$ is a partition of $\binom{V}{k+1}$.

The main idea of the proof is to inductively define a sequence of families of partitions $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ on V for $i \ge 0$, which will satisfy

(R_0.*1*) $\mathscr{R}_0 = \{\mathscr{R}_0^{(j)}\}_{j=1}^k$ is $(\eta_{\mathrm{RL}}, \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil}, \boldsymbol{a}^{\mathscr{R}_0})$-equitable and t_0-bounded,

(R_0.*2*) $\mathscr{R}_0 \prec \mathscr{O}$,

(R_i.*1*) $\mathscr{R}_i = \{\mathscr{R}_i^{(j)}\}_{j=1}^k$ is an $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i}), \boldsymbol{a}^{\mathscr{R}_i})$-equitable and t_i-bounded, and

(R_i.*2*) $\mathscr{R}_i \prec \mathscr{R}_0$.

Note that due to the fact that $\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \le \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_0})$ (cf. (4.115)), a family of partitions $\mathscr{R}_0$ which satisfies (R$_0$.*1*) and (R$_0$.*2*) also satisfies (R$_i$.*1*) and (R$_i$.*2*) for $i = 0$.

Moreover, we will show that if there is a hypergraph $H^{(k+1)} \in \mathscr{H}^{(k+1)}$ which is not $(\delta_{k+1,\mathrm{RL}}, *, r(\boldsymbol{a}^{\mathscr{R}_i}))$-regular w.r.t. $\mathscr{R}_i$, then $\mathscr{R}_{i+1}$ can be chosen in such a way that the index increases by $\delta^4_{k+1,\mathrm{RL}}/8$, More precisely we will show the following so-called *index pumping lemma*, which proof merges some ideas from [RS04] and [NRS06a, Cleaning Phase I].

Lemma 4.52 (Index pumping lemma). *Let $0 \le i < \lceil 8/\delta^4_{k+1,RL}\rceil$ be an integer and let $\mathscr{R}_0$ be a family of partitions satisfying* (R$_0$.*1*) *and* (R$_0$.*2*).

If $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ satisfies (R$_i$.*1*) *and* (R$_i$.*2*) *and fails to satisfy (H) of $RL(k+1)$ for $r(\boldsymbol{a}^{\mathscr{R}_i})$, then there exists a family of partitions $\mathscr{R}_{i+1} = \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}})$ satisfying* (R$_{i+1}$.*1*) *and* (R$_{i+1}$.*2*) *and*

$$ind(\mathscr{R}^{(k)}_{i+1}) \ge ind(\mathscr{R}^{(k)}_i) + \delta^4_{k+1,RL}/8\,, \tag{4.119}$$

where the index is defined with respect to $\mathscr{R}^{(1)}_0$ and $\mathscr{H}^{(k+1)}$.

Next we deduce RL$(k+1)$ (i.e., Lemma 4.34) from Lemma 4.52. We then give the proof of Lemma 4.52 in Section 4.5.3.

Proof of Lemma 4.34. Suppose all constants are fixed as in Section 4.5.2 and let V, $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$, and $\mathscr{H}^{(k+1)}$ satisfying (RL.a)–(RL.c) be given. We have to ensure the existence of a family of partitions $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}^{\mathscr{P}})$ on V satisfying

(RL.*P1*) $\mathscr{P}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_{RL}-bounded,

(RL.*P2*) $\mathscr{P} \prec \mathscr{O}$, and

(RL.*H*) $H^{(k+1)}_i$ is $(\delta_{k+1,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$ for every $i \in [s_{\mathrm{RL}}]$.

Construction of a family $\mathscr{R}_0$. In view of Lemma 4.52 we first need an appropriate family of partitions $\mathscr{R}_0$. We distinguish two cases depending on the size of $\eta^{\mathscr{O}}$.

Case 1 ($\eta^{\mathscr{O}} \le \eta_{\mathrm{RL}}$). In this case we set $\mathscr{R}_0 = \mathscr{O}$. It then follows from (RL.b) that $\mathscr{R}_0$ is $(\eta_{\mathrm{RL}}, \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}, \boldsymbol{a}^{\mathscr{R}_0})$-equitable, since $\mu_{\mathrm{RL}} \le \mu^* \le \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}$ by (4.117) and (4.118). Also $\mathscr{R}_0 = \mathscr{O}$ is o_{RL}-bounded by (RL.b) and, hence, it is t_0-bounded by (4.109). Therefore, $\mathscr{R}_0$ chosen this way satisfies (R$_0$.*1*). Moreover, (R$_0$.*2*) holds trivially. ◊

Case 2 ($\eta^{\mathscr{O}} > \eta_{\mathrm{RL}}$). We construct a refinement $\mathscr{R}_0$ of $\mathscr{O}$ so that

$$|\,\mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)\,| \ge (1 - \eta_{\mathrm{RL}})\binom{n}{k+1}.$$

We construct $\mathscr{R}_0 = \{\mathscr{R}^{(1)}_0, \dots, \mathscr{R}^{(k)}_0\}$ inductively. More precisely we show for every $j = 1, \dots, k$ that the following statement $(\mathfrak{S}_j)$ holds.

$(\mathfrak{S}_j)$ there exists a $(\eta_{\mathrm{RL}}, \mu^*, (a^{\mathscr{R}_0}_1, \dots, a^{\mathscr{R}_0}_j))$-equitable and t_0-bounded family of partitions $\mathscr{R}_0(j) = \{\mathscr{R}^{(1)}_0, \dots, \mathscr{R}^{(j)}_0\}$ on V, which refines $\mathscr{O}(j) = \{\mathscr{O}^{(1)}, \dots, \mathscr{O}^{(j)}\}$.

Since, $\mu^* \le \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}$ it then follows that there is a family of partitions $\mathscr{R}_0$ so that $(\mathrm{R}_0.\mathit{1})$ and $(\mathrm{R}_0.\mathit{2})$ are satisfied.

Induction start $j = 1$. We split each vertex class $W \in \mathscr{O}^{(1)}$ into $t_0/a_1^{\mathscr{O}}$ classes of size $n/(a_1^{\mathscr{O}} t_0)$, where t_0 is given in (4.109). Note that $t_0/a_1^{\mathscr{O}}$ is an integer by definition of t_0 and $o_{\mathrm{RL}} \ge a_1^{\mathscr{O}}$. Moreover, $n/(a_1^{\mathscr{O}} t_0)$ is an integer due to the choice of $t_{\mathrm{RL}} \ge t_0 > o_{\mathrm{RL}} \ge a_1^{\mathscr{O}}$ (cf. (4.118) and (4.109)) and (RL.a). This defines the partition $\mathscr{R}_0^{(1)}$ with $a_1^{\mathscr{R}_0} = t_0$. Note that

$$\left|\binom{V}{k+1} \setminus \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})\right| \le t_0 \binom{n/t_0}{2} n^{k-1} \le \frac{n^{k+1}}{2t_0} \overset{(4.109)}{\le} \eta_{\mathrm{RL}} \binom{n}{k+1}.$$

Consequently, $\mathscr{R}_0^{(1)}$ is an $(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0}))$-equitable, t_0-bounded refinement of $\mathscr{O}^{(1)}$. This establishes the induction start.

Induction step. Assume there exist a $(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0}, \dots, a_j^{\mathscr{R}_0}))$-equitable, t_0-bounded family of partitions $\mathscr{R}_0(j) = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(j)}\}$ refining $\mathscr{O}(j)$. We define $\mathscr{R}_0^{(j+1)}$ for each polyad $\hat{R}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$. We set $a_{j+1}^{\mathscr{R}_0} = a_{j+1}^{\mathscr{O}}$ and in view of statement $(\mathfrak{S}_{j+1})$ we have to show that for every $\hat{R}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$ there exists a partition $\{R_a^{(j+1)} \colon a \in [a_{j+1}^{\mathscr{R}_0}]\}$ of $\mathcal{K}_{j+1}(\hat{R}^{(j)})$ so that for every $a \in [a_{j+1}^{\mathscr{R}_0}]$ the following two assertions hold

(I) $R_a^{(j+1)}$ is $(\mu^*, 1/a_{j+1}^{\mathscr{R}_0})$-regular w.r.t. $\hat{R}^{(j)}$ and

(II) either $R_a^{(j+1)} \subseteq \mathrm{Cross}_{j+1}(\mathscr{R}_0^{(1)}) \setminus \mathrm{Cross}_{j+1}(\mathscr{O}^{(1)})$
or we have $R_a^{(j+1)} \subseteq O^{(j+1)}$ for some $O^{(j+1)} \in \mathscr{O}^{(j+1)}$.

So let $\hat{R}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$ and let $\boldsymbol{R}$ be the corresponding $(n/a_1^{\mathscr{R}_0}, j+1, j)$-complex, i.e., $\boldsymbol{R} = \boldsymbol{R}(J') = \{\hat{R}^{(h)}(J')\}_{h=1}^{j}$ for any $J' \in \mathcal{K}_{j+1}(\hat{R}^{(j)})$ (see (4.1)). From the induction assumption we infer that $\boldsymbol{R}$ is an $(\mu^*, (1/a_1^{\mathscr{R}_0}, \dots, 1/a_j^{\mathscr{R}_0}))$-regular complex. Therefore, by the choice of μ^* and $n_{\mathrm{RL}} \ge n^*$ in (4.117) and (4.118) we can apply Theorem 4.19 and infer that

$$|\mathcal{K}_{j+1}(\hat{R}^{(j)})| \ge \frac{1}{2} \prod_{h=2}^{j} \left(\frac{1}{a_h^{\mathscr{R}_0}}\right)^{\binom{j+1}{h}} \times \left(\frac{n}{a_1^{\mathscr{R}_0}}\right)^{j+1} \ge \frac{n^{j+1}}{2t_0^{2^{j+1}}}. \tag{4.120}$$

Case 2.1 ($\hat{R}^{(j)} \not\subseteq \mathrm{Cross}_j(\mathscr{O}^{(1)})$). In this case we simply apply the slicing lemma, Proposition 4.33, with

$$j_{\mathrm{SL}} = j+1, \quad s_{0,\mathrm{SL}} = o_{\mathrm{RL}}, \quad r_{\mathrm{SL}} = 1, \quad \delta_{0,\mathrm{SL}} = \frac{\mu^*}{3}, \quad \varrho_{0,\mathrm{SL}} = 1, \quad \text{and} \quad p_{0,\mathrm{SL}} = \frac{1}{o_{\mathrm{RL}}},$$

to $\hat{P}_{\mathrm{SL}}^{(j)} = \hat{R}^{(j)}$ and $P_{\mathrm{SL}}^{(j+1)} = \mathcal{K}_{j+1}(\hat{R}^{(j)})$ with

$$s_{\mathrm{SL}} = a_{j+1}^{\mathscr{O}}, \quad \delta_{\mathrm{SL}} = \frac{\mu^*}{3}, \quad \varrho_{\mathrm{SL}} = 1,$$

and

$$p_{\xi,\mathrm{SL}} = \frac{1}{a^{\mathscr{O}}_{j+1}} \quad \text{for every} \quad \xi \in [a^{\mathscr{O}}_{j+1}]\,.$$

It follows from (4.120) and the choice of n^* in (4.117) that all assumptions of Proposition 4.33 are satisfied for this choice of parameters. Consequently, there exist a partition of $\mathcal{K}_{j+1}(\hat{R}^{(j)})$ into $a^{\mathscr{O}}_{j+1}$ distinct $(n/a^{\mathscr{R}_0}_1, j+1, j+1)$-hypergraphs which are $(\mu^*, 1/a^{\mathscr{R}_0}_{j+1})$-regular w.r.t. $\hat{R}^{(j)}$, i.e., (I) holds. Moreover, since we assume $\hat{R}^{(j)} \not\subseteq \mathrm{Cross}_j(\mathscr{O}^{(1)})$ each of these $(n/a^{\mathscr{R}_0}_1, j+1, j+1)$-hypergraphs is contained in $\mathrm{Cross}_{j+1}(\mathscr{R}^{(1)}_0) \setminus \mathrm{Cross}_{j+1}(\mathscr{O}^{(1)})$ and (II) holds. ♢

Case 2.2 **($\hat{R}^{(j)} \subseteq \mathrm{Cross}_j(\mathscr{O}^{(1)})$).** Then there exists some $\hat{O}^{(j)} \in \hat{\mathscr{O}}^{(j)}$ such that $\hat{R}^{(j)} \subseteq \hat{O}^{(j)}$, since $\mathscr{R}_0(j) \prec \mathscr{O}(j)$ by induction assumption. Moreover, there exists a family

$$\{O^{(j+1)}_1, \dots, O^{(j+1)}_{a^{\mathscr{O}}_{j+1}}\} \subseteq \mathscr{O}^{(j+1)}$$

of $(\mu_{\mathrm{RL}}, 1/a^{\mathscr{O}}_{j+1})$-regular (w.r.t. $\hat{O}^{(j)}$) $(n/a^{\mathscr{O}}_1, j+1, j+1)$-hypergraphs which partition $\mathcal{K}_{j+1}(\hat{O}^{(j)})$. Due to (4.120), Proposition 4.29 yields that the hypergraph $\mathcal{K}_{j+1}(\hat{R}^{(j)}) \cap O^{(j+1)}_a$ is $(2t_0^{2^{j+1}}\mu_{\mathrm{RL}}, 1/a^{\mathscr{O}}_{j+1})$-regular w.r.t. $\hat{R}^{(j)}$ for every $a \in [a^{\mathscr{O}}_{j+1}]$. Therefore, from the choice of μ_{RL} in (4.118) we infer that

$$\{R^{(j+1)}_a = \mathcal{K}_{j+1}(\hat{R}^{(j)}) \cap O^{(j+1)}_a \colon\ a \in [a^{\mathscr{O}}_{j+1}]\}$$

is a partition of $\mathcal{K}_{j+1}(\hat{R}^{(j)})$ which satisfies (I). Moreover, (II) holds trivially. ♢

In both cases, Case 2.1 and Case 2.2, we defined a partition of $\mathcal{K}_{j+1}(\hat{R}^{(j)})$ which satisfies (I) and (II). Repeating the argument for every $\hat{R}^{(j)} \in \hat{\mathscr{R}}^{(j)}_0$ gives rise to $\mathscr{R}^{(j+1)}_0$ and establishes the induction step. Consequently, there exist a partition $\mathscr{R}_0$ which satisfies (R_0.*1*) and (R_0.*2*) in this case, Case 2. ♢

Having constructed and appropriate family of partitions $\mathscr{R}_0$, the rest of the proof of Lemma 4.34 is based on successive applications of Lemma 4.52. This idea was introduced by Szemerédi in [Sze78] and also used in [FR02, FRRT06, Gow07, Gre05, GT08, Koh97, RS04].

Since $\mathscr{R}_0$ was constructed in such a way that (R_0.*1*) and (R_0.*2*) hold, we note that due to

$$\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \overset{(4.115)}{\leq} \delta_{\mathrm{RL}}(t_0, \dots, t_0) \overset{(4.108)}{\leq} \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_0}),$$

and $t_0 \leq t_{\mathrm{RL}}$ (cf. (4.114) and (4.118)) the partition $\mathscr{P} = \mathscr{R}_0$ satisfies (RL.*P1*) and (RL.*P2*). If (RL.*H*) holds as well, then we are done.

Otherwise we iterate Lemma 4.52 and infer the existence of a sequence of partitions $\mathscr{R}_i$ for $i \geq 0$, which satisfy (R_i.*1*) and (R_i.*2*). It then follows from Fact 4.44 and (4.119) that there must be some $0 \leq i_0 \leq \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$ such that $\mathscr{R}_{i_0}$ also admits (RL.*H*) for $r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_{i_0}})$. Since $t_i \leq t_{\mathrm{RL}}$ (cf. (4.114) and (4.118)) and and $\mathscr{R}_i \prec \mathscr{R}_0 \prec \mathscr{O}$ (cf. (R_i.*2*) and (R_0.*2*)) for every $0 \leq i \leq \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$, $\mathscr{P} = \mathscr{R}_{i_0}$ satisfies (RL.*P1*), (RL.*P2*), and (RL.*H*). This concludes the proof of Lemma 4.34 based on Lemma 4.52. □

4.5.3 Proof of the index pumping lemma

We prove Lemma 4.52 in this section. The proof is based on Lemma 4.36 and the propositions from Section 4.5.1.

Proof of Lemma 4.52. Recall the definition of all the constants and functions in (4.109)–(4.118) in Section 4.5.2. Let $0 \le i < \lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil$ be some integer and suppose $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ satisfies (R_i.*1*) and (R_i.*2*) and fails to satisfy (H) of RL($k+1$) for $r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})$. In other words

($\neg H_i$) there exist some $s_0 \in [s_{\mathrm{RL}}]$ such that $H^{(k+1)}_{s_0}$ is $(\delta_{k+1,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i}))$-irregular w.r.t. $\mathscr{R}_i$.

Then V, $\mathscr{R}^{(1)}_0$, $\mathscr{H}^{(k+1)}$, and $\mathscr{R}^{(k)}_i$ satisfy the assumptions of Proposition 4.50 with $\delta = \delta_{k+1,\mathrm{RL}}$ and $r = r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})$, due to ($\mathrm{R}_0$.*1*) combined with (4.108) and ($\neg H_i$). Consequently, there exists a partition $\mathscr{X}^{(k)}$ of $\binom{V}{k}$ satisfying the conclusions (i)–(iii) of Proposition 4.50, i.e.,

(P.4.50.i) $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}_i \prec \mathscr{R}^{(k)}_0$ (cf. (R_i.*2*) for the second '$\prec$'),

(P.4.50.ii) $|\mathscr{X}^{(k)}| \le |\mathscr{R}^{(k)}_i| \times 2^{r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i}) \times |\hat{\mathscr{R}}^{(k)}_i|} \le s_i$ (due to the t_i-boundedness of $\mathscr{R}_i$ in (R_i.*1*), the monotonicity of $r_{\mathrm{RL}}(\cdot, \dots, \cdot)$ in (4.108), and the definition of s_i in (4.112)), and

(P.4.50.iii) $\mathrm{ind}(\mathscr{X}^{(k)}) \ge \mathrm{ind}(\mathscr{R}^{(k)}_i) + \delta^4_{k+1,\mathrm{RL}}/2$.

The next step is to apply RAL(k), Lemma 4.36 to V, $\mathscr{O} = \mathscr{R}_0$, and $\mathscr{H}^{(k)} = \mathscr{X}^{(k)}$, with constants o_i, s_i, η_i, ν_i, and the function $\varepsilon_i \colon \mathbb{N}^{k-1} \to (0,1]$ defined in (4.109)–(4.113). For this we have to check the assumptions of Lemma 4.36;

(RAL.a) $|V| = n \ge n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot))$, where the integer n is a multiple of $(t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot)))!$,

(RAL.b) $\mathscr{R}_0 = \mathscr{R}_0(k, \boldsymbol{a}^{\mathscr{R}_0})$ is a $(\eta', \mu', \boldsymbol{a}^{\mathscr{R}_0})$-equitable family of partitions (for some constant $\eta' > 0$ and $\mu' \le \mu_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot))$) and o_i-bounded, and

(RAL.c) $s' = |\mathscr{X}^{(k)}| \le s_i$ and $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}_0$.

Property (RAL.a) is implied by assumption (RL.a) and the fact that for $i < \lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil$

$$n_{\mathrm{RL}} \overset{(4.118)}{\ge} n_{\lceil 8/\delta^4_{k+1,\mathrm{RL}} \rceil} \overset{(4.116)}{\ge} n_{i+1} \overset{(4.116)}{\ge} n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot))$$

and that the same line of inequalities holds with n replaced by t.

It follows from the definition of o_i in (4.112) and (R_0.*1*) that $\mathscr{R}_0$ is o_i-bounded. Moreover, (R_0.*1*) and (4.115) imply the required equitability of $\mathscr{R}_0$, which yields (RAL.b).

Finally, (RAL.c) follows immediately from (P.4.50.i) and (P.4.50.ii).

Consequently, we can apply Lemma 4.36 to V, $\mathscr{O} = \mathscr{R}_0$, and $\mathscr{H}^{(k)} = \mathscr{X}^{(k)}$, with constants o_i, s_i, η_i, ν_i, and $\varepsilon_i \colon \mathbb{N}^{k-1} \to (0,1]$. Lemma 4.36 then asserts that there exist a family of partitions $\mathscr{S} = \mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$, and a partition $\mathscr{Y}^{(k)} = \{Y_1^{(k)}, \ldots, Y_{s'}^{(k)}\}$ of $\binom{V}{k}$ so that

(RAL.*S1*) $\mathscr{S}$ is t_{i+1}-bounded, $(\eta_i, \varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), \boldsymbol{a}^{\mathscr{S}})$-equitable family (since by (4.114) we have $t_{i+1} \geq t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \ldots, \cdot))$),

(RAL.*S2*) $\mathscr{S} \prec \mathscr{R}_0(k-1) = \{\mathscr{R}_0^{(j)}\}_{j=1}^{k-1}$

(RAL.*Y1*) $Y_\ell^{(k)}$ is perfectly $(\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}))$-regular w.r.t. $\mathscr{S}$ for every $\ell \in [s']$,

(RAL.*Y2*) $|Y_\ell^{(k)} \triangle X_\ell^{(k)}| \leq \nu_i n^k$ for every $\ell \in [s']$, and

(RAL.*Y3*) if $X_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ then $Y_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ for every $\ell \in [s']$ and $\mathscr{Y}^{(k)} \prec \mathscr{R}_0^{(k)} \prec \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$.

In particular, (P.4.50.*i*) and (RAL.*Y3*) show that both partitions $\mathscr{X}^{(k)}$ and $\mathscr{Y}^{(k)}$ refine $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, respectively. Hence, due to (RAL.*Y2*) and the first part of (RAL.*Y3*) the assumptions of Proposition 4.45 are satisfied for V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, $\mathscr{X}^{(k)}$, $\mathscr{Y}^{(k)}$, $s_{P.4.45} = s' \leq s_i$, and $\nu_{P.4.45} = \nu_i$ and, consequently, Proposition 4.45 yields

$$\begin{aligned}\mathrm{ind}(\mathscr{Y}^{(k)}) &\geq \mathrm{ind}(\mathscr{X}^{(k)}) - 3(k+1)s_{\mathrm{RL}} s_i^{k+1} \nu_i \\ &\overset{(4.112)}{\geq} \mathrm{ind}(\mathscr{X}^{(k)}) - \frac{\delta^4_{k+1,\mathrm{RL}}}{4} \overset{(\mathrm{P}.4.50.iii)}{\geq} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \frac{\delta^4_{k+1,\mathrm{RL}}}{4}\,. \end{aligned} \tag{4.121}$$

Our next goal is to construct a partition $\mathscr{Z}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ which forms a family of partitions together with $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$. This means, that such an $\mathscr{Z}^{(k)}$ has to satisfy two conditions – it must partition $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ and it must refine

$$\{\mathcal{K}_k(\hat{S}^{(k-1)}) \colon \hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}\,.$$

The partition $\mathscr{Y}^{(k)}$ fails to satisfy any of these two requirements. It partitions all of $\binom{V}{k}$ (rather than only $\mathrm{Cross}_k(\mathscr{S}^{(1)})$) and, more importantly, we cannot ensure that it refines $\{\mathcal{K}_k(\hat{S}^{(k-1)}) \colon \hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$. However, we easily "fix" these shortcomings of $\mathscr{Y}^{(k)}$ and define $\mathscr{Z}^{(k)}$ as follows

$$\mathscr{Z}^{(k)} = \{Y^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)}) \colon Y^{(k)} \in \mathscr{Y}^{(k)} \text{ and } \hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}\,. \tag{4.122}$$

For convenience we set for every $\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$

$$\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}) = \{Z^{(k)} \in \mathscr{Z}^{(k)} \colon Z^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)}) \neq \varnothing\}\,. \tag{4.123}$$

The partition $\mathscr{Z}^{(k)}$ has the following properties which we verify below.

(*Z1*) $\mathscr{Z}^{(k)}$ partitions $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ and $\mathscr{Z}^{(k)} \prec \{\mathcal{K}_k(\hat{S}^{(k-1)}) \colon \hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$, (*Z2*) $|\mathscr{Z}^{(k)}(\hat{S}^{(k-1)})| \leq s_i$ for every $\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$, and

(*Z3*) for every $\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ and $Z^{(k)} \in \mathscr{Z}^{(k)}(\hat{S}^{(k-1)})$ we have that $Z^{(k)}$ is $(\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}))$-regular w.r.t. $\hat{S}^{(k-1)}$,

(*Z4*) $\mathscr{Z}^{(k)} \prec \mathscr{R}_0^{(k)} \prec \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, and

(*Z5*) $\mathrm{ind}(\mathscr{Z}^{(k)}) \geq \mathrm{ind}(\mathscr{R}_i^{(k)}) + \delta_{k+1,\mathrm{RL}}^4/4$.

Property (*Z1*) follows from the fact that $\mathscr{Y}^{(k)}$ partitions all of $\binom{V}{k}$ and the definition of $\mathscr{Z}^{(k)}$ in (4.122). Assertion (*Z2*) is an immediate consequence of (4.123) and $|\mathscr{Y}^{(k)}| = s' \leq s_i$ (cf. (RAL.*c*)). We also note that (*Z3*) is simply a reformulation of (RAL.*Y1*).

Hence, it is only left to verify (*Z4*) and (*Z5*). First we consider (*Z4*). For that we first note that $\{\mathcal{K}_k(\hat{S}^{(k-1)})\colon \hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$ partitions a superset of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ due to (RAL.*S2*). Consequently, (*Z4*) follows from the definition of $\mathscr{Z}^{(k)}$ in (4.122) and (RAL.*Y3*).

Finally we focus on (*Z5*). For that we consider the restriction of $\mathscr{Y}^{(k)}$ on $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, i.e.,

$$\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} = \{Y^{(k)} \in \mathscr{Y}^{(k)}\colon Y^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\}.$$

It follows from Definition 4.43 that the index of $\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}$ w.r.t. $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ satisfies

$$\mathrm{ind}(\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}) = \mathrm{ind}(\mathscr{Y}^{(k)}) \overset{(4.121)}{\geq} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \frac{\delta_{k+1,\mathrm{RL}}^4}{4}. \tag{4.124}$$

On the other hand, in view of (*Z4*) the restriction of $\mathscr{Z}^{(k)}$ on $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$

$$\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} = \{Z^{(k)} \in \mathscr{Z}^{(k)}\colon Z^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\}$$

is a partition of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ and, therefore,

$$\mathrm{ind}(\mathscr{Z}^{(k)}) = \mathrm{ind}\big(\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big).$$

Moreover, we observe that

$$\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} \prec \mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}$$

due to (4.122). Finally, Proposition 4.47 then yields (*Z5*)

$$\begin{aligned}\mathrm{ind}(\mathscr{Z}^{(k)}) &= \mathrm{ind}\big(\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big)\\ &\geq \mathrm{ind}\big(\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big) \overset{(4.124)}{\geq} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \frac{\delta_{k+1,\mathrm{RL}}^4}{4}.\end{aligned}$$

Having verified (*Z1*)–(*Z5*) we come to the last part of the proof and define the family of partitions $\mathscr{R}_{i+1}$. The careful reader (who managed not to get lost in details so far) will note that due to (*Z1*) the partition $\mathscr{Z}^{(k)}$ together with $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ forms a family of partitions on V. Moreover, due to (RAL.*S2*) and (*Z4*) it satisfies (R_{i+1}.*2*) and due to (RAL.*S1*), (*Z2*), and (*Z3*) it "almost" satisfies (R_{i+1}.*1*). But unfortunately, the densities of the $Z^{(k)} \in \mathscr{Z}^{(k)}$ vary and thus this family of partitions $\mathscr{Z}^{(k)} \cup \mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ is not equitable. In the final step of this proof we derive $\mathscr{R}_{i+1}$ from $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}}) \cup \mathscr{Z}^{(k)}$ by "cleaning the imperfections" of $\mathscr{Z}^{(k)}$ mentioned above. For that we will use the following claim, which somewhat dry proof is based on repeated applications of Proposition 4.33.

Claim 4.53. *There exist a partition $\mathscr{T}^{(k)}$ of $Cross_k(\mathscr{S}^{(1)})$ such that*

(*T1*) $\mathscr{T}^{(k)} \prec \{\mathcal{K}_k(\hat{S}^{(k-1)}) \colon \hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$,

(*T2*) $|\{T^{(k)} \in \mathscr{T}^{(k)} \colon T^{(k)} \subseteq \mathcal{K}_k(\hat{S}^{(k-1)})\}| = f(s_i)$ *for every fixed* $\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$,

(*T3*) *every* $T^{(k)} \in \mathscr{T}^{(k)}$ *is* $(\delta_{RL}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i))$*-regular w.r.t. unique* $\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ *which satisfies* $T^{(k)} \subseteq \mathcal{K}_k(\hat{S}^{(k-1)})$,

(*T4*) $\mathscr{T}^{(k)} \prec \mathscr{R}_0^{(k)}$, *and*

(*T5*) $\mathscr{T}^{(k)}$ *is a* $(\delta^4_{k+1,RL}/8)$*-refinement of* $\mathscr{Z}^{(k)}$.

We first finish the proof of Lemma 4.52 and give the proof of Claim 4.53, which makes use of (*Z1*)–(*Z4*), afterwards. In order to conclude the proof of Lemma 4.52 we have to define a family of partitions $\mathscr{R}_{i+1}$ on V, which satisfies (R_{i+1}.*1*), (R_{i+1}.*2*), and (4.119). With this in mind we set

$$\boldsymbol{a}^{\mathscr{R}_{i+1}} = (a_1^{\mathscr{S}}, \dots, a_{k-1}^{\mathscr{S}}, f(s_i)),$$

$$\mathscr{R}_{i+1}^{(j)} = \begin{cases} \mathscr{S}^{(j)} & \text{for} \quad j \in [k-1] \\ \mathscr{T}^{(k)} & \text{for} \quad j = k \end{cases}, \quad \text{and} \quad \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}}) = \{\mathscr{R}_{i+1}^{(j)}\}_{j=1}^{k}.$$

We now first show that $\mathscr{R}_{i+1} = \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}})$ is a family of partitions on V. Due to the fact that $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ is a family of partitions on V, we only have to verify that $\mathscr{R}_{i+1}^{(k)} = \mathscr{T}^{(k)}$ fulfills both requirements of part (*ii*) of Definition 4.8. However, this is immediate from (*T1*) and (*T2*).

Next we consider (R_{i+1}.*1*). Note that (RAL.*S1*) (combined with (4.113)) and (*T3*) show that $\mathscr{R}_{i+1}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_{i+1}}), \boldsymbol{a}^{\mathscr{R}_{i+1}})$-equitable. Moreover, (RAL.*S1*) and the choice of $t_{i+1} \geq f(s_i)$ in (4.114) imply $\max_{j \in [k]} a_j^{\mathscr{R}_{i+1}} \leq t_{i+1}$. In other words, $\mathscr{R}_{i+1}$ is t_{i+1}-bounded and (R_{i+1}.*1*) holds.

The property (R_{i+1}.*2*) follows from (RAL.*S2*) and (*T4*) and (4.119) is a consequence of (*Z5*) and (*T5*), combined with Proposition 4.49.

Hence $\mathscr{R}_{i+1}$ has the desired properties and we conclude the proof of Lemma 4.52 based on Claim 4.53. □

Proof of Claim 4.53. We have to show that there is a partition $\mathscr{T}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ satisfying $(T1)$–$(T5)$. For technical reasons we first extend the partition $\mathscr{R}_0^{(k)}$ from a partition of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ to a partition of $\binom{V}{k}$ and we set

$$\widetilde{R}^{(k)} = \binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{R}_0^{(1)}) \quad \text{and} \quad \widetilde{\mathscr{R}}_0^{(k)} = \mathscr{R}_0^{(k)} \cup \widetilde{R}^{(k)} . \tag{4.125}$$

In view of $(T1)$ and $(T4)$ it seems natural to define $\mathscr{T}^{(k)}$ separately for every pair

$$\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}, \quad R^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)} \quad \text{satisfying} \quad \mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)} \neq \varnothing . \tag{4.126}$$

In fact, we will prove the following claim.

Claim 4.53′. *Suppose the hypergraphs $\hat{S}^{(k-1)}$ and $R^{(k)}$ satisfy (4.126). Then there exists a partition $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ of $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$ satisfying the following properties*

$(T2')$

$$|\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})| = \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if} \quad R^{(k)} \neq \widetilde{R}^{(k)} ,^{5} \\ f(s_i) & \text{if} \quad R^{(k)} = \widetilde{R}^{(k)} , \end{cases}$$

$(T3')$ *every $T^{(k)} \in \mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ is $(\delta_{RL}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i))$-regular w.r.t. $\hat{S}^{(k-1)}$, and*

$(T5')$

$$\Big|\bigcup \{T^{(k)} \colon T^{(k)} \in \mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)}) \text{ and } T^{(k)} \not\subseteq Z^{(k)} \ \forall Z^{(k)} \in \mathscr{Z}^{(k)}\}\Big| \leq \tfrac{\delta_{k+1,RL}^4}{8} |\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}| .$$

Before we verify Claim 4.53′, we deduce Claim 4.53 from it. So let $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ be given for every $\hat{S}^{(k-1)}$, $R^{(k)}$ satisfying (4.126). We then set

$$\mathscr{T}^{(k)} = \bigcup \Big\{ \mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)}) \colon \ \hat{S}^{(k-1)} \text{ and } R^{(k)} \text{ satisfy (4.126)} \Big\} .$$

Clearly, $\mathscr{T}^{(k)}$ is a partition of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$, since $\widetilde{\mathscr{R}}_0^{(k)}$ is a partition of $\binom{V}{k}$ (cf. (4.125)). Furthermore, $(T1)$ and $(T4)$ are immediate since we constructed $\mathscr{T}^{(k)}$ separately on $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$. Moreover, it is easy to see that $(T2)$, $(T3)$, and $(T5)$ are implied by its "prime" counterpart.

This finishes the reduction of Claim 4.53 to Claim 4.53′, which is the last missing piece in the proof of the implication $\mathrm{RAL}(k) \Longrightarrow \mathrm{RL}(k+1)$. □

[5] Note that $f(s_i)/a_k^{\mathscr{R}_0}$ is an integer since $a_k^{\mathscr{R}_0} \leq t_0$ (cf. (R_0.*1*)) and due to the fact that the definition of the function $f(\cdot)$ in (4.111) ensures that $f(s_i)$ is a multiple of $(t_0)!$.

Below we prove Claim 4.53′. The proof resembles ideas from [NRS06a, Section 5]. The main tool in that proof is the somewhat technical slicing lemma, Proposition 4.33 and we first give an informal outline to convey the idea.

Suppose $\hat{S}^{(k-1)}$ and $R^{(k)}$ satisfy (4.126). Let $\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ be the collection of those partition classes $Z^{(k)}$ of $\mathscr{Z}^{(k)}$ which are contained in $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$, i.e.,

$$\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)}) = \{Z^{(k)} \in \mathscr{Z}^{(k)} \colon\ Z^{(k)} \subseteq \mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}\}\,. \tag{4.127}$$

Note that due to (*Z1*) and (*Z4*)

$$\{Z^{(k)} \colon\ Z^{(k)} \in \mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})\} \text{ partitions } \mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}\,. \tag{4.128}$$

Indeed by (*Z1*), $\mathscr{Z}^{(k)}$ has each of its partition classes completely within or outside $\mathcal{K}_k(\hat{S}^{(k-1)})$ and by (*Z4*) the same is true for $R^{(k)}$.

We will use the slicing lemma twice. In the first round we apply the slicing lemma separately to each $Z^{(k)} \in \mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ to slice it in such a way that all but at most one slice ("leftover" part) has density $1/f(s_i)$ w.r.t. $\hat{S}^{(k-1)}$. On the other hand, we infer from the choice of $\mu_{i+1} \geq \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}$ and (R_0.*1*) that $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$ is still $(\delta', 1/a_k^{\mathscr{R}_0})$-regular with $\delta' \ll \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$ (cf. (4.115)). (In the special case $R^{(k)} = \widetilde{R}^{(k)}$ we have $(\delta', 1)$-regularity for any $\delta' > 0$.) Consequently, the union of the earlier produced "leftovers" must have a density very close to a multiple of $1/f(s_i)$, since it is $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$ minus regular pieces of density $1/f(s_i)$. Therefore, we can use the slicing lemma again (second round) to "recycle" the "leftovers", splitting it into regular pieces of density $1/f(s_i)$ and the "recycled" partition will satisfy (*T2′*) and (*T3′*). Finally, we will show that it also exhibits (*T5′*) since we chose $f(\cdot)$ in (4.111) in such a way that $|\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})| \times 1/f(s_i) \leq s_i/f(s_i) \ll \delta_{k+1,\mathrm{RL}}/8$, which is an upper bound on the density of the union of the "leftovers".

Below we give the technical details of the plan outlined above.

Proof of Claim 4.53′. Let $\hat{S}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ and $R^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)}$ satisfying (4.126) be given. We start with a few observations. From the choice of the function ε_i in (4.113) and n_i in 4.116 combined with (*S1*) we infer by Theorem 4.19 that

$$|\mathcal{K}_k(\hat{S}^{(k-1)})| \geq \frac{1}{2} \prod_{j=2}^{k-1} \left(\frac{1}{a_j^{\mathscr{S}}}\right)^{\binom{k}{j}} \times \left(\frac{n}{a_1^{\mathscr{S}}}\right)^{k} \geq \frac{n^k}{2t_{i+1}^{2^k}}\,. \tag{4.129}$$

Suppose $R^{(k)} \neq \widetilde{R}^{(k)}$. Let $\hat{R}^{(k-1)}$ be the polyad in $\hat{\mathscr{R}}_0^{(k-1)}$ such that $R^{(k)} \subseteq \mathcal{K}_k(\hat{R}^{(k-1)})$. Since, $\mathscr{S}^{(k-1)} \prec \mathscr{R}_0^{(k-1)}$ (cf. (RAL.*S2*)) and $R^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)}) \neq \varnothing$ (cf. (4.126)) we have that $\hat{S}^{(k-1)} \subseteq \hat{R}^{(k-1)}$. Consequently, we infer from an application of Proposition 4.29, combined with (4.129), and (R_0.*1*) that if $R^{(k)} \neq \widetilde{R}^{(k)}$ then

$$R^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)}) \text{ is } (2t_{i+1}^{2^k}\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}, 1/a_k^{\mathscr{R}_0})\text{-regular w.r.t. } \hat{S}^{(k-1)}\,.$$

Moreover, if $R^{(k)} = \widetilde{R}^{(k)}$, then assumption (4.126) yields that $\mathcal{K}_k(\hat{S}^{(k-1)}) \subseteq R^{(k)}$ and $R^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)})$ is $(\delta', 1)$-regular w.r.t. $\hat{S}^{(k-1)}$ for every $\delta' > 0$. Therefore, in view of

$$\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \leq \mu_{i+1} \overset{(4.115)}{\leq} \frac{\delta_{\mathrm{RL}}(t_{i+1}, \dots, t_{i+1}, f(s_i))}{12 t_{i+1}^{2^k}} \overset{(\mathrm{RAL}.S1)}{\leq} \frac{\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))}{12 t_{i+1}^{2^k}}$$

we have for every $R^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)}$ that

$$R^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)}) \text{ is } (\tfrac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), d_{R^{(k)}})\text{-regular w.r.t. } \hat{S}^{(k-1)}\,, \tag{4.130}$$

where

$$d_{R^{(k)}} = \begin{cases} 1/a_k^{\mathscr{R}_0} & \text{if} \quad R^{(k)} \neq \widetilde{R}^{(k)} \\ 1 & \text{if} \quad R^{(k)} = \widetilde{R}^{(k)} \end{cases}. \tag{4.131}$$

Furthermore, we infer from (4.130) and $d_{R^{(k)}} \geq 1/a_k^{\mathscr{R}_0} \geq 1/t_0$ (cf. ($\mathrm{R}_0.1$)) that

$$d(R^{(k)} | \hat{S}^{(k-1)}) \geq \frac{1}{t_0} - \frac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) \overset{(4.110)}{>} \max\left\{\frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))\,, \frac{2}{3t_0}\right\}. \tag{4.132}$$

Recall the definition of $\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ from (4.127) and consider an enumeration $\{Z_1^{(k)}, \dots, Z_z^{(k)}\}$ of its members. Clearly, $\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)}) \subseteq \mathscr{Z}^{(k)}(\hat{S}^{(k-1)})$ (cf. (4.123)) and due to (*Z2*) we have

$$z \leq s_i\,. \tag{4.133}$$

We apply the slicing lemma to every member $Z_j^{(k)}$ of $\mathscr{Z}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$. For that we have to satisfy the assumptions of the slicing lemma among which we have to ensure that $d(Z_j^{(k)} | \hat{S}^{(k-1)})$ is not "too small". However, since the $Z_j^{(k)}$ arose from an application of RAL(k), we only have limited control over their densities, which leads to the following definition

$$Z_{\mathrm{THIN}} = \left\{ j \in [z] \colon\; d(Z_j^{(k)} | \hat{S}^{(k-1)}) < \tfrac{1}{f(s_i)} \right\}. \tag{4.134}$$

Moreover, for every $j \in [z]$ we set

$$\zeta_j = \lfloor f(s_i) \times d(Z_j^{(k)} | \hat{S}^{(k-1)}) \rfloor\,. \tag{4.135}$$

Clearly, $\zeta_j > 0$ if and only if $j \notin Z_{\mathrm{THIN}}$. We now apply the slicing lemma, Proposition 4.33, for every $j \in [z] \setminus Z_{\mathrm{THIN}}$ separately with

$$j_{\mathrm{SL}} = k\,, \quad s_{0,\mathrm{SL}} = s_{4.38}\,, \quad r_{\mathrm{SL}} = 1\,, \quad \delta_{0,\mathrm{SL}} = \varepsilon_i(\overbrace{t_{i+1}, \dots, t_{i+1}}^{(k-1)\text{-times}})\,,$$

$$\varrho_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)}\,, \quad \text{and} \quad p_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)}\,,$$

to $\hat{P}^{(k-1)}_{\mathrm{SL}} = \hat{S}^{(k-1)}$ and $P^{(k)}_{\mathrm{SL}} = Z^{(k)}_j$ with

$$s_{\mathrm{SL}} = \zeta_j\,, \quad \delta_{\mathrm{SL}} = \varepsilon_i(\boldsymbol{a}^{\mathscr{S}})\,, \quad \varrho_{\mathrm{SL}} = d(Z^{(k)}_j|\hat{S}^{(k-1)})\,,$$

and

$$p_{\xi,\mathrm{SL}} = \frac{1/f(s_i)}{d(Z^{(k)}_j|\hat{S}^{(k-1)})} \quad \text{for every} \quad \xi \in [\zeta_j]\,.$$

It follows from (4.129) that the assumption (i) of Proposition 4.33 is satisfied for $\hat{P}^{(k-1)}_{\mathrm{SL}} = \hat{S}^{(k-1)}$. Moreover, (ii) is a consequence of $(Z3)$ (yielding the $(\delta_{\mathrm{SL}}, \varrho_{\mathrm{SL}}, r_{\mathrm{SL}})$-regularity of $P^{(k)}_{\mathrm{SL}} = Z^{(k)}_j$), the definition of Z_{THIN} (yielding $\varrho_{\mathrm{SL}} \geq \varrho_{0,\mathrm{SL}}$), $(RAL.S1)$ and the monotonicity of the function ε_i (yielding $\delta_{\mathrm{SL}} \geq \delta_{0,\mathrm{SL}}$), and the choice of ε_i in (4.113) (yielding $\varrho_{0,\mathrm{SL}} \geq 2\delta_{\mathrm{SL}}$). Furthermore, assumption (iii) of Proposition 4.33 is a consequence of the fact that $d(Z^{(k)}_j|\hat{S}^{(k-1)}) \leq 1$ (yielding $p_{\xi,\mathrm{SL}} \geq p_{0,\mathrm{SL}}$ for $\xi \in [\zeta_j]$ and the choice of the integer parameter ζ_j in (4.135) (yielding $\sum_{\xi\in[\zeta_j]} p_{\xi,\mathrm{SL}} \leq 1$).

Having verified the assumptions of Proposition 4.33 for all $j \in [z] \setminus Z_{\mathrm{THIN}}$, we infer that for every such j there exists a family $\{T^{(k)}_{j,0}, T^{(k)}_{j,1}, \dots, T^{(k)}_{j,\zeta_j}\}$ such that

$$\left\{T^{(k)}_{j,0}, T^{(k)}_{j,1}, \dots, T^{(k)}_{j,\zeta_j}\right\} \text{ partitions } Z^{(k)}_j\,, \tag{4.136}$$

$$T^{(k)}_{j,\xi} \text{ is } (3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), 1/f(s_i))\text{-regular w.r.t. } \hat{S}^{(k-1)} \text{ for every } \xi = 1, \dots, \zeta_j\,, \tag{4.137}$$

and

$$T^{(k)}_{j,0} \text{ is } (3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), d_{j,0})\text{-regular} \tag{4.138}$$

for some

$$0 \leq d_{j,0} \leq d(Z^{(k)}_j|\hat{S}^{(k-1)}) - \frac{\zeta_j}{f(s_i)} \overset{(4.135)}{\leq} \frac{1}{f(s_i)}\,.$$

Unfortunately, the "leftover" hypergraph $T^{(k)}_{j,0}$ might not be empty and has a density differing from $1/f(s_i)$. Moreover, in general Z_{THIN} is not empty and we have to recycle the "leftovers" $T^{(k)}_{j,0}$ with $j \notin Z_{\mathrm{THIN}}$ and the hypergraphs $Z^{(k)}_j$ with $j \in Z_{\mathrm{THIN}}$. For that we consider their union

$$U^{(k)} = \bigcup_{j\in[z]\setminus Z_{\mathrm{THIN}}} T^{(k)}_{j,0} \cup \bigcup_{j\in Z_{\mathrm{THIN}}} Z^{(k)}_j\,. \tag{4.139}$$

Clearly, $U^{(k)}$ is the complement of $\bigcup_{j\in[z]\setminus Z_{\mathrm{THIN}}} \bigcup_{\xi\in[\zeta_j]} T^{(k)}_{j,\xi}$ in $R^{(k)} \cap \mathcal{K}_k(\hat{S}^{(k-1)})$. Consequently, in view of (4.137), $|Z_{\mathrm{THIN}}| \leq z \leq s_i$ (cf. (4.133)), and (4.130) a consecutive application of Proposition 4.30 and Proposition 4.28 yields that

$$U^{(k)} \text{ is } \Big(\tfrac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})s_i, d_{R^{(k)}} - \frac{\sum_{j\notin Z_{\mathrm{THIN}}}\zeta_j}{f(s_i)}\Big)\text{-regular w.r.t. } \hat{S}^{(k-1)}\,. \tag{4.140}$$

Recall, that $d_{R^{(k)}}$ is an integer multiple of $1/f(s_i)$. (This is obvious if $R^{(k)} = \widetilde{R}^{(k)}$ since $f(s_i)$ is an integer-valued function. Moreover, if $R^{(k)} \neq \widetilde{R}^{(k)}$, then $d_{R^{(k)}} = 1/a_k^{\mathscr{R}_0}$ which is a multiple of $1/f(s_i)$ since $(t_0)!$ divides $f(s_i)$ (cf. (4.111)) and $t_0 \geq a_k^{\mathscr{R}_0}$ (cf. (R$_0$.*1*)).) Consequently,

$$d_{R^{(k)}} - \frac{\sum_{j \notin Z_{\mathrm{THIN}}} \zeta_j}{f(s_i)} = \frac{u}{f(s_i)} \text{ for some integer } 0 \leq u \leq f(s_i)\,. \tag{4.141}$$

This observation and the choice of the function ε_i in (4.113) allows us to rewrite (4.140)

$$U^{(k)} \text{ is } \Big(\tfrac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \tfrac{u}{f(s_i)}\Big)\text{-regular w.r.t. } \hat{S}^{(k-1)}\,. \tag{4.142}$$

The further treatment of $U^{(k)}$ depends on the value of and we consider two cases.

Case 1 $(u = 0)$**.** Note that the assumption $u = 0$ and (4.142) not necessarily implies that $U^{(k)} = \varnothing$. It rather yields, that

$$d(U^{(k)} | \hat{S}^{(k-1)}) \leq \tfrac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))\,.$$

On the other hand, by (4.132)

$$d(R^{(k)} | \hat{S}^{(k-1)}) > \tfrac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))\,.$$

Therefore, from (4.128), we infer that $Z_{\mathrm{THIN}} \neq [z]$ and there exist some $j_0 \in [z] \setminus Z_{\mathrm{THIN}}$ with $\zeta_{j_0} \geq 1$ and hence $T_{j_0,1}^{(k)}$ exists. We then define, the promised partition $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ as follows

$$\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)}) = \Big\{T_{j,\xi}^{(k)} \colon j \in [z] \setminus Z_{\mathrm{THIN}},\, j \neq j_0, \text{ and } \xi \in [\zeta_j]\Big\} \\ \cup \Big\{T_{j_0,\xi}^{(k)} \colon \xi = 2, \dots, \zeta_j\Big\} \cup \Big\{T_{j_0,1}^{(k)} \cup U^{(k)}\Big\}\,.$$

It follows from the definition of $U^{(k)}$ in (4.139) in conjunction with (4.136) and (4.128) that $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ defined above partitions $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$. We conclude this case with the verification of properties (*T2′*), (*T3′*), and (*T5′*).

First we consider (*T2′*). Clearly, $|\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})| = \sum_{j \in [z] \setminus Z_{\mathrm{THIN}}} \zeta_j$. So in view of (4.141), we infer from the assumption $u = 0$ in this case, that

$$|\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})| = \sum_{j \in [z] \setminus Z_{\mathrm{THIN}}} \zeta_j = d_{R^{(k)}} f(s_i) \overset{(4.131)}{=} \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if } R^{(k)} \neq \widetilde{R}^{(k)} \\ f(s_i) & \text{if } R^{(k)} = \widetilde{R}^{(k)} \end{cases},$$

which is (*T2′*).

Since $\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leq \frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$ (cf. (4.113)), (4.137) guarantees (*T3′*) for all members of $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ with exception $T_{j_0}^{(k)} \cup U^{(k)}$. Consequently, verifying (*T3′*) reduces to showing that

$$T^{(k)}_{j_0,1} \cup U^{(k)} \text{ is } (\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i))\text{-regular w.r.t. } \hat{S}^{(k-1)} . \tag{4.143}$$

However, this follows from (4.137) and (4.142) by Proposition 4.30, since $u = 0$ and since by the choice in (4.113) we have $3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \le \frac{2}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$.

Finally, we consider ($T5'$). Here we note that due to the definition of the partition $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ it suffices to show that

$$d(T^{(k)}_{j_0,1} \cup U^{(k)} | \hat{S}^{(k-1)}) \le \frac{\delta^4_{k+1,\mathrm{RL}}}{8} d(R^{(k)} | \hat{S}^{(k-1)}) . \tag{4.144}$$

For that we first derive from (4.143) combined with the definition of the function $f(\cdot)$ in (4.111) and the bound from (4.110) that

$$d(T^{(k)}_{j_0,1} \cup U^{(k)} | \hat{S}^{(k-1)}) \overset{(4.143)}{\le} \frac{1}{f(s_i)} + \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) \le \frac{\delta^4_{k+1,\mathrm{RL}}}{12 t_0} . \tag{4.145}$$

Therefore, the estimate (4.144) follows from (4.145) and (4.132). This concludes the proof of Claim 4.53′ in this case. ♢

Case 2 ($u > 0$)**.** Recall that $U^{(k)}$ is $(\frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \frac{u}{f(s_i)})$-regular with respect to $\hat{S}^{(k-1)}$ by (4.142). In this case we are going to apply the slicing lemma to "recycle" the edges of $U^{(k)}$, i.e., to partition it into regular pieces of density $1/f(s_i)$. More precisely, we apply Proposition 4.33 with

$$j_{\mathrm{SL}} = k , \quad s_{0,\mathrm{SL}} = f(s_i) , \quad r_{\mathrm{SL}} = 1 , \quad \delta_{0,\mathrm{SL}} = \tfrac{1}{3}\delta_{\mathrm{RL}}(\overbrace{t_{i+1}, \dots, t_{i+1}}^{(k-1)\text{-times}}, f(s_i)) ,$$

$$\varrho_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)} , \quad \text{and} \quad p_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)} ,$$

to $\hat{P}^{(k-1)}_{\mathrm{SL}} = \hat{S}^{(k-1)}$ and $P^{(k)}_{\mathrm{SL}} = U^{(k)}$ with

$$s_{\mathrm{SL}} = u , \quad \delta_{\mathrm{SL}} = \frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) , \quad \varrho_{\mathrm{SL}} = \frac{u}{f(s_i)} ,$$

and

$$p_{\xi,\mathrm{SL}} = \frac{1}{u} \quad \text{for every} \quad \xi \in [u] .$$

It follows from (4.129) that the assumption (i) of Proposition 4.33 is satisfied for $\hat{P}^{(k-1)}_{\mathrm{SL}} = \hat{S}^{(k-1)}$. Moreover, property ($ii$) is a consequence of (4.142) (yielding the $(\delta_{\mathrm{SL}}, \varrho_{\mathrm{SL}}, r_{\mathrm{SL}})$-regularity of $P^{(k)}_{\mathrm{SL}} = U^{(k)}_j$), the assumption of the case $u \ge 1$ (yielding $\varrho_{\mathrm{SL}} \ge \varrho_{0,SL}$), (RAL.*S1*) and the monotonicity (cf. (4.108)) of the function δ_{RL} (yielding $\delta_{\mathrm{SL}} \ge \delta_{0,\mathrm{SL}}$), and of (4.110) (yielding $\varrho_{0,\mathrm{SL}} \ge 2\delta_{\mathrm{SL}}$). Furthermore, note that $p_{\xi,\mathrm{SL}} \ge p_{0,\mathrm{SL}}$, since $u \leq f(s_i)$ (cf. (4.141)) and that that

$$\sum_{\xi \in [u]} p_{\xi,\mathrm{SL}} = 1\,. \tag{4.146}$$

Consequently, assumption (*iii*) of Proposition 4.33 holds for the choice of parameters above.

Having verified the assumptions of Proposition 4.33, we infer that there exists a family $\{U_1^{(k)}, \dots, U_u^{(k)}\}$ (note that due to (4.146) there is "leftover" class $U_0^{(k)}$) such that

$$\{U_1^{(k)}, \dots, U_u^{(k)}\} \text{ partitions } U^{(k)} \tag{4.147}$$

$$U_\xi^{(k)} \text{ is } (\delta_{k,\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i))\text{-regular w.r.t. } \hat{S}^{(k-1)} \text{ for every } \xi \in [u]\,. \tag{4.148}$$

We finally define the required family $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ in a straightforward manner

$$\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)}) = \Big\{T_{j,\xi}^{(k)} \colon\ j \in [z] \setminus Z_{\mathrm{THIN}} \text{ and } \xi \in [\zeta_j]\Big\} \cup \Big\{U_1^{(k)}, \dots, U_u^{(k)}\Big\}\,. \tag{4.149}$$

Again it directly follows from the definition of $U^{(k)}$ in (4.139) in conjunction with (4.147) and (4.128) that $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ defined above partitions $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$ and it is left to verify (*T2′*), (*T3′*), and (*T5′*) for this partition.

First we consider (*T2′*). By the definition of $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ we have

$$\begin{aligned}|\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})| &= \sum_{j \in [z] \setminus Z_{\mathrm{THIN}}} \zeta_j + u \\ &\overset{(4.141)}{=} d_{R^{(k)}} f(s_i) \overset{(4.131)}{=} \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if } \; R^{(k)} \neq \widetilde{R}^{(k)} \\ f(s_i) & \text{if } \; R^{(k)} = \widetilde{R}^{(k)} \end{cases},\end{aligned}$$

which is (*T2′*).

Property (*T3′*) is immediate from (4.148) and (4.137) combined with the choice of the function ε_i in (4.113), which easily ensures $3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leq \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$.

Finally, we discuss (*T5′*). Due to (4.136) and due to the definition of $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ in (4.149), the left-hand side of (*T5′*) is bounded by $|U^{(k)}|$ and thus it suffices to show that

$$d(U^{(k)}|\hat{S}^{(k-1)}) \leq \frac{\delta_{k+1,\mathrm{RL}}^4}{8} d(R^{(k)}|\hat{S}^{(k-1)})\,. \tag{4.150}$$

From the definition of $U^{(k)}$ in (4.139), combined with (4.138) and the definition of Z_{THIN} in (4.134) we infer that

$$\begin{aligned}d(U^{(k)}|\hat{S}^{(k-1)}) &\leq |[z] \setminus Z_{\mathrm{THIN}}|\Big(\frac{1}{f(s_i)} + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})\Big) + |Z_{\mathrm{THIN}}|\frac{1}{f(s_i)} \\ &\overset{(4.133)}{\leq} \frac{s_i}{f(s_i)} + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})s_i\,,\end{aligned}$$

which in view of the definition of $f(\cdot)$ in (4.111) and (4.113) yields

$$d(U^{(k)}|\hat{S}^{(k-1)}) \leq \frac{\delta^4_{k+1,\mathrm{RL}}}{12t_0}.$$

Therefore, (4.150) follows from the last inequality and (4.132). This verifies (*Z5*′), which finishes the proof of Claim 4.53′ in this case. ◊

In both cases the partition $\mathscr{T}^{(k)}(\hat{S}^{(k-1)}, R^{(k)})$ of $\mathcal{K}_k(\hat{S}^{(k-1)}) \cap R^{(k)}$ has properties (*T2*′), (*T3*′), and (*T5*′), as required in Claim 4.53′. □

4.6 Proofs of the counting lemmas

In this section we prove the counting lemmas, Theorem 4.17 and Theorem 4.18, which complement the hypergraph regularity lemmas Theorem 4.12 and Theorem 4.15, respectively. The proof of Theorem 4.17 will be a consequence of the results from Section 4.2.1, i.e., the dense counting lemma (Theorem 4.19) and dense extension lemma (Corollary 4.26).

Proof of Theorem 4.17. Given integers $\ell \geq k \geq 2$ and positive constants γ and d_k set

$$\nu = \frac{d_k \gamma}{16\binom{\ell}{k}}. \tag{4.151}$$

After fixing ν the constant d_0 is displayed and we set

$$\gamma_{\mathrm{DEL}} = \frac{\gamma}{8\binom{\ell}{k}} \times \min\{d_0, d_k\}^{2^\ell}, \tag{4.152}$$

and then for $h = k$ and $F^{(k)} = K^{(k)}_\ell$ Corollary 4.26 yields positive constants

$$\varepsilon_{\mathrm{DEL}} = \varepsilon_{\mathrm{DEL}}(K^{(k)}_\ell, \gamma_{\mathrm{DEL}}, \min\{d_0, d_k\}) \\ \text{and } m_{\mathrm{DEL}} = m_{\mathrm{DEL}}(K^{(k)}_\ell, \gamma_{\mathrm{DEL}}, \min\{d_0, d_k\}). \tag{4.153}$$

We finally set $\varepsilon = \min\{\varepsilon_{\mathrm{DEL}}, \frac{d_k}{2}\}$ and $m_0 = m_{\mathrm{DEL}}$.

Let now $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$, $G^{(k)}$, and $H^{(k)}$ satisfying assumptions (*i*)–(*iii*) of Theorem 4.17 be given. Hence $\{R^{(j)}\}_{j=1}^{(k-1)} \cup \{G^{(k)}\}$ is an $(\varepsilon_{\mathrm{DEL}}, \boldsymbol{d})$-regular (m, ℓ, k)-complex with $\boldsymbol{d} = (d_2, \dots, d_k)$ and $d_j \geq \min\{d_0, d_k\}$ for $j = 1, \dots, k$. Observe that the choice of γ_{DEL} in (4.152) yields

$$\gamma_{\mathrm{DEL}} \leq \frac{\gamma}{8\binom{\ell}{k}} \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \leq \frac{\gamma}{8\binom{\ell}{k}} \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}}. \tag{4.154}$$

By Definition 4.20 we may view $\{R^{(j)}\}_{j=1}^{k-1} \cup \{G^{(k)}\}$ as an $(\varepsilon_{\mathrm{DEL}}, \boldsymbol{d}, K^{(k)}_\ell)$-regular complex. By the choice of constants in (4.153), we therefore can apply the dense extension lemma, Corollary 4.26, to $G^{(k)}$ and infer that

$$|G^{(k)}| = \binom{\ell}{k} \times (1 \pm \gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{k}{j}} \times m^k \,, \tag{4.155}$$

and, more importantly, that all but $\gamma_{\mathrm{DEL}}|G^{(k)}|$ edges $e \in G^{(k)}$ obey

$$\mathrm{ext}_{G^{(k)}}\,(e, K_\ell^{(k)}) = (1 \pm \gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} \,. \tag{4.156}$$

In view of the last assertion let $X \subseteq G^{(k)}$ be the set of exceptional edges in $G^{(k)}$. Consequently,

$$|X| \le \gamma_{\mathrm{DEL}}|G^{(k)}| \,, \tag{4.157}$$

and we infer

$$\begin{aligned} |\mathcal{K}_\ell(G^{(k)})| &= \frac{1}{\binom{\ell}{k}} \sum_{e \in G^{(k)}} \mathrm{ext}_{G^{(k)}}\,(e, K_\ell^{(k)}) \ge \frac{1}{\binom{\ell}{k}} \sum_{e \in G^{(k)} \setminus X} \mathrm{ext}_{G^{(k)}}\,(e, K_\ell^{(k)}) \\ &\overset{(4.156)}{\ge} \frac{1}{\binom{\ell}{k}} |G^{(k)} \setminus X| \times (1-\gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} \\ &\overset{(4.157)}{\ge} \frac{1}{\binom{\ell}{k}} (1-\gamma_{\mathrm{DEL}})|G^{(k)}| \times (1-\gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} \\ &\overset{(4.155)}{\ge} (1-\gamma_{\mathrm{DEL}})^3 \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell \ge \left(1 - \frac{\gamma}{2}\right) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell \,, \end{aligned} \tag{4.158}$$

where we used $\gamma_{\mathrm{DEL}} \le \gamma/6$ in the last inequality. We also note that (4.157) and (4.154) imply

$$|X| \le \frac{\gamma}{8\binom{\ell}{k}} \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times |G^{(k)}| \,. \tag{4.159}$$

Having estimated the number of cliques in $G^{(k)}$ we are going to bound the corresponding quantity in $H^{(k)}$. First observe that

$$|\mathcal{K}_\ell(H^{(k)})| \;\ge\; |\mathcal{K}_\ell(H^{(k)} \cap G^{(k)})| \;\ge\; |\mathcal{K}_\ell(G^{(k)})| - \sum_{e \in G^{(k)} \setminus H^{(k)}} \mathrm{ext}_{G^{(k)}}\,(e, K_\ell^{(k)}) \,. \tag{4.160}$$

Since the first term of the last estimate has been estimated (cf. (4.158)), we will now focus on the second. Since $G^{(k)}$ and $H^{(k)}$ are ν-close by assumption (*iii*) of Theorem 4.17 we have

$$|G^{(k)} \setminus H^{(k)}| \le \nu|\mathcal{K}_k(R^{(k-1)})| \le \frac{\nu|G^{(k)}|}{d_k - \varepsilon} \le \frac{2\nu}{d_k}|G^{(k)}| \,, \tag{4.161}$$

where we appealed to the (ε, d_k)-regularity of $G^{(k)}$ in the second inequality and $\varepsilon \le d_k/2$ in the last one. Consequently,

$$\begin{aligned}\sum_{e \in G^{(k)} \setminus H^{(k)}} &\mathrm{ext}_{G^{(k)}}\left(e, K_\ell^{(k)}\right) \\ &\overset{(4.156)}{\le} \left|(G^{(k)} \setminus H^{(k)}) \setminus X\right| (1+\gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} + |X| m^{\ell-k} \\ &\overset{(4.161)}{\le} \frac{2\nu}{d_k} |G^{(k)}| (1+\gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} + |X| m^{\ell-k} \qquad (4.162) \\ &\overset{(4.159)}{\le} \left(\frac{2\nu}{d_k}(1+\gamma_{\mathrm{DEL}}) + \frac{\gamma}{8\binom{\ell}{k}}\right) |G^{(k)}| \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} \\ &\overset{(4.155)}{\le} \frac{\gamma}{2} \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^{\ell},\end{aligned}$$

where we also used $\gamma_{\mathrm{DEL}} < 1$ and (4.151) in the last step. Then, (4.158) and (4.162) combined with (4.160), yields

$$|\mathcal{K}_\ell(H^{(k)})| \ge (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^{\ell},$$

which concludes the proof of Theorem 4.17. □

We now deduce Theorem 4.18 from Theorem 4.17. Theorem 4.18 gives a lower bound on the number of cliques in a (δ_k, d_k, r)-regular hypergraph $H^{(k)}$. In order to apply Theorem 4.17 we have to find an ε-regular $G^{(k)}$, which is ν-close to $H^{(k)}$ (cf. Definition 4.16). Such a regular approximation will be provided by Lemma 4.38.

Proof of Theorem 4.18. Let $\ell \ge k \ge 2$ be integers and γ and d_k be positive reals, given by Theorem 4.18. We first have to fix δ_k. For that let

$$\nu_{4.17} = \nu\big(\text{Thm.}4.17(\ell, k, \tfrac{\gamma}{2}, d_k)\big), \qquad (4.163)$$

be given by Theorem 4.17. We set δ_k

$$\delta_k = \frac{\nu_{4.17}}{24}. \qquad (4.164)$$

After displaying δ_k we get $d_{k-1}, \ldots, d_2 > 0$ satisfying $\frac{1}{d_i} \in \mathbb{N}$ for $i = 2, \ldots, k-1$ and have to fix constants δ, r, and m_0. For that we first use Theorem 4.17, which gives

$$\begin{aligned}\varepsilon_{4.17} &= \varepsilon\big(\text{Thm.}4.17(\ell, k, \tfrac{\gamma}{2}, d_0 = \min\{d_2, \ldots, d_{k-1}, d_k\})\big), \\ m_{4.17} &= m_0\big(\text{Thm.}4.17(\ell, k, \tfrac{\gamma}{2}, d_0 = \min\{d_2, \ldots, d_{k-1}, d_k\})\big).\end{aligned} \qquad (4.165)$$

As mentioned earlier, we intend to apply Lemma 4.38. For that we now fix the constants

$$s_{4.38} = 2\,,\ \nu_{4.38} = \nu_{4.17}\,,\ \varepsilon_{4.38} = \tfrac{1}{2}\varepsilon_{4.17}\,,\ \text{and } \boldsymbol{d}_{4.38} = (d_2, \dots, d_{k-1}) \tag{4.166}$$

to obtain the constants

$$\delta_{4.38}\,, \qquad \xi_{4.38}\,, \qquad t_{4.38}\,, \quad \text{and} \quad m_{4.38}\,.$$

Finally, we fix δ, r, and m_0 required by Theorem 4.18 to

$$\delta = \min\left\{\tfrac{1}{2}\varepsilon_{4.17}\,, \tfrac{1}{2}\delta_{4.38}\right\}, \qquad r = t_{4.38}^{2^k}\,, \quad \text{and} \tag{4.167}$$

$$m_0 = \max\left\{m_{4.17} + (t_{4.38})!\,, m_{4.38} + (t_{4.38})!\,, \tfrac{2}{\gamma}\ell(t_{4.38})!\right\}. \tag{4.168}$$

Having fixed all constants, let $m \geq m_0$, along with an $(\delta, (d_2, \dots, d_{k-1}))$-regular $(m, \ell, k-1)$-complex $\boldsymbol{R} = \{R^{(j)}\}_{j=1}^{k-1}$, and a hypergraph $H^{(k)} \subseteq \mathcal{K}_k(R^{(k-1)})$, satisfying $H^{(k)}$ is (δ_k, d_k, r)-regular w.r.t. $R^{(k-1)}[\Lambda_k]$ for every $\Lambda_k \in \binom{[\ell]}{k}$, be given.

First we obtain an $(\widetilde{m}, \ell, k-1)$-complex $\widetilde{\boldsymbol{R}} = \{\widetilde{R}^{(j)}\}_{j=1}^{k-1}$ and a hypergraph $\widetilde{H}^{(k)} \subseteq \mathcal{K}_k(\widetilde{R}^{(k-1)})$ from $\boldsymbol{R}$ and $H^{(k)}$, respectively, by removing at most $(t_{4.38})!$ vertices from each vertex class so that

$$(t_{4.38})! \quad \text{divides} \quad \widetilde{m} \quad \text{and} \quad m - (t_{4.38})! \leq \widetilde{m} \leq m\,. \tag{4.169}$$

Since we remove only constantly many vertices, we may assume w.l.o.g. that $\widetilde{\boldsymbol{R}}$ is a $(2\delta, (d_2, \dots, d_{k-1}))$-regular complex and $\widetilde{H}^{(k)}$ is $(2\delta_k, d_k, r)$-regular w.r.t. $\widetilde{R}^{(k-1)}[\Lambda_k]$ for every $\Lambda_k \in \binom{[\ell]}{k}$ and

$$d(\widetilde{H}^{(k)} | \widetilde{R}^{(k-1)}[\Lambda_k]) = d(H^{(k)} | R^{(k-1)}[\Lambda_k]) \pm o(1) = d_k \pm \varepsilon_{4.38}\,. \tag{4.170}$$

Now we want to apply Lemma 4.38 $\binom{\ell}{k}$ times for every $\Lambda_k \in \binom{[\ell]}{k}$, with the constants chosen in (4.166) to

$$\widetilde{\boldsymbol{R}}[\Lambda_k] = \{\widetilde{R}^{(j)}[\Lambda_k]\}_{j=1}^{k-1}\,, \qquad F_{\Lambda_k}^{(k)} = \mathcal{K}_k(\widetilde{R}^{(k-1)}[\Lambda_k])\,,$$

and

$$\left\{\widetilde{H}_{1,\Lambda_k}^{(k)} = \widetilde{H}^{(k)} \cap F_{\Lambda_k}^{(k)}\,,\ \widetilde{H}_{2,\Lambda_k}^{(k)} = F_{\Lambda_k}^{(k)} \setminus \widetilde{H}_{1,\Lambda_k}^{(k)}\right\}.$$

We now verify that $\widetilde{\boldsymbol{R}}[\Lambda_k]$, $F_{\Lambda_k}^{(k)}$, $\widetilde{H}_{1,\Lambda_k}^{(k)}$, and $\widetilde{H}_{1,\Lambda_k}^{(k)}$ satisfy the assumptions (a)–(d) of Lemma 4.38. In fact, (a) follows from $m \geq m_0$, (4.168), and (4.169).

Similarly, (b) follows from the fact that $\widetilde{\boldsymbol{R}}$ is a $(2\delta, (d_2, \dots, d_{k-1}))$-regular $(\widetilde{m}, \ell, k-1)$-complex and the choice of $\delta \leq \delta_{4.38}/2$ in (4.167).

Regarding property (c), we note that by definition $F_{\Lambda_k}^{(k)}$ is ξ-regular w.r.t. $\widetilde{R}^{(k-1)}[\Lambda_k]$ for every $\xi > 0$.

Finally, we consider (d). Clearly, $\{\widetilde{H}^{(k)}_{1,\Lambda_k}, \widetilde{H}^{(k)}_{2,\Lambda_k}\}$ is a partition of $F^{(k)}_{\Lambda_k}$. Moreover, $\widetilde{H}^{(k)}$ is $(2\delta_k, d_k, r)$-regular w.r.t. $\widetilde{R}^{(k-1)}$ and, hence, from the choice of δ_k in (4.164), $\nu_{4.17}$ in (4.166), and r in (4.167) we have that $\widetilde{H}^{(k)}_{1,\Lambda_k}$ is $(\nu_{4.38}/12, d_k, t^{2^k}_{4.38})$-regular w.r.t. $\widetilde{R}^{(k-1)}[\Lambda_k]$. The regularity of $\widetilde{H}^{(k)}_{2,\Lambda_k}$ then follows from Proposition 4.28.

Having verified the assumptions of Lemma 4.38, we repeatedly apply Lemma 4.38 for every $\Lambda_k \in \binom{[\ell]}{k}$ and infer that for each $\Lambda_k \in \binom{[\ell]}{k}$ there exist an

$$(\varepsilon_{4.38}, d(\widetilde{H}^{(k)}_{1,\Lambda_k}|\widetilde{R}^{(k-1)}[\Lambda_k]))\text{-regular hypergraph} \quad \widetilde{G}^{(k)}_{1,\Lambda_k}$$

which satisfies

$$|\widetilde{G}^{(k)}_{1,\Lambda_k} \triangle \widetilde{H}^{(k)}_1| \leq \nu_{4.38}|\mathcal{K}_k(\widetilde{R}^{(k-1)}[\Lambda_k])| \,.$$

Moreover, since $d(\widetilde{H}^{(k)}_{1,\Lambda_k}|\widetilde{R}^{(k-1)}[\Lambda_k]) = d(\widetilde{H}^{(k)}|\widetilde{R}^{(k-1)}[\Lambda_k]) = d_k \pm \varepsilon_{4.38}$ for every $\Lambda_k \in \binom{[\ell]}{k}$ (cf. (4.170)) setting

$$\widetilde{G}^{(k)} = \bigcup_{\Lambda_k \in \binom{[\ell]}{k}} \widetilde{G}^{(k)}_{1,\Lambda_k} \,,$$

gives rise to a sub-hypergraph of $\mathcal{K}_k(\widetilde{R}^{(k-1)})$, which is $\nu_{4.38}$-close to $\widetilde{H}^{(k)}$ and which is $(2\varepsilon_{4.38}, d_k)$-regular w.r.t. $\widetilde{R}^{(k-1)}[\Lambda_k]$ for every $\Lambda_k \in \binom{[\ell]}{k}$. Since, $2\varepsilon_{4.38} = \varepsilon_{4.17}$ and $\nu_{4.38} = \nu_{4.17}$ (cf. (4.166)) we can apply Theorem 4.17 to $\widetilde{\boldsymbol{R}}$, $\widetilde{G}^{(k)}$, and $\widetilde{H}^{(k)}$, which yields by the choices in (4.163) and (4.165) that

$$|\mathcal{K}_\ell(\widetilde{H}^{(k)})| \geq \left(1 - \frac{\gamma}{2}\right) \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times \widetilde{m}^\ell \,, \tag{4.171}$$

and, consequently, since $H^{(k)} \supseteq \widetilde{H}^{(k)}$ we have

$$\begin{aligned}
|\mathcal{K}_\ell(H^{(k)})| &\overset{(4.171)}{\geq} \left(1 - \frac{\gamma}{2}\right) \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times \widetilde{m}^\ell \\
&\overset{(4.169)}{\geq} \frac{1-\gamma}{1-\frac{\gamma}{2}} \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times (m - (t_{4.38})!)^\ell \\
&\overset{(4.168)}{\geq} (1-\gamma) \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times m^\ell \,.
\end{aligned}$$

□

5 Property testing and the removal lemma

In this Chapter we prove Theorem 1.19. The proof combines ideas from the work of Alon, Fischer, Krivelevich, and M. Szegedy [AFKS00] and from the recent work of Lovász and B. Szegedy [LS05] and is based on the hypergraph regularity lemma of Rödl and Skokan [RS04] and the corresponding counting lemma of Nagle, Rödl, and Schacht [NRS06a]. We introduce those lemmas in the next section.

5.1 The Rödl-Skokan lemma

We mainly follow the notation developed in Chapter 4. In contrast to the notions of regularity considered in Chapter 4, here we use the concepts of (δ, d, r)-regularity on all layers and not only on the k-th layer (compare Definition 5.2 below with Definition 4.6).

We extend the notion of (δ, d, r)-regular (m, j, j)-hypergraphs from Definition 4.13 to (m, ℓ, j)-hypergraphs.

Definition 5.1 **((δ, d, r)-regular).** *For m, $\ell \geq j \geq 1$ we say an (m, ℓ, j)-hypergraph $H^{(j)}$ is (δ, d, r)-**regular** (resp. $(\delta, \geq d, r)$-**regular**) w.r.t. an $(m, \ell, j-1)$-hypergraph $H^{(j-1)}$ if for every $\Lambda_j \in \binom{\ell}{j}$, the restriction $H^{(j)}[\Lambda_j] = H^{(j)}[\bigcup_{\lambda \in \Lambda_j} V_\lambda]$ is (δ, d, r)-regular (resp. $(\delta, \geq d, r)$-regular) w.r.t. the restriction $H^{(j-1)}[\Lambda_j] = H^{(j-1)}[\bigcup_{\lambda \in \Lambda_j} V_\lambda]$.*

We now extend the notion of (δ, d, r)-regularity from hypergraphs to complexes.

Definition 5.2 **($(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex).** *Let $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_h)$ be a vector of positive reals and let $\boldsymbol{d} = (d_2, \ldots, d_h)$ be a vector of non-negative reals. We say an (m, ℓ, h)-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{h}$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-**regular** (resp. $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r)$-**regular**) if*

(i) $H^{(2)}$ is $(\delta_2, d_2, 1)$-regular (resp. $(\delta_2, \geq d_2, 1)$-regular) w.r.t. $H^{(1)}$ and

(ii) $H^{(j)}$ is (δ_j, d_j, r)-regular (resp. $(\delta_j, \geq d_j, 1)$-regular) w.r.t. $H^{(j-1)}$ for every $j = 3, \ldots, h$.

5.1.1 Counting lemma

The following theorem was proved by Nagle, Rödl, and Schacht in [NRS06a]. It was one of the key ingredients for the proof of the removal lemma, Theorem 1.4, and will also play a crucial rôle here.

Theorem 5.3 (Counting lemma). *For all integers $2 \leq k \leq \ell$ the following is true: $\forall \gamma > 0$ $\forall d_k > 0$ $\exists \delta_k > 0$ $\forall d_{k-1} > 0$ $\exists \delta_{k-1} > 0$ $\ldots$ $\forall d_2 > 0$ $\exists \delta_2 > 0$ and there are* *integers r and m_0 so that, with $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $m \geq m_0$, whenever $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ is a $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r)$-regular (m, ℓ, k)-complex, then*

$$|\mathcal{K}_\ell(H^{(k)})| \geq (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell .$$

□

Since Theorem 1.19 concerns induced copies of hypergraphs an induced version of the counting lemma, which is a simple corollary of Theorem 5.3, will be useful. For the statement of that version we need the following definition.

Definition 5.4 (**$(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular, induced $(m, F^{(k)})$-complex**)**.** *Let $F^{(k)}$ be a k-uniform hypergraph with $V(F^{(k)}) = [\ell]$. Let $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ be a vector of positive reals and let $\boldsymbol{d} = (d_2, \dots, d_k)$ be a vector of non-negative reals. We say an (m, ℓ, k)-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ with vertex partition $V_1 \cup \dots \cup V_k$ is a* **$(\boldsymbol{\delta}, \geq\boldsymbol{d}, r)$-regular, *induced* $(m, F^{(k)})$-*complex*** *if*

(*i*) *the complex $\{H^{(j)}\}_{j=1}^{k-1}$ is a $(\boldsymbol{\delta}', \geq\boldsymbol{d}', r)$-regular $(m, \ell, k-1)$-complex with $\boldsymbol{\delta}' = (\delta_2, \dots, \delta_{k-1})$ and $\boldsymbol{d}' = (d_2, \dots, d_{k-1})$,*

(*ii*) *for every k-tuple in $K = \{\lambda_1, \dots, \lambda_k\} \in \binom{\ell}{k}$ we have*

  (*a*) *if $K \in E(F^{(k)})$, then the (m, k, k)-hypergraph $H^{(k)}[K] = H^{(k)}[\bigcup_{j=1}^{k} V_{\lambda_j}]$ is $(\delta_k, \geq d_k, r)$-regular w.r.t. $H^{(k-1)}[K]$*

  (*b*) *if K is not an edge in $F^{(k)}$, then the (m, k, k)-hypergraph complement*

$$\mathcal{K}_k(H^{(k-1)}[K]) \setminus H^{(k)}[K]$$

  is $(\delta_k, \geq d_k, r)$-regular w.r.t. $H^{(k-1)}[K]$.

We then state the induced version of Theorem 5.3.

Corollary 5.5. *For all integers $2 \leq k \leq \ell$ the following is true: $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ and there are integers r and m_0 so that for every $m \geq m_0$ and for every k-uniform hypergraph $F^{(k)}$ with vertex set $[\ell]$ the following holds.*

Let $\boldsymbol{d} = (d_2, \dots, d_k)$, $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$, and let $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ be a $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r)$-regular, induced $(m, F^{(k)})$-complex with vertex partition $V_1 \cup \dots \cup V_k$. Then $H^{(k)}$ contains at least

$$(1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell$$

induced copies of $F^{(k)}$.

Proof. It follows from Definition 5.4, that if $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ is a $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$-regular, induced $(m, F^{(k)})$-complex, then

$$\widetilde{\boldsymbol{H}} = \{H^{(1)}, \ldots, H^{(k-1)}, \widetilde{H}^{(k)}\}$$

is a $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$-regular (m, ℓ, k)-complex, where $\widetilde{H}^{(k)}$ is defined by setting for every $K \in \binom{\ell}{k}$

$$\widetilde{H}^{(k)}[K] = \begin{cases} H^{(k)}[K] & \text{if} \quad K \in F^{(k)}, \\ \mathcal{K}_k(H^{(k-1)}[K]) \setminus H^{(k)}[K] & \text{if} \quad K \notin F^{(k)}. \end{cases}$$

Moreover, every clique $K_k^{(\ell)}$ in $\widetilde{H}^{(k)}$ corresponds to an induced copy of $F^{(k)}$ in $H^{(k)}$ and, hence, Corollary 5.5 follows from Theorem 5.3 applied to $\widetilde{\boldsymbol{H}}$. □

5.1.2 Regularity lemma

In this section we introduce some more notation needed for the statement of the hypergraph regularity lemma, Theorem 5.7, from [RS04].

The following two definitions describe the "regularity" properties of the partition the regularity lemma shall provide. While the first definition deals with regularity properties of the auxiliary structure, the second definition describes how $H^{(k)}$ interacts with the partition.

Definition 5.6 ($(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$**-equitable**)**.** *Suppose V is a set of n vertices, $\mu > 0$, $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1}) \in (0,1]^{k-2}$ and $\boldsymbol{d} = (d_2, \ldots, d_{k-1}) \in [0,1]^{k-2}$ are vectors of reals and r is a positive integer.*

*We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V is $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$**-equitable** if:*

(*a*) $|\binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})| \leq \mu\binom{n}{k}$,

(*b*) *$\mathscr{P}^{(1)} = \{V_i \colon i \in [a_1]\}$ is an equitable vertex partition, i.e., $|V_1| \leq \cdots \leq |V_{a_1}| \leq |V_1| + 1$, and*

(*c*) *for all but at most $\mu\binom{n}{k}$ k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ the complex $\boldsymbol{P}(K)$ (see (4.1)) is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(n/a_1, k, k-1)$-complex.*

Next we state the regularity lemma for hypergraphs of Rödl and Skokan.

Theorem 5.7 (Rödl-Skokan lemma)**.** *Let $k \geq 2$ be a fixed integer. For every positive integer S, all positive μ and δ_k and functions $\delta_j \colon (0,1]^{k-j} \to (0,1]$ for $j = 2, \ldots k-1$ and $r \colon \mathbb{N} \times (0,1]^{k-2} \to \mathbb{N}$ there are integers T_0 and n_0 and $d_0 > 0$ so that the following holds.*

For every k-uniform hypergraph $H^{(k)}$ satisfying $|V(H^{(k)})| = n \geq n_0$ and every S-bounded family of partitions $\mathscr{Q} = \mathscr{Q}(k-1, \boldsymbol{a}^{\mathscr{Q}})$ with an equitable vertex partition, i.e., the vertex partition $\mathscr{Q}^{(1)} = \{V_1, \ldots, V_{a_1^{\mathscr{Q}}}\}$ satisfies $|V_1| \leq \cdots \leq |V_{a_1}| \leq |V_1| + 1$, there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ and a vector $\boldsymbol{d} = (d_2, \ldots, d_{k-1}) \in (0,1]^{k-2}$ so *that for*

$$\boldsymbol{\delta} = \boldsymbol{\delta}(\boldsymbol{d}) = (\delta_2, \dots, \delta_{k-1}), \text{ where } \delta_j = \delta_j(d_j, \dots, d_{k-1}) \text{ for } j = 2, \dots, k-1, \\ \text{and } r = r(a_1^{\mathscr{P}}, d_2, \dots, d_{k-1})$$

the following holds:

(*i*) $\mathscr{P}$ *is* $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$*-equitable and* T_0*-bounded,*

(*ii*) $H^{(k)}$ *is* $(\delta_k, *, r)$*-regular w.r.t.* $\mathscr{P}$ *(cf. Definition 4.14 in Chapter 4),*

(*iii*) $\mathscr{P} \prec \mathscr{Q}$*, i.e.,* $\mathscr{P}^{(j)} \prec \mathscr{Q}^{(j)}$ *for every* $j = 1, \dots, k-1$*, and*

(*iv*) $d_j \geq d_0$ *for every* $j = 2, \dots, k-1$*.*

□

Theorem 5.7 slightly differs from the stated hypergraph regularity lemma of Rödl and Skokan from [RS04]. However, a proof of Theorem 5.7 follows along the lines of [RS04]. We discuss the the five small differences below.

1. In the definition of family of partitions (Definition 4.8), we require that for every $j = 2, \dots, k-1$ and every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ there are *precisely* a_j partition classes in $\mathscr{P}^{(j)}$, which decompose $\mathcal{K}_j(\hat{P}^{(k-1)})$. In [RS04] a_j is only an upper bound of the number of partition classes contained in $\mathcal{K}_j(\hat{P}^{(k-1)})$. We may think of simply adding some artificial empty classes to $\mathscr{P}^{(j)}$ to have precisely a_j classes for every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$.

2. By Definition 5.6 part (*b*) we require that the vertex classes of $\mathscr{P}^{(1)}$ differ in size by at most 1. We can require this additional assertion, provided the initial vertex partition of $\mathscr{Q}$ has the same property, since it is well know that such an assertion holds for the graph regularity lemma of Szemerédi [Sze78] and since the hypergraph regularity lemma in [RS04] is proved by induction on the uniformity. For more details we refer to [RS04, Remark 7.19].

3. We also use a slightly different notation for the boundedness of a partition. More precisely the lemma in [RS04] admits a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ such that $|\hat{\mathscr{P}}^{(k-1)}| \leq T_0$. However, this clearly implies by Definition 4.8 that $\max_{j \in [k-1]} a_j \leq T_0$, i.e., $\mathscr{P}$ is T_0-bounded as stated in (*i*) of Theorem 5.7.

4. Another difference concerns assertion (*iii*) in Theorem 5.7. Recall that the proof of Szemerédi's regularity lemma relies on a procedure in which a given non-regular vertex partition $V_0 \cup V_1 \cup \dots \cup V_s$ will be "almost" refined by a partition $W_0 \cup W_1 \cup \dots \cup W_t$. Here "almost" refinement means that only the "exceptional" class W_0 may not be contained in V_0, while for every other class W_j there exist some $V_i \supseteq W_j$. However the initial vertex partition $U_1 \cup \dots \cup U_r$ is completely arbitrary and one can insist that the partitions obtained in the proof always refine the initial

one, if one allows not only one "exceptional" class, but one exceptional class, say $U_{i,0} \subseteq U_i$, for each $i \in [r]$, i.e., one exceptional class for every vertex class from the initial partition.

Similar adjustments can be made in the proof of the hypergraph regularity lemma from [RS04], this way we will have for every $j = 1, \ldots, k-1$ and every $Q^{(j)} \in \mathscr{Q}^{(j)}$ (of the given partition) always precisely one exceptional class $Q_0^{(j)}$.

We also note that such an argument was carried out in [RS04, Corollary 12.1], where the additional assertion (*iii*) of Theorem 5.7 was proved in the similar case when $\mathscr{Q}$ is replaced by an $(\ell, k-1)$-complex $\boldsymbol{G} = \{G^{(j)}\}_{j=1}^{k-1}$ and "refinement" means for every $j = 1, \ldots, k-1$ and every $P^{(j)}$ either $P^{(j)} \subseteq G^{(j)}$ or $P^{(j)} \cap G^{(j)} = \varnothing$. The proof for a bounded partition $\mathscr{Q}$ instead of a complex $\boldsymbol{G}$ is the same and follows the lines of the proof of [RS04, Corollary 12.1].

5. The last difference concerns (*iv*). This condition was not "built in" the regularity lemma of [RS04], but was explicitly proved, e.g., in [RS06, Claim 6.1]. We outline the simple proof here.

   First recall that by Definition 5.6 the number of non-crossing k-tuples, as well as, the number of k-tuples in irregular polyads is bounded by $\mu\binom{n}{k}$ for each reason. Therefore if $\mu < 1/8$ (an assumption one can clearly make without loss of generality) there are at least $(1-2\mu)\binom{n}{k} > (3/4)\binom{n}{k}$ k-tuples in regular polyads. Now all those k-tuples have its $\binom{k}{j}$ j-tuples $(2 \le j < k)$ in (d_j, δ_j, r)-regular (j, j)-hypergraphs from $\mathscr{P}^{(j)}$. Since the number of such hypergraphs is bounded by $T_0^{2^j} \le T_0^{2^k}$ we infer by the (d_j, δ_j, r)-regularity that $T_0^{2^k}(d_j + \delta_j)\binom{n}{k} \ge \frac{3}{4}\binom{n}{k}$, which provided $\delta_j(d_j, \ldots, d_{k-1}) \le d_j/2$ (an assumption one can clearly make without loss of generality) implies $d_j \ge 1/(2T_0^{2^k}) =: d_0$.

5.2 Auxiliary lemmas

5.2.1 Cluster hypergraphs.

An important part of the argument in the proof of Theorem 1.19 will be to compare hypergraphs of very different sizes to find two of "similar structure." For that we will use the hypergraph regularity lemma. Suppose hypergraphs of different size were regularized by Theorem 5.7 with the *same* input parameters. Then sizes of all of the families of partitions corresponding to each of the hypergraphs are bounded by the same T_0. Let us assume for now that all the partitions have the same size or more precisely have the same vector $\boldsymbol{a}$. Then we would like to say that two hypergraphs have the same structure, if there densities are similar on "every pair of corresponding polyads," for an appropriate bijection between the polyads of two partitions.

The similar idea of comparing "cluster graphs" corresponding to graphs of various sizes was used by Lovász and B. Szegedy [LS05]. The structure of partition yielded by the hypergraph regularity lemma is unfortunately more complicated than that for

Szemerédi's regularity lemma. In Section 5.2.1 we first introduce the notion of a *labeled* family of partitions, which in the graph case corresponds to a labeling of the vertex classes of the regular partition. Then, in Section 5.2.1, we develop the notion, which will later allow us to identify hypergraphs of the same structure, which is similar to the edge weights of the cluster graph.

Labeled partitions

It will be convenient to consider labeled families of partitions. Let $\mathscr{P}(k-1,\boldsymbol{a})$ be a family of partitions on V (see Definition 4.8). Consider an arbitrary numbering of the vertex classes of $\mathscr{P}^{(1)}$, i.e., $\mathscr{P}^{(1)} = \{V_i\colon\ i \in [a_1]\}$. For $j = 2,\dots,k-1$ let $\varphi^{(j)}\colon\ \mathscr{P}^{(j)} \to [a_j]$ be a labeling such that for every polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ the members of

$$\{P^{(j)} \in \mathscr{P}^{(j)}\colon\ P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}$$

are numbered from 1 to a_j in an arbitrary way.

This way, we obtain for every k-tuple $K = \{v_1,\dots,v_k\} \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ an integer vector $\hat{\boldsymbol{x}}_K = (\boldsymbol{x}_K^{(1)},\dots,\boldsymbol{x}_K^{(k-1)})$, where

$$\boldsymbol{x}_K^{(1)} = (\alpha_1 < \dots < \alpha_k) \text{ so that w.l.o.g. } K \cap V_{\alpha_i} = \{v_i\} \tag{5.1}$$

and for $j = 2,\dots,k-1$ we set

$$\boldsymbol{x}_K^{(j)} = \Big(\varphi^{(j)}(P^{(j)})\colon\ \{v_\lambda\colon\ \lambda \in \Lambda\} \in P^{(j)}\Big)_{\Lambda \in \binom{[k]}{j}} \tag{5.2}$$

Let $\binom{[a_1]}{k}_< = \{(\alpha_1,\dots,\alpha_k)\colon\ 1 \le \alpha_1 < \dots < \alpha_k \le a_1\}$ be the set of all "naturally" ordered k-element subsets of $[a_1]$ and set

$$\hat{A}(k-1,\boldsymbol{a}) = \binom{[a_1]}{k}_< \times \prod_{j=2}^{k-1} \underbrace{[a_j] \times \dots \times [a_j]}_{\binom{k}{j}\text{-times}} \tag{5.3}$$

for the address space of all k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$. The definitions above yield $\hat{\boldsymbol{x}}_K \in \hat{A}(k-1,\boldsymbol{a})$ for every $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$. Moreover, for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ we have

$$\hat{\boldsymbol{x}}_K = \hat{\boldsymbol{x}}_{K'} \text{ for all } K, K' \in \mathcal{K}_k(\hat{P}^{(k-1)}) \tag{5.4}$$

hence, for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ with $\mathcal{K}_k(\hat{\mathscr{P}}^{(k-1)}) \neq \varnothing$ we may set

$$\hat{\boldsymbol{x}}(\hat{P}^{(k-1)}) = \hat{\boldsymbol{x}}_K \text{ for some } K \in \mathcal{K}_k(\hat{P}^{(k-1)})\,. \tag{5.5}$$

Let

$$\hat{\mathscr{P}}^{(k-1)}_{\neq\varnothing} = \{\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}\colon\ \mathcal{K}_k(\hat{\mathscr{P}}^{(k-1)}) \neq \varnothing\}$$

and

$$\hat{A}_{\neq\varnothing} = \{\hat{\boldsymbol{x}} \in \hat{A}(k-1,\boldsymbol{a})\colon \exists\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\neq\varnothing} \text{ such that } \hat{\boldsymbol{x}}(\hat{P}^{(k-1)}) = \hat{\boldsymbol{x}}\}\,.$$

It is easy to see that the definition in (5.5) establishes a bijection between $\hat{\mathscr{P}}^{(k-1)}_{\neq\varnothing}$ and $\hat{A}_{\neq\varnothing}$.

Moreover, since $|\hat{\mathscr{P}}^{(k-1)}| = |\hat{A}(k-1,\boldsymbol{a})|$ (see (4.3) and (5.3)) this bijection can be extended to a bijection between $\hat{\mathscr{P}}^{(k-1)}$ and $\hat{A}(k-1,\boldsymbol{a})$. The inverse bijection maps $\hat{\boldsymbol{x}} \mapsto \hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$ and in the case $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})) \neq \varnothing$, i.e., $\hat{\boldsymbol{x}} \in \hat{A}_{\neq\varnothing}$ then

$$\boldsymbol{P}(\hat{\boldsymbol{x}}) = \boldsymbol{P}(K) \text{ for some } K \in \hat{P}^{(k-1)}(\hat{\boldsymbol{x}})\,,$$

is well defined due to (5.4). Note that $\boldsymbol{P}(\hat{\boldsymbol{x}}) = \{P^{(j)}\}_{j=1}^{k-1}$ is a $(k,k-1)$-complex with $P^{(k-1)} = \hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$. For later reference we summarize the above.

Definition 5.8 (labeled family of partitions). *Suppose $k \geq 2$ is an integer and $\boldsymbol{a} = (a_1,\dots,a_{k-1})$ is a vector of positive integers. We say*

$$\hat{A}(k-1,\boldsymbol{a}) = \binom{[a_1]}{k}_{<} \times \prod_{j=2}^{k-1} \underbrace{[a_j]\times\cdots\times[a_j]}_{\binom{k}{j}\text{-times}}\,,$$

is the ***address space****.*

For a family of partitions $\mathscr{P}(k-1,\boldsymbol{a})$ on some vertex set $V = V_1 \cup\dots\cup V_{a_1}$ we say a set of mappings $\boldsymbol{\varphi} = \{\varphi^{(2)},\dots,\varphi^{(k-1)}\}$, $\varphi^{(j)}\colon \mathscr{P}^{(j)} \to [a_j]$ for every $j = 2,\dots,k-1$ is an ***a-labeling*** *if for every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ we have*

$$\varphi^{(j)}\Big(\Big\{P^{(j)} \in \mathscr{P}^{(j)}\colon\ P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\Big\}\Big) = [a_j]\,.$$

Then $\hat{\boldsymbol{x}}_K = (\boldsymbol{x}_K^{(1)},\dots,\boldsymbol{x}_K^{(k-1)}) \in \hat{A}(k-1,\boldsymbol{a})$ defined in (5.1) and (5.2) defines an equivalence relation on $\mathrm{Cross}_k(\mathscr{P}^{(1)})$ (see (5.4)).

Consequently, such a labeling $\boldsymbol{\varphi}$ defines a bijection between $\hat{A}_{\neq\varnothing}$ and $\hat{\mathscr{P}}^{(k-1)}_{\neq\varnothing}$ (see paragraph below (5.5)) which can be extended to a bijection between $\hat{A}(k-1,\boldsymbol{a})$ and $\hat{\mathscr{P}}^{(k-1)}$ such that

(a) $\hat{\boldsymbol{x}} \in \hat{A}(k-1,\boldsymbol{a}) \mapsto \hat{P}^{(k-1)}(\hat{\boldsymbol{x}}) \in \hat{\mathscr{P}}^{(k-1)}$ and

(b) if $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})) \neq \varnothing$, then $\boldsymbol{P}(\hat{\boldsymbol{x}}) = \boldsymbol{P}(K)$ for some $K \in \hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$ is well defined,

(c) $K \in \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\boldsymbol{x}}_K))$ for every $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$, and

(d) $\boldsymbol{P}(\hat{\boldsymbol{x}}) = \{P^{(j)}\}_{j=1}^{k-1}$ is a $(k,k-1)$-complex with $P^{(k-1)} = \hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$.

Similarity of hypergraphs

The following definition will enable us to compare hypergraphs of different sizes. Very roughly speaking, we will think of two hypergraphs of being "similar" if there exists an integer vector $\boldsymbol{a}$ so that for each of them there exists an $\boldsymbol{a}$-labeled family of partitions on there respective vertex sets such that for every $\hat{\boldsymbol{x}} \in \hat{A}(k-1, \boldsymbol{a})$ the hypergraphs have the similar density on the respective polyad with address $\hat{\boldsymbol{x}}$.

Definition 5.9 (**$(d_{\boldsymbol{a},k}, \varepsilon)$-partition**)**.** *Suppose $\varepsilon > 0$, $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers, $\hat{A}(k-1, \boldsymbol{a})$ is an address space, $d_{\boldsymbol{a},k}\colon \hat{A}(k-1, \boldsymbol{a}) \to [0, 1]$ is a* ***density function****, and $H^{(k)}$ is a k-uniform hypergraph.*

We say an $\boldsymbol{a}$-labeled family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on $V(H^{(k)})$ is a $(d_{\boldsymbol{a},k}, \varepsilon)$- ***partition of*** *$H^{(k)}$ if for every $\hat{\boldsymbol{x}} \in \hat{A}(k-1, \boldsymbol{a})$*

$$d\big(H^{(k)} \big| \hat{P}^{(k-1)}(\hat{\boldsymbol{x}})\big) = d_{\boldsymbol{a},k}(\hat{\boldsymbol{x}}) \pm \varepsilon\,.$$

The concepts above allow to define an object similar to the *cluster graph* in the context of Szemerédi's regularity lemma. For a given $\delta > 0$ Szemerédi's regularity lemma provides a partition of the vertex set $V = V_1 \cup \dots \cup V_t$ of a given graph G, so that all but δt^2 pairs (V_i, V_j) are $(\delta, *, 1)$-regular. For many applications of that lemma it suffices to "reduce" the whole graph to a weighted graph on $[t]$, where the weight of the edge ij corresponds to the density of the bipartite subgraph of G induced on (V_i, V_j) (usually it will also be useful to mark those edges which correspond to irregular pairs). With that notion of cluster graph, one may say that two graphs G_1 and G_2 have the same structure if they admit a regular partition in the same number of parts so that the weights (densities) of the cluster graphs are essentially equal or deviate by at most ε.

The notion of address space extends the concept of the vertex labeling of the cluster graph in the context of the hypergraph regularity lemma. This way the function $d_{\boldsymbol{a},k}$ plays the rôle of the edge weights of the cluster graph. As we considered two graphs to be similar if they admit a regular partition with essentially the same cluster graph, we will view hypergraphs $H_1^{(k)}$ and $H_2^{(k)}$ to be ε-similar if there exists an integer vector $\boldsymbol{a}$ (and hence an address space $\hat{A}(k-1, \boldsymbol{a})$) and a density function function $d_{\boldsymbol{a},k}$ such that there is a "regular" $(d_{\boldsymbol{a},k}, \varepsilon)$-partition $\mathscr{P}_1(k-1, \boldsymbol{a})$ of $H_1^{(k)}$ and a "regular" $(d_{\boldsymbol{a},k}, \varepsilon)$-partition $\mathscr{P}_2(k-1, \boldsymbol{a})$ of $H_2^{(k)}$.

The following lemma, which is a simple corollary of the regularity lemma for hypergraphs, roughly states, that for any given infinite sequence $(H_i^{(k)})_{i=1}^{\infty}$ of hypergraphs and partitions, there exists a sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of "similar" hypergraphs (see (iv) of Lemma 5.10) on a "regular family of partitions" (see (i) and (ii)), which refine the original partitions (see (iii)).

Lemma 5.10. *Let $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ be a vector of positive integers. Suppose $(H_i^{(k)})_{i=1}^{\infty}$ is a sequence of hypergraphs such that $n_i = |V(H_i^{(k)})| \to \infty$ and for every $i \in \mathbb{N}$ there is a family of partitions $\mathscr{Q}_i = \mathscr{Q}_i(k-1, \boldsymbol{a})$ on $V(H_i^{(k)})$ with an equitable vertex partition, $\mathscr{Q}^{(1)} = \{V_1, \dots, V_{a_1}\}$ satisfying $|V_1| \le \dots \le |V_{a_1}| \le |V_1| + 1$. Then the following is true.*

For all positive constants μ and δ_k and functions

$$\delta_j\colon\ (0,1]^{k-j} \to (0,1] \text{ for } j=2,\dots k-1 \text{ and } r\colon\ \mathbb{N}\times(0,1]^{k-2}\to\mathbb{N}$$

there exist an integer vector $\boldsymbol{b}=(b_1,\dots,b_{k-1})$, an address space $\hat{A}(k-1,\boldsymbol{b})$, a density function $d_{\boldsymbol{b},k}\colon\ \hat{A}(k-1,\boldsymbol{b})\to[0,1]$, some $d_0>0$ and a sub-sequence $(H^{(k)}_{j_i})_{i=1}^{\infty}$ of $(H^{(k)}_i)_{i=1}^{\infty}$ such that for every $i\in\mathbb{N}$ there is a vector $\boldsymbol{d}_{j_i}=(d_{j_i,2},\dots,d_{j_i,k-1})\in[0,1]^{k-2}$ and a $\boldsymbol{b}$-labeled family of partitions $\mathscr{P}_{j_i}=\mathscr{P}_{j_i}(k-1,\boldsymbol{b})$ on $V(H^{(k)}_{j_i})$ such that

(*i*) *$\mathscr{P}_{j_i}$ is $(\mu,\boldsymbol{\delta}(\boldsymbol{d}_{j_i}),\boldsymbol{d}_{j_i},r(b_1,\boldsymbol{d}_{j_i}))$-equitable,*

(*ii*) *$H^{(k)}_{j_i}$ is $(\delta_k,*,r(b_1,\boldsymbol{d}_{j_i}))$-regular w.r.t. $\mathscr{P}_{j_i}$,*

(*iii*) *$\mathscr{P}_{j_i}\prec\mathscr{Q}_{j_i}$,*

(*iv*) *$\min\{d_{j_i,2},\dots,d_{j_i,k-1}\}\geq d_0$, and*

(*v*) *$\mathscr{P}_{j_i}$ is a $(d_{\boldsymbol{b},k},\mu)$-partition of $H^{(k)}_{j_i}$.*

Since we introduced three concepts related to the partitions, before we start with the proof, we briefly recall their meaning. Part (*i*) of Lemma 5.10 describes the *regularity* properties of the auxiliary partition $\mathscr{P}_{j_i}$ (see Definition 5.6) and (*ii*) describes the regularity of the hypergraph $H^{(k)}_{j_i}$ w.r.t. the partition $\mathscr{P}_{j_i}$ (see Definition 4.14). Finally (*v*) states that the *densities* of all H_{j_i} ($i\in\mathbb{N}$) on polyads with the same address are essentially the same and described by the function $d_{\boldsymbol{b},k}$ (see Definition 5.9).

Proof. Note that for given input parameters $S=\max_{j\in[k-1]}a_j$, μ, and δ_k and functions δ_j and r the regularity lemma, Theorem 5.7, guarantees for every $i\in\mathbb{N}$ the existence of a family of partitions $\mathscr{P}_i$ on $H^{(k)}_i$ with properties (*i*)–(*iv*) for some $\boldsymbol{b}=\boldsymbol{b}_i$ (which may depend on i) and some d_0 (independent of i).

The proof of Lemma 5.10 relies on the observation that it suffices to consider only finitely many choices for the integer vector $\boldsymbol{b}$ and for the density function $d_{\boldsymbol{b},k}$ (in view of (*v*)), which implies that for an infinite sub-sequence of $(H^{(k)}_i)_{i=1}^{\infty}$ those choices must be the same. It is obvious, that there are only finitely many choices for $\boldsymbol{b}$ as Theorem 5.7 gives an upper bound T_0 on $\max_{j\in[k-1]}b_j$, which is independent of $H^{(k)}_i$. However, the function $d_{\boldsymbol{b},k}$ is real-valued and we have to use an appropriate discretization. In view of Definition 5.8, one possible discretization is to consider intervals in $[0,1]$ of length about 2μ. More precisely, let $\mu_0\in(0,1]$ such that $\lceil 1/(2\mu)\rceil=1/(2\mu_0)$ and for every $\boldsymbol{b}$ consider special density functions

$$d_{\boldsymbol{b},k}\colon\ \hat{A}(k-1,\boldsymbol{b})\to\{(2j-1)\mu_0\colon\ j=1,\dots,1/(2\mu_0)\}\,. \tag{5.6}$$

Clearly, for every $\boldsymbol{b}$ there are only finitely many such density functions and, on the other hand, for any hypergraph $H^{(k)}_i$ and any $\boldsymbol{b}_i$-labeled family of partitions $\mathscr{P}_i(k-1,\boldsymbol{b}_i)$ there exist at least one such special function $\boldsymbol{d}_{\boldsymbol{b}_i,k}$ so that (*v*) holds.

Summarizing, since any given $S = \max_{j\in[k-1]} a_j$ and input parameters μ, δ_k and functions δ_j and r after an application of Theorem 5.7 to an S-bounded $\mathscr{Q}_i$ and $H_i^{(k)}$ the entries of the resulting $\boldsymbol{b}_i$ is bounded by T_0 there exist some particular vector $\boldsymbol{b}$ and an infinite sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ and a sequence of partitions $(\mathscr{P}_{j_i})_{i=1}^{\infty}$ such that properties (i)–(iv) hold. Considering then only density functions $d_{\boldsymbol{b},k}$ as in (5.6), we infer the existence of some function $d_{\boldsymbol{b},k}$ and the existence of some infinite sub-sequences of $(H_{j_i}^{(k)})_{i=1}^{\infty}$ and $(\mathscr{P}_{j_i})_{i=1}^{\infty}$ such that (v) holds. □

5.2.2 Index of a partition

In this section we recall the notion of *index* (or *mean-square density*) of a family of partition, which plays a crucial rôle in the proofs of the aforementioned (hyper)graph regularity lemmas (cf. Definition 4.43 in Chapter 4).

Definition 5.11 (index). *Let $H^{(k)}$ be a k-uniform hypergraph on n vertices and $\mathscr{P}$ be a family of partitions on $V(H^{(k)})$. The* **index of $\mathscr{P}$ w.r.t.** *$H^{(k)}$ is defined by*

$$\operatorname{ind}(\mathscr{P}|H^{(k)}) = \frac{1}{\binom{n}{k}} \sum \Big\{ d^2(H^{(k)}|\hat{P}^{(k-1)})|\mathcal{K}_k(\hat{P}^{(k-1)})| \colon \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \Big\} .$$

As an immediate consequence from the definition of index we have

$$0 \le \operatorname{ind}(\mathscr{P}|H^{(k)}) \le 1 \tag{5.7}$$

for every hypergraph $H^{(k)}$ and every family of partitions $\mathscr{P}$ on $V(H^{(k)})$. The following is a simple consequence of the Cauchy–Schwarz inequality.

Fact 5.12. *If $H^{(k)}$ is a k-uniform hypergraph and $\mathscr{P} \prec \mathscr{Q}$ are two refining families of partitions on $V(H^{(k)})$, then $\operatorname{ind}(\mathscr{P}|H^{(k)}) \ge \operatorname{ind}(\mathscr{Q}|H^{(k)})$.*

A proof of Fact 5.12 can be found in [RS04, Lemma 10.3].

The main lemma of this section, Lemma 5.14, considers two refining partitions, with "almost" the same index. For the statement of that lemma we need the following definition.

Definition 5.13 (ν-misbehaved). *Let $\nu > 0$ and $\mathscr{P} \prec \mathscr{Q}$ be two refining families of partitions on the same vertex set. We say a polyad $\hat{Q}^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$ is* **ν-misbehaved** *w.r.t. $\mathscr{P}$, if*

$$\begin{aligned}\sum \Big\{ |\mathcal{K}_k(\hat{P}^{(k-1)})| \colon \; &\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}, \; \hat{P}^{(k-1)} \subseteq \hat{Q}^{(k-1)}, \\ &\text{and } |d(H^{(k)}|\hat{P}^{(k-1)}) - d(H^{(k)}|\hat{Q}^{(k-1)})| > \nu \Big\} \ge \nu |\mathcal{K}_k(\hat{Q}^{(k-1)})| .\end{aligned} \tag{5.8}$$

We denote by $\mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)$ the set of all ν-misbehaved polyads $\hat{Q}^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$.

The following is the main lemma of the section. It asserts that if the index of two refining partitions is "close," then there are only few misbehaved polyads in the coarser partition (cf. Lemma 2.6 in Chapter 2).

Lemma 5.14. *Let ε, $\nu > 0$, $H^{(k)}$ be a k-uniform hypergraph on n vertices and $\mathscr{P} \prec \mathscr{Q}$ be two refining families of partitions on $V(H^{(k)})$. If $\operatorname{ind}(\mathscr{P}|H^{(k)}) \leq \operatorname{ind}(\mathscr{Q}|H^{(k)}) + \varepsilon$, then*

$$\sum \left\{|\mathcal{K}_k(\hat{Q}^{(k-1)})|\colon\ \hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)\right\} \leq \frac{2\varepsilon}{\nu^3}\binom{n}{k}.$$

The proof of Lemma 5.14 relies on the defect form of the Cauchy–Schwarz inequality, Lemma 2.4.

Proof of Lemma 5.14. Let $\hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)$ be fixed and let all the hypergraphs $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ with $\hat{P}^{(k-1)} \subseteq \hat{Q}^{(k-1)}$ be indexed by some set I and set for every $i \in I$

$$d_i = d(H^{(k)}|\hat{P}_i^{(k-1)}) \quad \text{and} \quad \sigma_i = |\mathcal{K}_k(\hat{P}_i^{(k-1)})|\,.$$

Clearly, with $\sigma_I = |\mathcal{K}_k(\hat{Q}^{(k-1)})|$ and $d = d(H^{(k)}|\hat{Q}^{(k-1)})$ we have $\sum_{i\in I}\sigma_i = \sigma_I$ and

$$|H^{(k)} \cap \mathcal{K}_k(\hat{Q}^{(k-1)})| = d\sigma_I = \sum_{i\in I} d_i\sigma_i\,. \tag{5.9}$$

Moreover, (5.8) corresponds to $\sum\{\sigma_j\colon\ |d_j - d| > \nu\} \geq \nu\sigma_I$ and, consequently, for some $J \subseteq I$ we obtain

$$\left|\sum_{j\in J}\frac{\sigma_j}{\sigma_J}d_j - \sum_{i\in I}\frac{\sigma_i}{\sigma_I}d_i\right| = \left|\sum_{j\in J}\frac{\sigma_j}{\sigma_J}d_j - d\right| \geq \nu\,.$$

where σ_J is defined as $\sigma_J = \sum_{j\in J}\sigma_j$ and J satisfies

$$\sigma_J \geq \frac{\nu}{2}\sigma_I$$

Therefore, Lemma 2.4 implies

$$\sum_{i\in I}\sigma_i d_i^2 \geq \sigma_I\left(\sum_{i\in I}\frac{\sigma_i}{\sigma_I}d_i\right)^2 + \frac{\nu^3}{2}\sigma_I\,. \tag{5.10}$$

Summarizing, due to (5.10) and (5.9) we showed for all $\hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)$ that

$$\begin{aligned}\sum\left\{d^2(H^{(k)}|\hat{P}^{(k-1)})|\mathcal{K}_k(\hat{P}^{(k-1)})|\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}, \hat{P}^{(k-1)} \subseteq \hat{Q}^{(k-1)}\right\}&\\ \geq d^2(H^{(k)}|\hat{Q}^{(k-1)})|\mathcal{K}_k(\hat{Q}^{(k-1)})| + \frac{\nu^3}{2}|\mathcal{K}_k(\hat{Q}^{(k-1)})|\,.&\end{aligned}$$

Hence, we infer from Lemma 2.4 (applied to every $\hat{Q}^{(k-1)} \notin \mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)$ with $\alpha = 0$) and the last inequality (applied to every $\hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)$) that

$$\operatorname{ind}(\mathscr{P}|H^{(k)}) \geq \operatorname{ind}(\mathscr{Q}|H^{(k)}) + \frac{\nu^3}{2\binom{n}{k}}\sum\left\{|\mathcal{K}_k(\hat{Q}^{(k-1)})|\colon\ \hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q}, \nu)\right\}$$

and hence the assumption of Lemma 5.14 implies

$$\sum\left\{|\mathcal{K}_k(\hat{Q}^{(k-1)})|:\ \hat{Q}^{(k-1)}\in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu)\right\}\le \frac{2\varepsilon}{\nu^3}\binom{n}{k}.$$

□

5.3 Proof of the general removal lemma

5.3.1 Proof of Theorem 1.19

In our argument we will assume that Theorem 1.19 fails. This means that there exists a family of k-uniform hypergraphs $\mathscr{F}$ and a constant $\eta>0$ such that for every c, C, and n_0 there exists a hypergraph $H^{(k)}$ on $n\ge n_0$ vertices which is η-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and which for every $\ell\le C$ contains at most cn^ℓ induced copies of $F^{(k)}$ for every $F^{(k)}\in\mathscr{F}$ on ℓ vertices. Applying this assumption successively with $c=1/i$ and $C=i$ for $i\in\mathbb{N}$ yields the following fact.

Fact 5.15. *If Theorem 1.19 fails for a family of k-uniform hypergraphs $\mathscr{F}$ and $\eta>0$, then there exists a sequence of hypergraphs $(H_i^{(k)})_{i=1}^\infty$ with $n_i=|V(H_i^{(k)})|\to\infty$ such that for every $i\in\mathbb{N}$*

(*i*) *$H_i^{(k)}$ is η-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and*

(*ii*) *for every $\ell\le i$ and every $F^{(k)}\in\mathscr{F}$ with $|V(F^{(k)})|=\ell$ the number of induced copies of $F^{(k)}$ in $H_i^{(k)}$ is less than n_i^ℓ/i.*

The same assumption was considered by Lovász and B. Szegedy in [LS05]. While they derived a contradiction based on the properties of a "limit object" of a carefully chosen sub-sequence of $(H_i^{(k)})_{i=1}^\infty$ the existence of which was established in [LS04], here we will only consider hypergraphs of the sequence $(H_i^{(k)})_{i=1}^\infty$. More precisely, the following, main lemma in the proof of Theorem 1.19, will locate two special hypergraphs $I^{(k)}=H_i^{(k)}$ and $J^{(k)}=H_j^{(k)}$ in the sequence from which we derive a contradiction.

Lemma 5.16. *Suppose Theorem 1.19 fails for $\mathscr{F}$ and $\eta>0$. Then there exists a hypergraph $I=I^{(k)}$ on ℓ vertices, an integer vector $\boldsymbol{a}=(a_1,\dots,a_{k-1})$, a density function $d_{\boldsymbol{a},k}\colon\ \hat{A}(k-1,\boldsymbol{a})\to[0,1]$, and a family of partitions $\mathscr{Q}_I=\mathscr{Q}_I(k-1,\boldsymbol{a})$ on $V(I^{(k)})$ such that*

(*I1*) *$\mathscr{Q}_I$ is a $(d_{\boldsymbol{a},k},\eta/24)$-partition of $I^{(k)}$,*

(*I2*) *$|\mathrm{Cross}_k(\mathscr{Q}_I^{(1)})|\ge(1-\frac{\eta}{24})\binom{\ell}{k}$, and*

(*I3*) *$I^{(k)}$ is η-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.*

Furthermore, there exists a hypergraph $J=J^{(k)}$ on $n\ge\ell$ vertices, a family of partitions $\mathscr{Q}_J=\mathscr{Q}_J(k-1,\boldsymbol{a})$ on $V(J^{(k)})$, an integer vector $\boldsymbol{b}=(b_1,\dots,b_{k-1})$, and a family of partitions $\mathscr{P}_J=\mathscr{P}_J(k-1,\boldsymbol{b})$ on $V(J^{(k)})$ such that

(J1) $\mathscr{Q}_J$ *is a* $(d_{\boldsymbol{a},k}, \eta/24)$*-partition of* $J^{(k)}$ *and*

(J2) $\mathscr{P}_J \prec \mathscr{Q}_J$.

Moreover, there exists an ℓ*-set* $L \in \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})$ *such that*

(L1) $|L \cap V_i| = |U_i|$ *where* $\mathscr{Q}_I^{(1)} = \{U_i \colon i \in [a_1]\}$ *and* $\mathscr{Q}_J^{(1)} = \{V_i \colon i \in [a_1]\}$,

(L2)

$$\left|\left\{K \in \binom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)}) \colon \right.\right.$$
$$\left.\left. |d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))| > \frac{\eta}{12}\right\}\right| \leq \frac{4\eta}{9}\binom{\ell}{k},$$

(L3) any k*-uniform hypergraph* $G^{(k)}$ *with vertex set* L *and with the property*

$$K \in G^{(k)} \Rightarrow d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \geq \frac{\eta}{12}$$

and

$$K \notin G^{(k)} \Rightarrow d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \leq 1 - \frac{\eta}{12},$$

belongs to $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

For the proof of Lemma 5.16 we will successively chose sub-sequences of $(H_i^{(k)})_{i=1}^\infty$ (see Fact 5.15), with each sequence being a sub-sequence of the previous. The sub-sequences will be obtained by Lemma 5.10 and after finitely many iterations we will select $I^{(k)}$ and $J^{(k)}$ from the "most current" sub-sequence (from which properties (*I1-I3*) and (*J1-J2*) will follow). We stop the iterations when the last sub-sequence $(H_{j_i}^{(k)})_{i=1}^\infty$ satisfies for every $i \in \mathbb{N}$

$$\mathrm{ind}(\mathscr{P}_{j_i}|H_{j_i}^{(k)}) \leq \mathrm{ind}(\mathscr{Q}_{j_i}|H_{j_i}^{(k)}) + \varepsilon \tag{5.11}$$

for some appropriately chosen $\varepsilon = \varepsilon(\eta)$. Clearly, we will reach this situation after at most $1/\varepsilon$ iterations (see (5.7) and Fact 5.12). By Lemma 5.14, we will infer from (5.11) that a randomly selected ℓ-tuple from the set of all ℓ-tuples satisfying (*L1*) admits (*L2*). Moreover, if we select $J^{(k)}$ "far enough" in the sequence, then (*ii*) of Fact 5.15 will be the key for proving (*L3*). We give the precise details in Section 5.3.2 and below we derive Theorem 1.19 from Lemma 5.16.

Proof of Theorem 1.19. The proof is by contradiction. Suppose there exists a family of k-uniform hypergraphs $\mathscr{F}$ and some $\eta > 0$ so that Theorem 1.19 fails. We apply Lemma 5.16 which yields hypergraphs $I^{(k)}$ (on ℓ vertices) and $J^{(k)}$ (on n vertices) and an ℓ-set $L \subseteq V(J^{(k)})$. In view of property (*L3*) we will define a hypergraph $G^{(k)}$ on the vertex set L. In order to obtain the desired contradiction we will "compare" the ℓ-vertex hypergraph $G^{(k)}$ with the ℓ-vertex hypergraph $I^{(k)}$. For that we need some bijection ψ from L to $V(I^{(k)})$. We will chose some bijection ψ which "agrees" with the labellings of $\mathscr{Q}_J$ and $\mathscr{Q}_I$, i.e., we require that for any k-tuple $K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ the address $\hat{\boldsymbol{x}}_K$ (see Definition 5.8) of K w.r.t. the $\boldsymbol{a}$-labeled partition $\mathscr{Q}_J$ coincides with the address $\hat{\boldsymbol{x}}_{\psi(K)}$ of $\psi(K)$ w.r.t. the $\boldsymbol{a}$-labeled partition $\mathscr{Q}_I$. More precisely, fix a bijection $\psi\colon L \to V(I^{(k)})$ such that for every $K \in \binom{L}{k}$ the following holds: if $K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ then

$$\psi(K) \in \mathrm{Cross}_k(\mathscr{Q}_I^{(1)}) \quad \text{and} \quad \hat{\boldsymbol{x}}_K = \hat{\boldsymbol{x}}_{\psi(K)}\,. \tag{5.12}$$

For a subset of $E \subseteq \binom{L}{k}$ we set $\psi(E) = \{\psi(K)\colon\ K \in E\}$.

We then define the hypergraph $G^{(k)}$ on L by

$$K \in G^{(k)} \quad \iff \quad \begin{cases} \text{either} & d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \geq \frac{\eta}{12} \text{ and } \psi(K) \in I^{(k)} \\ \text{or} & d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) > 1 - \frac{\eta}{12}\,. \end{cases} \tag{5.13}$$

for every k-tuple $K \in \binom{L}{k}$. Consequently, by (*L3*) of Lemma 5.16

$$G^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})\,. \tag{5.14}$$

It is left to show

$$|I^{(k)} \triangle \psi(G^{(k)})| \leq \eta \binom{\ell}{k}, \tag{5.15}$$

which due to (5.14) contradicts (*I3*) of Lemma 5.16, i.e., (5.15) contradicts the fact that $I^{(k)}$ is η-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

We cover the symmetric difference $I^{(k)} \triangle \psi(G^{(k)})$ by four sets $D_1, \ldots, D_4$ defined by

$$\begin{aligned} D_1 &= \binom{V(I^{(k)})}{k} \setminus \mathrm{Cross}_k(\mathscr{Q}_I^{(1)})\,, \\ D_2 &= \psi\Big(\{K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})\colon \\ &\qquad\qquad |d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))| > \eta/12\}\Big)\,, \\ D_3 &= I^{(k)} \cap \bigcup \Big\{\mathcal{K}_k(\hat{Q}_I^{(k-1)})\colon\ d(I^{(k)}|\hat{Q}_I^{(k-1)}) < \eta/4\Big\}\,, \end{aligned}$$

and

$$D_4 = \binom{L}{k} \setminus \Big(I^{(k)} \cap \bigcup \Big\{\mathcal{K}_k(\hat{Q}_I^{(k-1)})\colon\ d(I^{(k)}|\hat{Q}_I^{(k-1)}) > 1 - \eta/4\Big\}\Big)\,.$$

We first show that indeed $I^{(k)} \triangle \psi(G^{(k)}) \subseteq D_1 \cup \cdots \cup D_4$. For that first consider some $K' \in I^{(k)} \setminus \psi(G^{(k)})$ and set $K = \psi^{-1}(K')$. By the definition of $G^{(k)}$ in (5.13) we have

$d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) < \frac{\eta}{12}$. Then it is easy to show that if $K' \notin D_1 \cup D_2$ then $K' \in D_3$. Indeed, we have:

$$\begin{aligned} K' \in I^{(k)} \setminus \big(\psi(G^{(k)}) \cup D_1 \cup D_2\big) \\ \stackrel{(5.13)}{\Longrightarrow} d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) < \frac{\eta}{12} \\ \stackrel{K' \notin D_1 \cup D_2}{\Longrightarrow} d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) < \frac{\eta}{6}\,. \end{aligned} \tag{5.16}$$

Due to (*J1*) and (*I1*) of Lemma 5.16, $\mathscr{Q}_J$ and $\mathscr{Q}_I$ are $(d_{\boldsymbol{a},k}, \eta/24)$-partitions having the same density function $d_{\boldsymbol{a},k}\colon \hat{A}(k-1,\boldsymbol{a}) \to [0,1]$.

Hence, on the one hand, we infer $d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) = d_{\boldsymbol{a},k}(\hat{\boldsymbol{x}}_K) \pm \eta/24$ and, on the other hand, due to (5.12) and $K = \psi^{-1}(K')$, we have

$$d(I^{(k)}|\hat{Q}_I^{(k-1)}(K')) = d_{\boldsymbol{a},k}(\hat{\boldsymbol{x}}_K) \pm \eta/24\,.$$

Thus,

$$|d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) - d(I^{(k)}|\hat{Q}_I^{(k-1)}(K'))| \le \eta/12$$

and the right-hand side of (5.16) implies

$$\begin{aligned} K' \in I^{(k)} \setminus \big(\psi(G^{(k)}) \cup D_1 \cup D_2\big) \stackrel{(5.16)}{\Longrightarrow} d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) < \frac{\eta}{6} \\ \stackrel{(J1)\&(I1)}{\Longrightarrow} d(I^{(k)}|\hat{Q}_I^{(k-1)}(K')) < \frac{\eta}{6} + \frac{\eta}{12} = \frac{\eta}{4} \implies K' \in D_3\,. \end{aligned}$$

Similarly, for $K' \in \psi(G^{(k)}) \setminus I^{(k)}$ and $K = \psi^{-1}(K')$ we infer by similar arguments as above:

$$\begin{aligned} K' \in \psi(G^{(k)}) \setminus \big(I^{(k)} \cup D_1 \cup D_2\big) \\ \stackrel{(5.13)}{\Longrightarrow} d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) > 1 - \frac{\eta}{12} \stackrel{K' \notin D_1 \cup D_2}{\Longrightarrow} d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) > 1 - \frac{\eta}{6} \\ \stackrel{(J1)\&(I1)}{\Longrightarrow} d(I^{(k)}|\hat{Q}_I^{(k-1)}(K')) > 1 - \frac{\eta}{4} \implies K' \in D_4\,. \end{aligned}$$

Consequently, $I^{(k)} \triangle \psi(G^{(k)}) \subseteq D_1 \cup \dots \cup D_4$ and from (*I2*) of Lemma 5.16 we infer $|D_1| = |\binom{V(I^{(k)})}{k} \setminus \mathrm{Cross}_k(\mathscr{Q}_I^{(1)})| \le \eta\binom{\ell}{k}/24$ and (*L2*) implies $|D_2| \le 4\eta\binom{\ell}{k}/9$. Finally, the definitions of D_3 and D_4 yield $|D_3| \le \eta\binom{\ell}{k}/4$ and $|D_3| \le \eta\binom{\ell}{k}/4$. Summarizing the above, we obtain

$$|I^{(k)} \triangle \psi(G^{(k)})| \le |D_1| + |D_2| + |D_3| + |D_4| \le \Big(\frac{\eta}{24} + \frac{4\eta}{9} + \frac{\eta}{4} + \frac{\eta}{4}\Big)\binom{\ell}{k} < \eta\binom{\ell}{k}\,.$$

Thus we proved (5.15), which together with (5.14) yields a contradiction to (*I3*) of Lemma 5.16. □

5.3.2 Proof of Lemma 5.16

Since the proof is a bit technical, we will first give a sketch. The proof of Lemma 5.16 is based on iterative applications of Lemma 5.10. Given an infinite sequence of hypergraphs $(H_i^{(k)})_{i=1}^{\infty}$ each with a partition $\mathscr{Q}_i(k-1,\boldsymbol{a})$ (where $\boldsymbol{a}$ is the same for every $i \in \mathbb{N}$) and "measures of precision" (constants μ, δ_k and functions $\delta_{k-1},\dots,\delta_2$), Lemma 5.10 guarantees a vector $\boldsymbol{b}$ and a function $d_{\boldsymbol{b},k}\colon \hat{A}(k-1,\boldsymbol{b}) \to [0,1]$, a subsequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of $(H_i^{(k)})_{i=1}^{\infty}$ and $\boldsymbol{b}$-labeled partitions $\mathscr{P}_{j_i}(k-1,\boldsymbol{b}) \prec \mathscr{Q}_{j_i}$ such that

(*a*) $\mathscr{P}_{j_i}$ is "sufficiently regular" and

(*b*) $\mathscr{P}_{j_i}$ is $(d_{\boldsymbol{b},k},\mu)$-partition of $H_{j_i}^{(k)}$.

We will show that after at most $1/\varepsilon$ iterations, we will get two consecutive partitions $\mathscr{P}_{j_i} \prec \mathscr{Q}_{j_i}$ with the refining polyads having similar densities, more precisely $\mathscr{P}_{j_i}$ and $\mathscr{Q}_{j_i}$ will satisfy the assumptions of Lemma 5.14. We then set $J^{(k)}$ equal to $H_{j_i}^{(k)}$ (for some appropriately chosen i) and $I^{(k)}$ equal to the smallest hypergraph of the last sequence $(H_i^{(k)})_{i=1}^{\infty}$ (to which we applied Lemma 5.10 in the last application). Then Lemma 5.14 will imply that a random ℓ-tuple (chosen uniform at random from all ℓ-sets satisfying (*L1*)) will exhibit property (*L2*). Moreover, since by part (*ii*) of Fact 5.15, which holds since we assume that Theorem 1.19 fails, $J^{(k)} = H_{j_i}^{(k)}$ contains only a "few" induced copies of forbidden hypergraphs $F^{(k)} \in \mathscr{F}$ and, hence, the counting lemma (in form of Corollary 5.5) will yield (*L3*) of Lemma 5.16.

Proof of Lemma 5.16. Let $\mathscr{F}$ be a family of k-uniform hypergraphs and $\eta > 0$ and suppose Theorem 1.19 fails for $\mathscr{F}$ and η. By Fact 5.15 there exist a sequence of hypergraphs $(H_i^{(k)})_{i=1}^{\infty}$ with $n_i = |V(H_i^{(k)})| \to \infty$ admitting properties (*i*) and (*ii*) of Fact 5.15. Without loss of generality we may assume that

$$|V(H_i^{(k)})|^k = n_i^k \le \tfrac{3}{2} n_i \times \cdots \times (n_i - k + 1) \tag{5.17}$$

for every $i \in \mathbb{N}$. In the proof we need an auxiliary constant ε defined by

$$\varepsilon = \frac{1}{6}\left(\frac{\eta}{12}\right)^4 . \tag{5.18}$$

We want to iterate Lemma 5.10. This lemma locates a sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of hypergraphs satisfying (*i*)–(*v*) of Lemma 5.10 within a sequence of hypergraphs $(H_i^{(k)})_{i=1}^{\infty}$. Note that in particular property (*i*) of the sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ yields (among other things) the assumption allowing the next iteration, i.e., after an appropriate renaming and relabeling (*i*) implies that there exist an integer vector $\boldsymbol{a}$ and for every $i \in \mathbb{N}$ there is a family of partitions $\mathscr{Q}_i = \mathscr{Q}_i(k-1,\boldsymbol{a})$ on $V(H_i^{(k)})$ each of them having an equitable vertex partition (see (*b*) of Definition 5.6).

For the first iteration let $\boldsymbol{a} = (1, \dots, 1) \in \mathbb{N}^{k-1}$ and for every $i \in \mathbb{N}$ let $\mathscr{Q}_i = \mathscr{Q}_i(k-1, \boldsymbol{a})$ be the trivial partition $\mathscr{Q}_i = \{\{V(H_i^{(k)}\}, \{\varnothing\}, \dots, \{\varnothing\}\}$ on $V(H_i^{(k)})$ with only one vertex class and $\mathrm{Cross}_j(\mathscr{Q}_i^{(1)})$ being empty for $j \geq 2$.

More generally, suppose that after $p-1 \geq 0$ iterations we are given an integer vector $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ and sequences $(H_i^{(k)})_{i=1}^{\infty}$ and $(\mathscr{Q}_i)_{i=1}^{\infty}$ such that $\mathscr{Q}_i = \mathscr{Q}_i(k-1, \boldsymbol{a})$ is a family of partitions on $V(H_i^{(k)})$. We will now choose μ, δ_k, and functions δ_j ($j = 2, \dots, k-1$) and r with which we want to apply Lemma 5.10 in the p-th iteration. For that set

$$\ell_p = |V(H_1^{(k)})|, \qquad \mu = \min\left\{\tfrac{\eta}{24}, \tfrac{\ell^k}{9k!}\right\}, \tag{5.19}$$

and

$$\delta_k = \min\left\{\tfrac{\ell^k}{9k!}, \delta_k(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12))\right\}, \tag{5.20}$$

where $\delta_k(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12))$ is given by the "induced counting lemma," Corollary 5.5, applied for hypergraphs on ℓ_p vertices with $\gamma = 1/2$ and $d_k = \eta/12$. Similarly, for $j = 2, \dots, k-1$ let $\delta_j \colon (0,1]^{k-j} \to (0,1]$ be the function in variables $D_j, \dots, D_{k-1}$ given by Corollary 5.5 for ℓ_p, $\gamma = 1/2$, and $d_k = \eta/12$, i.e., for $j = 2, \dots, k-1$ we set

$$\delta_j(D_j, \dots, D_{k-1}) = \delta_j(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12, D_{k-1}, \dots, D_j)) \tag{5.21}$$

and

$$r(B_1, D_2, \dots, D_{k-1}) = r(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12, D_{k-1}, \dots, D_2)) \tag{5.22}$$

where we make no use of the variable B_1 in the definition of r. For those choices Lemma 5.10 yields an integer vector $\boldsymbol{b}$, a density function $d_{\boldsymbol{b},k} \colon \hat{A}(k-1, \boldsymbol{b}) \to [0,1]$, a constant $d_0 > 0$, a sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of $(H_i^{(k)})_{i=1}^{\infty}$, and for every $i \in \mathbb{N}$ a $\boldsymbol{b}$-labeled family partitions $\mathscr{P}_{j_i} = \mathscr{P}_{j_i}(k-1, \boldsymbol{b})$ on $H_{j_i}^{(k)}$ satisfying (i)–(v) of Lemma 5.10. We consider the index (see Definition 5.11) of the partitions $\mathscr{P}_{j_i}$ and define

$$S_p = \left\{ i \in \mathbb{N} \colon \ \mathrm{ind}(\mathscr{P}_{j_i} | H_{j_i}^{(k)}) \leq \mathrm{ind}(\mathscr{Q}_{j_i} | H_{j_i}^{(k)}) + \varepsilon \right\},$$

where ε was defined in (5.18). We distinguish two cases.

If S_p is finite then we iterate Lemma 5.10 and apply it in the next iteration (after an appropriate relabeling) to the infinite sub-sequence

$$(H_{j_i}^{(k)})_{i \in \mathbb{N} \setminus S_p} \quad \text{with} \quad \ell_{p+1} = |V(H_{\min \mathbb{N} \setminus S_p}^{(k)})|.$$

If, on the other hand, S_p is infinite, then we stop iterating. Note that in each iteration the index of $\mathscr{P}_{j_i}$ compared to the index of $\mathscr{Q}_{j_i}$ with respect to $H_{j_i}^{(k)}$ increases by a fixed ε (chosen independent of p) for every $i \in \mathbb{N} \setminus S_p$. Hence, in view of (5.7), after at most

$1/\varepsilon$ iterations the above procedure ends with an infinite set S_p.

Let $\boldsymbol{a}$, $(H_i^{(k)})_{i=1}^{\infty}$, $(\mathscr{Q}_i)_{i=1}^{\infty}$, $\boldsymbol{b}$, $d_{\boldsymbol{b},k}\colon \hat{A}(k-1,\boldsymbol{b}) \to [0,1]$, $d_0 > 0$, $(H_{j_i}^{(k)})_{i=1}^{\infty}$, $(\mathscr{P}_{j_i})_{i=1}^{\infty}$, and S_{p_0} be the the input and outcome of that "final," say p_0-the, iteration of Lemma 5.10. In other words for every $i \in S_{p_0}$ we have a $\boldsymbol{b}$-labeled family partitions $\mathscr{P}_{j_i} = \mathscr{P}_{j_i}(k-1,\boldsymbol{b})$ on $H_{j_i}^{(k)}$ satisfying

(L5.10.i) $\mathscr{P}_{j_i}$ is $(\mu, \boldsymbol{\delta}(\boldsymbol{d}_{j_i}), \boldsymbol{d}_{j_i}, r(b_1, \boldsymbol{d}_{j_i}))$-equitable,

(L5.10.ii) $H_{j_i}^{(k)}$ is $(\delta_k, *, r(b_1, \boldsymbol{d}_{j_i}))$-regular w.r.t. $\mathscr{P}_{j_i}$,

(L5.10.iii) $\mathscr{P}_{j_i} \prec \mathscr{Q}_{j_i}$,

(L5.10.iv) $\min\{d_{j_i,2}, \dots, d_{j_i,k-1}\} \geq d_0$, and

(L5.10.v) $\mathscr{P}_{j_i}$ is a $(d_{\boldsymbol{b},k}, \mu)$-partition of $H_{j_i}^{(k)}$,

where μ, δ_k, and functions δ_j and r were chosen in (5.19), (5.20), (5.21) and (5.22) depending on $\ell_{p_0} = \ell = |V(H_1^{(k)})|$. Moreover, by the definition of S_{p_0} for every $i \in S_{p_0}$ we have

$$\operatorname{ind}(\mathscr{P}_{j_i}|H_{j_i}^{(k)}) \leq \operatorname{ind}(\mathscr{Q}_{j_i}|H_{j_i}^{(k)}) + \varepsilon \tag{5.23}$$

Without loss of generality we may assume that $p_0 \geq 1$ and due to the choice of μ_{p_0-1} in (5.19) and from properties (i) and (v) of the penultimate iteration of Lemma 5.10 there exist a density function $d_{\boldsymbol{a},k}\colon \hat{A}(k-1,\boldsymbol{a}) \to [0,1]$ such that for every $i \in \mathbb{N}$

$$|\operatorname{Cross}_k(\mathscr{Q}_i^{(1)})| \geq \left(1 - \frac{\eta}{24}\right)\binom{|V(H_i^{(k)})|}{k} \tag{5.24}$$

$$\mathscr{Q}_i^{(1)} \text{ is an equitable vertex partition (see } (b) \text{ of Definition 5.6)} \tag{5.25}$$

$$\text{and } \mathscr{Q}_i \text{ is a } (d_{\boldsymbol{a},k}, \eta/24)\text{-partition of } H_i^{(k)}\,. \tag{5.26}$$

Next we choose the special hypergraphs $I^{(k)}$ and $J^{(k)}$ and verify properties ($I1$-$I3$) and ($J1$-$J2$). Then we will focus on ($L1$-$L3$). We set $I^{(k)}$ equal to the first hypergraph in the given sequence for the last iteration, i.e.,

$$I^{(k)} = H_1^{(k)}\,, \quad \ell = \ell_{p_0} = |V(I^{(k)})|\,, \text{and} \quad \mathscr{Q}_I = \mathscr{Q}_I(k-1,\boldsymbol{a}) = \mathscr{Q}_1(k-1,\boldsymbol{a})\,. \tag{5.27}$$

Note, however, that due to the relabeling in every iteration $H_1^{(k)}$ in (5.27) can be different from the first hypergraph in the sequence $(H_i^{(k)})_{i=1}^{\infty}$ originally obtained by Fact 5.15, which holds since by assumption of Lemma 5.16 Theorem 1.19 fails.

Next we select $J^{(k)}$ from the last sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$. It will be essential for our proof that the selected $J^{(k)}$ contains only a "few" induced copies of forbidden hypergraphs $F^{(k)} \in \mathscr{F}$ on ℓ or less vertices. For that we define the auxiliary constant

$$\alpha = \frac{1}{2}\left(\frac{\eta}{12}\right)^{\binom{\ell}{k}} \prod_{h=2}^{k-1} d_0^{\binom{\ell}{h}} \times \left(\frac{1}{b_1}\right)^{\ell}, \tag{5.28}$$

where d_0 is given by Lemma 5.10 (see ($L5.10.iv$)). We consider the subset $S^*_{p_0} \subseteq S_{p_0}$ with the property that for every $i \in S^*_{p_0}$

$$\#\Big\{F^{(k)} \overset{\text{ind}}{\subseteq} H^{(k)}_{j_i}\Big\} < \alpha |V(H^{(k)}_{j_i})|^{|V(F^{(k)})|} \tag{5.29}$$

for all $F^{(k)} \in \mathscr{F}$ with $|V(F^{(k)})| \leq \ell$. In fact $S^*_{p_0}$ is an infinite subset of S_{p_0}, since S_{p_0} is infinite and since $(H^{(k)}_{j_i})_{i=1}^{\infty}$ is a sub-sequence of the original sequence of hypergraphs, which satisfy in particular (ii) of Fact 5.15. For technical reasons we also want the hypergraph $J^{(k)}$ to be large and we select i_0 in $S^*_{p_0}$ sufficiently large, so that

$$\frac{1}{b_1}|V(H^{(k)}_{j_{i_0}})| \geq m_0(\mathrm{ICL}(\ell, \gamma = 1/2, d_k = \eta/12, d_{k-1} = d_0, \dots, d_2 = d_0))\,, \tag{5.30}$$

where $m_0(\mathrm{ICL}(\ell, \gamma = 1/2, d_k = \eta/12, d_{k-1} = d_0, \dots, d_2 = d_0))$ is given by Corollary 5.5. We then set $J^{(k)} = H^{(k)}_{j_{i_0}}$, $n = |V(J^{(k)})|$

$$\begin{aligned} \boldsymbol{d}_J &= (d_{J,2}, \dots, d_{J,k-1}) = (d_{j_{i_0},2}, \dots, d_{j_{i_0},k-1})\,,\\ \mathscr{Q}_J &= \mathscr{Q}_J(k-1, \boldsymbol{a}) = \mathscr{Q}_{j_{i_0}}(k-1, \boldsymbol{a})\,, \end{aligned}$$

and

$$\mathscr{P}_J = \mathscr{P}_J(k-1, \boldsymbol{b}) = \mathscr{P}_{j_{i_0}}(k-1, \boldsymbol{b})\,.$$

Properties ($I1$-$I3$) and ($J1$-$J2$) of Lemma 5.16 are immediate for those choices of $I^{(k)}$ and $J^{(k)}$. Indeed ($I1$) and ($J1$) follow from (5.26) and ($I2$) is satisfied due to (5.24). Property ($I3$) follows from part (i) of Fact 5.15 and, finally, ($J2$) is a consequence of ($L5.10.iii$).

It is left to prove the existence of an ℓ-set $L \in \mathrm{Cross}_\ell(\mathscr{P}^{(1)}_J)$ which displays properties ($L1$-$L3$). For that we consider a random ℓ-set from $V(J^{(k)})$. More precisely, let the labeled vertex partitions of $\mathscr{Q}_I$ and $\mathscr{Q}_J$ be $\mathscr{Q}^{(1)}_I = \{U_1, \dots, U_{a_1}\}$ and $\mathscr{Q}^{(1)}_J = \{V_1, \dots, V_{a_1}\}$. We select an ℓ-set uniformly at random from the probability space

$$\Omega = \prod_{i=1}^{a_1} \binom{V_i}{|U_i|}\,,$$

i.e., we select precisely $|U_i|$ vertices from V_i for every $i = 1, \dots, a_1$. Due to that particular choice of L, it displays property ($L1$). In view of the other "desired" properties of L we consider the following "bad" events

$$\begin{aligned}
&B_1\colon\ L \notin \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})\,,\\
&B_2\colon\ \exists K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\colon\ \boldsymbol{P}_J(K) \text{ is not a}\\
&\qquad\qquad (\boldsymbol{\delta}(\boldsymbol{d}_J), \boldsymbol{d}_J, r(b_1, \boldsymbol{d}_J))\text{-regular } (n/b_1, k, k-1)\text{-complex}\,,\\
&B_3\colon\ \exists K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\colon\ J^{(k)} \text{ is not}\\
&\qquad\qquad (\delta_k, *, r(b_1, \boldsymbol{d}_J))\text{-regular w.r.t. } \hat{P}_J(K)\,,
\end{aligned}$$

and

$$\begin{aligned}
B_4\colon\ \Big|\Big\{&K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})\colon\\
&|d(J^{(k)}|\hat{P}_J^{(k-1)}(K) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))| > \eta/12\Big\}\Big| > \frac{4\eta}{9}\binom{\ell}{k}\,.
\end{aligned}$$

Next we estimate the probabilities of the events $B_1, \dots, B_4$. For that the following observation will be useful.

Fact 5.17. *For every* $K \in \mathrm{Cross}_k(\mathscr{P}_J^{(1)})$

$$ext_L(K) := |\{L \in \Omega\colon\ K \subseteq L\}| = (1 \pm o(1)) \left(\frac{\ell}{n}\right)^k \binom{n/a_1}{\ell/a_1}^{a_1},$$

where $o(1) \to 0$ *as both* ℓ *and* n *tend to infinity and* a_1 *is fixed.*

Proof. Recall that by the definition of Ω, $\mathrm{ext}_L(K)$ is counting for a fixed k-set K the number of ℓ-sets L each of which contain K and for every $i \in [a_1]$ intersect the set V_i in $|U_i| = \ell/a_1$. This number is smallest if $K \subseteq V_i$ for some $i \in [a_1]$ and largest when $|K \cap V_i| \leq 1$ for every $i \in [a_1]$. Consequently and (5.25) we have for every $K \in \binom{V(J^{(k)})}{k}$

$$\binom{n/a_1 - k}{\ell/a_1 - k}\binom{n/a_1}{\ell/a_1}^{a_1-1} \leq \mathrm{ext}_L(K) \leq \binom{n/a_1 - 1}{\ell/a_1 - 1}^k \binom{n/a_1}{\ell/a_1}^{a_1-k},$$

and straightforward calculations yield Fact 5.17. □

Without loss of generality we assume that ℓ and n are sufficiently large, so that for every $K \in \binom{V(J^{(k)})}{k}$

$$\mathrm{ext}_L(K) = \left(1 \pm \frac{1}{3}\right)\left(\frac{\ell}{n}\right)^k \binom{n/a_1}{\ell/a_1}^{a_1}. \tag{5.31}$$

(This can easily be achieved by focusing on only sufficiently large hypergraphs in the sub-sequence $(H_{j_i}^{(k)})_{i=1}^\infty$ in the iteration procedure.) We now turn our attention to the events $B_1, \dots, B_4$ and prove upper bounds on the probabilities of those "bad" events. We start with $B_1 \cup B_2 \cup B_3$, i.e., we estimate the events that there is some k-set $K \subset L$

such that either $K \notin \mathrm{Cross}_k(\mathscr{P}_J^{(1)})$ or $\boldsymbol{P}_J(K)$ (see (4.1)) is not regular of $J^{(k)}$ is not regular w.r.t. $\hat{P}_J(K)$.

By $(L5.10.i)$ we have

$$|\{K \in \binom{V(J^{(k)})}{k} \colon K \notin \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\}| \le \mu \binom{n}{k}.$$

Moreover, it follows from $(L5.10.i)$ that

$$|\{K \in \mathrm{Cross}_k(\mathscr{P}_J^{(1)}) \colon \boldsymbol{P}_J(K) \text{ is not } (\delta(\boldsymbol{d}_J), \boldsymbol{d}_J, r(b_1, \boldsymbol{d}_J))\text{-regular}|\} \le \mu \binom{n}{k}$$

and from $(L5.10.ii)$ that

$$|\{K \in \mathrm{Cross}_k(\mathscr{P}_J^{(1)}) \colon J^{(k)} \text{ is not } (\delta_k, *, r(b_1, \boldsymbol{d}_J))\text{-regular w.r.t. } \hat{P}_J(K)|\} \le \delta_k \binom{n}{k}.$$

Due to (5.31) each k-tuple $K \in \binom{V(J^{(k)})}{k}$ extends to at most

$$\frac{4}{3}\left(\frac{\ell}{n}\right)^k \binom{n/a_1}{\ell/a_1}^{a_1}$$

different ℓ-sets $L \in \Omega$. Consequently, the number of pairs (L, K), with $L \in \Omega$ and $K \in \binom{L}{k}$ which is *bad*, i.e., $K \notin \mathrm{Cross}_k(\mathscr{P}_J^{(1)})$, or $\boldsymbol{P}_J(K)$ is not regular, or $J^{(k)}$ is not regular w.r.t. $\hat{P}_J(K)$, is at most

$$(2\mu + \delta_k)\binom{n}{k} \times \frac{4}{3}\left(\frac{\ell}{n}\right)^k \binom{n/a_1}{\ell/a_1}^{a_1}. \tag{5.32}$$

As $|\Omega| = \binom{n/a_1}{\ell/a_1}^{a_1}$, the expected number of bad k-tuples K in $\binom{L}{k}$ for a random ℓ-set L is at most

$$(2\mu + \delta_k) \times \frac{4\ell^k}{3k!} < \frac{1}{2} \tag{5.33}$$

(see (5.19), (5.20), and (5.27)). Therefore, by Markov's inequality we have

$$\mathbb{P}(B_1 \cup B_2 \cup B_3) < \frac{1}{2}. \tag{5.34}$$

Next, we consider B_4. We use the abortion criteria for the iteration of Lemma 5.10, i.e., we use (5.23). By Lemma 5.14 we infer from (5.23) that

$$\left|\left\{K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)}) \colon \hat{Q}_J^{(k-1)}(K) \in \mathrm{MB}_{\mathscr{P}_J}(\mathscr{Q}_J, \eta/12)\right\}\right| \le \frac{2\varepsilon}{(\eta/12)^3}\binom{n}{k}.$$

We say a k-tuple $K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ *misbehaves* if

$$\left|d(J^{(k)}|\hat{P}_J^{(k-1)}(K) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))\right| > \frac{\eta}{12}\,.$$

Hence for every $\hat{Q}_J^{(k-1)} \notin \mathrm{MB}_{\mathscr{P}_J}(\mathscr{Q}_J, \frac{\eta}{12})$ it follows from the definition of $\mathrm{MB}_{\mathscr{P}_J}(\mathscr{Q}_J, \frac{\eta}{12})$ (see Definition 5.13) that

$$\left|\{K \in \mathcal{K}_k(\hat{Q}_J^{(k-1)})\colon\ K \text{ misbehaves}\}\right| \le \frac{\eta}{12}|\mathcal{K}_k(\hat{Q}_J^{(k-1)})|\,.$$

Therefore, the combination of the last two estimates yields

$$\left|\{K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})\colon\ K \text{ misbehaves}\}\right| \le \left(\frac{2\varepsilon}{(\eta/12)^3} + \frac{\eta}{12}\right)\binom{n}{k} \overset{(5.18)}{\le} \frac{\eta}{9}\binom{n}{k}\,.$$

Consequently, similar calculations as in (5.32) and (5.33) give that for randomly chosen ℓ-set $L \in \Omega$ the expected number of misbehaved k-tuples $K \in \binom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ is at most

$$\frac{\eta}{9} \times \frac{4\ell^k}{3k!} \overset{(5.17)}{\le} \frac{2\eta}{9}\binom{\ell}{k}\,, \tag{5.35}$$

since $\ell = |V(I^{(k)})|$ and $I^{(k)}$ is a hypergraph from the sequence $(H_i^{(k)})_{i=1}^{\infty}$.

Recalling that B_4 is the event that a random ℓ-set $L \in \Omega$ contains more than $(4\eta/9)\binom{\ell}{k}$ misbehaved k-tuples we infer from (5.35) by Markov's inequality

$$\mathbb{P}(B_4) \le \frac{1}{2}\,. \tag{5.36}$$

From (5.34) and (5.36) we infer that there exist a "good" ℓ-set, i.e., there exist an ℓ-set $L \in \Omega \setminus (B_1 \cup \dots \cup B_4)$. We now show that such an ℓ-set has the desired properties (*L1*-*L3*) of Lemma 5.16.

First, since $L \notin B_1$ we have $L \in \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})$ as required. Moreover, (*L1*) holds by definition of Ω and (*L2*) is equivalent to $L \notin B_4$.

Finally, we focus on property (*L3*). Let a hypergraph $G^{(k)}$ with vertex set L be given as in (*L3*). Let $\boldsymbol{P}_J(L) = \{P_J^{(j)}(L)\}_{j=1}^k$ be defined for $j \in [k]$ by

$$P_J^{(j)}(L) = \begin{cases} \bigcup\left\{P_J^{(j)}(K)\colon\ K \in \binom{L}{k}\right\} & \text{if } j = 1\dots,k-1\,,\\ \bigcup\left\{J^{(k)} \cap \mathcal{K}_k(\hat{P}_J^{(k-1)}(K))\colon\ K \in \binom{L}{k}\right\} & \text{if } j = k\,. \end{cases} \tag{5.37}$$

Since $L \notin B_2 \cup B_3$ the complex $\boldsymbol{P}_J(K)$ is a $(\boldsymbol{\delta}(\boldsymbol{d}_J), \boldsymbol{d}_J, r(b_1, \boldsymbol{d}_J))$-regular $(n/b_1, k, k-1)$-complex for every $K \in \binom{L}{k}$ and $J^{(k)}$ is $(\delta_k, *, r(b_1, \boldsymbol{d}_J))$-regular w.r.t. $\hat{P}_J^{(k-1)}(K)$. Furthermore, the assumptions on $G^{(k)}$ in (*L3*) imply $d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \ge \eta/12$ for $K \in G^{(k)}$ and $d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \le 1 - \eta/12$ for $K \notin G^{(k)}$. Consequently, the definition of $\boldsymbol{P}_J(L)$ in (5.37) yields that $\boldsymbol{P}_J(L)$ is a $(\boldsymbol{\delta}', {\ge}\boldsymbol{d}', r(b_1, \boldsymbol{d}_J))$-regular, induced $(n/b_1, G^{(k)})$-

complex with $\boldsymbol{\delta}' = (\boldsymbol{\delta}(\boldsymbol{d}_J), \delta_k)$ and $\boldsymbol{d}' = (\boldsymbol{d}_J, \eta/12)$. Due to the choice of δ_k, and the functions δ_j and r in (5.20), (5.21), and (5.22) and due to (5.30) we can apply the "induced" counting lemma, Corollary 5.5. It follows that $J^{(k)}$ contains at least

$$\frac{1}{2}\left(\frac{\eta}{12}\right)^{\binom{\ell}{k}} \prod_{j=2}^{k-1} d_{J,j}^{\binom{\ell}{j}} \times \left(\frac{n}{b_1}\right)^{\ell} \overset{(L5.10.iv)}{\geq} \frac{1}{2}\left(\frac{\eta}{12}\right)^{\binom{\ell}{k}} \prod_{j=2}^{k-1} d_0^{\binom{\ell}{j}} \times \left(\frac{n}{b_1}\right)^{\ell} \overset{(5.28)}{=} \alpha|n|^{\ell}$$

induced copies of $G^{(k)}$. Then the choice of $J^{(k)}$ due to (5.29) implies that $G^{(k)} \notin \mathscr{F}$. Similarly, for every subset $L' \subseteq L$ we infer from Corollary 5.5 applied to $\boldsymbol{P}_J(L')$ that the number of induced copies of $G^{(k)}[L']$ in $J^{(k)}$ is at least

$$\frac{1}{2}\left(\frac{\eta}{12}\right)^{\binom{|L'|}{k}} \prod_{j=2}^{k-1} d_{J,j}^{\binom{|L'|}{j}} \times \left(\frac{n}{b_1}\right)^{|L'|}$$

$$\textit{subsection} \overset{(L5.10.iv)}{\geq} \frac{1}{2}\left(\frac{\eta}{12}\right)^{\binom{\ell}{k}} \prod_{j=2}^{k-1} d_0^{\binom{\ell}{j}} \times \frac{n^{|L'|}}{b_1^{\ell}} \overset{(5.28)}{=} \alpha|n|^{|L'|}\,.$$

Hence, the choice of $J^{(k)}$ in view of (5.29) implies $G^{(k)}[L'] \notin \mathscr{F}$. Since we inferred $G^{(k)}[L'] \notin \mathscr{F}$ for any $L' \subseteq L$ we have $G^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$, which is (*L3*) of Lemma 5.16. We thus showed that L displays all required properties and this concludes the proof of Lemma 5.16. □

6 Sparse partition universal graphs

In 1983, Chvátal, Rödl, Szemerédi, and Trotter proved that for any Δ there exists B so that, for any n, any 2-coloring of the edges of the complete graph K_N with $N \geq Bn$ vertices yields a monochromatic copy of any graph H that has n vertices and maximum degree Δ. In this Chapter prove that the complete graph may be replaced by a sparser graph G that has N vertices and $O(N^{2-1/\Delta} \log^{1/\Delta} N)$ edges, with $N = \lfloor B'n \rfloor$ for some constant B' that depends only on Δ. Consequently, the so called *size-Ramsey number* of any H with n vertices and maximum degree Δ is $O(n^{2-1/\Delta} \log^{1/\Delta} n)$. Our approach is based on random graphs; in fact, we show that the classical Erdős–Rényi random graph with the numerical parameters above satisfies a stronger partition property with high probability, namely, that any 2-coloring of its edges contains a monochromatic *universal graph* for the class of graphs on n vertices and maximum degree Δ.

The main tool in our proof is the regularity method, adapted to a suitable sparse setting. The novel ingredient developed here is an embedding strategy that allows one to embed bounded degree graphs of linear order in certain quasi-random graphs. Crucial to our proof is a rather surprising phenomenon, namely, the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed.

In Section 6.1 we recall some basic facts about regularity, including the results on inheritance of regularity proved in [GKRS07] (see Section 6.2). In Section 6.3.3, the results on the hereditary nature of regularity, in the form that is required here, are derived from the results quoted in Section 6.2. Other relevant results on random graphs are given in Sections 6.3.1 and 6.3.2. The proof of Theorem 1.23 is given in Section 6.4.

6.1 The sparse regularity lemma

Let $G = (V, E)$ be a graph. Suppose $0 < p \leq 1$, $\eta > 0$ and $K > 1$. For two disjoint subsets X, Y of V, we let $e_G(X, Y)$ be the number of edges of G with one endpoint in X and the other endpoint in Y. Furthermore, we let

$$d_{G,p}(X, Y) = \frac{e_G(X, Y)}{p|X||Y|},$$

which we refer to as the *p-density* of the pair (X, Y). We say that G is an (η, K)-*bounded* graph with respect to density p if for all pairwise disjoint sets X, $Y \subset V$, with $|X|$, $|Y| \geq \eta|V|$, we have

$$e_G(X, Y) \leq Kp|X||Y|\,.$$

For $\varepsilon > 0$ fixed and X, $Y \subset V$, $X \cap Y = \varnothing$, we say that the pair (X, Y) is (ε, p)-*regular* if for all $X' \subset X$ and $Y' \subset Y$ with

$$|X'| \geq \varepsilon|X| \quad \text{and} \quad |Y'| \geq \varepsilon|Y|,$$

we have

$$|d_{G,p}(X,Y) - d_{G,p}(X',Y')| \leq \varepsilon.$$

Note that for $p = 1$ we get the well-known definition of ε-regularity [Sze78].

Let $\dot{\bigcup}_{j=0}^{t} V_j$ be a partition of V. We call V_0 the *exceptional class*. This partition is called (ε, t)-*equitable* if $|V_0| \leq \varepsilon|V|$ and $|V_1| = \cdots = |V_t|$.

We say that an (ε, t)-equitable partition $\dot{\bigcup}_{j=0}^{t} V_j$ of V is (ε, G, p)-*regular* if all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq k$, are (ε, p)-regular. Now we state a variant of the Szemerédi's regularity lemma [Sze78] for sparse graphs, which was observed independently by Kohayakawa and Rödl (see, e.g., [Koh97, KR03b]).

Theorem 6.1 (Sparse regularity lemma). *For any $\varepsilon > 0$, $K > 1$, and $t_0 \geq 1$, there exist constants T_0, η, and N_0 such that any graph G with at least N_0 vertices that is (η, K)-bounded with respect to density $0 < p \leq 1$ admits an (ε, t)-equitable (ε, G, p)-regular partition of its vertex set with $t_0 \leq t \leq T_0$.* □

6.2 The hereditary nature of sparse regularity

We shall also use the fact that ε-regularity is typically inherited on "small" (sublinear) subsets. This was essentially observed for the classical notion of (dense) regular pairs by Duke and Rödl [DR85] and for sparse regular pairs in [GKRS07, KR03a]. Here we shall use a result from [GKRS07] regarding the hereditary nature of (ε, α, p)-*denseness* (or "one sided-regularity").

Definition 6.2. *Let α, $\varepsilon > 0$, and $0 < p \leq 1$ be given and let $G = (V, E)$ be a graph. For sets X, $Y \subset V$, $X \cap Y = \varnothing$, we say that the pair (X, Y) is (ε, α, p)-dense if for all $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have*

$$d_{G,p}(X', Y') \geq \alpha - \varepsilon.$$

It follows immediately from the definition that (ε, α, p)-denseness is inherited on large sets, i.e., that for a (ε, α, p)-dense pair (X, Y) and any sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \mu|X|$ and $|Y'| \geq \mu|Y|$ the pair (X', Y') is $(\varepsilon/\mu, \alpha, p)$-dense. The following result from [GKRS07] states that this "denseness-property" is even inherited on randomly chosen subsets of much smaller size with overwhelming probability.

Theorem 6.3 ([GKRS07, Theorem 3.6]). *For every α, $\beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon') > 0$ and $L = L(\alpha, \varepsilon')$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < p < 1$, every (ε, α, p)-dense pair (X, Y) in a graph G satisfies that the number of sets $X' \subseteq X$ with $|X'| = m \geq L/p$ such that (X', Y) is an $(\varepsilon', \alpha, p)$-dense pair is at least $(1 - \beta^m)\binom{|X|}{m}$.* □

The following is a direct consequence of Theorem 6.3, which we obtain by applying it first to X and then to subsets of Y.

Corollary 6.4 ([GKRS07, Corollary 3.8]). *For every α, $\beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon') > 0$ and $L = L(\alpha, \varepsilon')$ such that for any $0 < \varepsilon \le \varepsilon_0$ and $0 < p < 1$, every (ε, α, p)-dense pair (X, Y) in a graph G satisfies that the number of pairs (X', Y') of sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = m_1 \ge L/p$ and $|Y'| = m_2 \ge L/p$ such that (X', Y') is an $(\varepsilon', \alpha, p)$-dense pair is at least $(1 - \beta^{\min\{m_1, m_2\}})\binom{|X|}{m_1}\binom{|Y|}{m_2}$.* □

6.3 Properties of the random graph

In this section we shall verify a few properties of random graphs that will be useful for the proof of Theorem 1.21.

6.3.1 Uniform edge distribution

We start with a well known fact, which follows easily from the properties of the binomial distribution, concerning the edge distribution of $G(N, p)$.

Definition 6.5. *For an integer N and $0 < p \le 1$ we define the family of graphs $U_{N,p}$ on $[N] = \{1, \ldots, N\}$ with uniform edge distribution by*

$$\begin{aligned} U_{N,p} := \Big\{ G\colon\ & V(G) = [N] \text{ and} \\ & \forall\, U, W \subset V(G) \text{ with } U \cap W = \varnothing,\ |U| \ge \tfrac{N}{\log N}, \\ & \text{and } |W| \ge \tfrac{N}{\log N} \text{ we have } e_G(U, W) = (1 \pm \tfrac{1}{\log N}) p |U||W| \Big\}. \end{aligned}$$

The following proposition follows directly from the Chernoff bound for binomially distributed random variables.

Proposition 6.6. *If $p = p(N) \gg (\log N)^4/N$, then $\mathbb{P}(G(N, p) \in U_{N,p}) = 1 - o(1)$.* □

In Proposition 6.6 and in the remainder of this chapter, $o(1)$ denotes a function that tends to 0 as $N \to \infty$. We also use the symbols $\ll$ and $\gg$; e.g., we write $f(N) \ll g(N)$ to mean that $f(N)/g(N) \to 0$ as $N \to \infty$.

6.3.2 Expansion properties of neighbourhoods

For a graph $G = (V, E)$ and an integer $k \ge 1$, we define the auxiliary, bipartite graph $\Gamma(k, G) = (\binom{V}{k} \cup V, E_{\Gamma(k,G)})$ by

$$(K, v) \in E_{\Gamma(k,G)} \iff \{w, v\} \in E(G) \text{ for all } w \in K\,. \tag{6.1}$$

Proposition 6.8, given below, states that if G is the random graph $G(N, p)$, then the graph $\Gamma(k, G)$ has no "dense patches". More precisely, we consider the following property.

Definition 6.7. *Let integers N and $k \ge 1$ and reals $\xi > 0$ and $0 < p \le 1$ be given. We say that a graph $G = (V, E)$ with $V = [N]$ has the neighbourhood expansion property $E^k_{N,p}(\xi)$ if for every $U \subseteq V$ and every family $F_k \subseteq \binom{V \setminus U}{k}$ of pairwise disjoint k-sets with* (*i*) $|F_k| \le \xi N$ and

(*ii*) $|U| \le |F_k|$

we have

$$e_{\Gamma(k,G)}(F_k, U) \le p^k |F_k||U| + 6\xi N p^k |F_k| \,. \tag{6.2}$$

We show that for appropriate p the random graph $G(N,p)$ asymptotically almost surely has property $E^k_{N,p}(\xi)$.

Proposition 6.8. *For every integer $k \ge 1$ and real $\xi > 0$, there exists $C > 1$ such that if $p > C(\log N/N)^{1/k}$, then $\mathbb{P}(G(N,p) \in E^k_{N,p}(\xi)) = 1 - o(1)$.*

Proof. For given k and ξ we let C be a constant satisfying

$$C^k > k/\xi \,.$$

Let F_k and U satisfy (*i*) and (*ii*) of Definition 6.7. We consider two cases depending on the size of F_k.

Case 1 ($|F_k| \ge N/\log N$)**.** *Note that for fixed F_k and U the edges of*

$$\Gamma[F_k, U] = \Gamma(k, G(N,p))[F_k, U]$$

appear independently with probability p^k. Thus $e_\Gamma(F_k, U)$ is a binomial random variable with distribution $\mathrm{Bi}(p^k, |F_k||U|)$. From Chernoff's inequality

$$\mathbb{P}(X \ge \mathbb{E}X + t) \le \exp(-t)$$

for a binomial random variable X and $t \ge 6\mathbb{E}X$ (see e.g. [JŁR00, Corollary 2.4]), we infer

$$\mathbb{P}\big(e_\Gamma(F_k, U) > p^k|F_k||U| + 6\xi N p^k |F_k|\big) \le \exp\big(-6\xi N p^k |F_k|\big)\,,$$

since we have $|U| \le |F_k| \le \xi N$ from (i) and (ii) of Definition 6.7.

Moreover, the number of choices for F_k (satisfying the assumptions of this case) and U is at most $\sum_{f=N/\log N}^{\xi N} N^{kf}$ and 2^N, respectively, and since

$$\sum_{f=N/\log N}^{\xi N} N^{kf} 2^N \exp(-6\xi N p^k f) \to 0$$

as $N \to \infty$ follows from the choice of $C^k > k/\xi$ and $p > C(\log N/N)^{1/k}$, the proposition is established in this case.

Case 2 ($|F_k| < N/\log N$)**.** *The analysis in this case is very similar to the first, but instead of Chernoff's inequality we use that*

$$\mathbb{P}(X \ge t) \le q^t \binom{M}{t} \le \left(\frac{eqM}{t}\right)^t$$

for a binomial random variable $X \sim \mathrm{Bi}(q, M)$. *Consequently,*

$$\mathbb{P}\big(e_\Gamma(F_k, U) \geq p^k|F_k||U| + 6\xi N p^k|F_k|\big) \leq \mathbb{P}\big(e_\Gamma(F_k, U) \geq 6\xi N p^k|F_k|\big) \leq \left(\frac{\mathrm{e}|U|}{6\xi N}\right)^{6\xi N p^k|F_k|} \leq \exp\big(-6\xi N p^k|F_k|\ln(2\xi N/|U|)\big)\,.$$

In this case, the number of choices for the pair (F_k, U) *is at most*

$$\sum_{f=1}^{N/\log N} \sum_{u=1}^{f} N^{kf}\binom{N}{u}\,.$$

Consequently, from the union bound we infer that the probability that there exists a family F_k *and a set* U *with* $|U| \leq |F_k| < N/\log N$ *such that* $e_\Gamma(F_k, U) \geq p^k|F_k||U| + 6\xi N p^k|F_k|$ *is at most*

$$\sum_{f=1}^{N/\log N} \sum_{u=1}^{f} \exp\big(kf\ln N + u\ln(eN/u) - 6\xi N p^k f\ln(2\xi N/u)\big) \to 0\,,$$

as $N \to \infty$ *since* $p^k N \gg \log N/\log\log N$.

This concludes the proof of Proposition 6.8. □

6.3.3 Hereditary nature of (ε, α, p)-denseness

In this section we shall show that in the random graph $G(N, p)$ all sufficiently large (not necessarily induced) 3-partite subgraphs, say, with vertex set $X \cup Y \cup Z$, in which all the three pairs (X, Y), (X, Z) and (Y, Z) are (ε, α, p)-dense, have the following property: The (ε, α, p)-denseness of the pair (Y, Z) is "typically" inherited on the one-sided neighbourhood $(N(x) \cap Y, Z)$ as well as on the two-sided neighbourhood $(N(x) \cap Y, N(x) \cap Z)$ for $x \in X$. Below we introduce classes B_p^{I} and B_p^{II} of "bad" tripartite graphs, which fail to have the above one-sided and two-sided property (for similar concepts see [KR03a]).

Definition 6.9. *Let integers* m_1, m_2, *and* m_3 *and reals* $\alpha, \varepsilon', \varepsilon, \mu > 0$, *and* $0 < p \leq 1$ *be given.*

(I) *Let* $B_p^I(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ *be the family of tripartite graphs with vertex set* $X \cup Y \cup Z$, *where* $|X| = m_1$, $|Y| = m_2$, *and* $|Z| = m_3$, *satisfying*

  (*a*) (X, Y) *and* (Y, Z) *are* (ε, α, p)*-dense pairs and*

  (*b*) *there exists* $X' \subseteq X$ *with* $|X'| \geq \mu|X|$ *such that* $(N(x) \cap Y, Z)$ *is not an* $(\varepsilon', \alpha, p)$*-dense pair for every* $x \in X'$.

(II) *Let* $B_p^{II}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ *be the family of tripartite with vertex set* $X \cup Y \cup Z$, *where* $|X| = m_1$, $|Y| = m_2$, *and* $|Z| = m_3$, *satisfying*

  (*a*) (X, Y), (X, Z), *and* (Y, Z) *are* (ε, α, p)*-dense pairs and* (*b*) *there exists* $X' \subseteq X$ *with* $|X'| \geq \mu|X|$ *such that* $(N(x) \cap Y, N(x) \cap Z)$ *is not an* $(\varepsilon', \alpha, p)$*-dense pair for every* $x \in X'$*.*

Definition 6.10. *For integers* N *and* $\Delta \geq 2$ *and reals* α, γ, ε', ε, $\mu > 0$ *and* $0 < p \leq 1$ *we say that a graph* $G = (V, E)$ *with* $V = [N]$ *has the denseness property* $D^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$*, if* G *contains no member from*

$$B^I_p(m^I_1, m^I_2, m^I_3, \alpha, \varepsilon', \varepsilon, \mu) \cup B^{II}_p(m^{II}_1, m^{II}_2, m^{II}_3, \alpha, \varepsilon', \varepsilon, \mu)$$

with $m^I_1, m^I_3 \geq \gamma p^{\Delta-1}N$ *and* $m^I_2, m^{II}_1, m^{II}_2, m^{II}_3 \geq \gamma p^{\Delta-2}N$ *as a (not necessarily induced) subgraph.*

Proposition 6.11. *For an integer* $\Delta \geq 2$ *and positive reals* α, ε', *and* μ *there exists*

$$\varepsilon = \varepsilon(\Delta, \alpha, \varepsilon', \mu) > 0 \tag{6.3}$$

such that for every $\gamma > 0$ *there exists* $C > 1$ *such that if* $p > C(\log N/N)^{1/\Delta}$*, then*

$$\mathbb{P}(G(N,p) \in D^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)) = 1 - o(1).$$

Proof of Proposition 6.11

We first verify Proposition 6.11 for the special case in which $m^{\mathrm{I}}_1 = pm^{\mathrm{I}}_2 = m^{\mathrm{I}}_3$ and $m^{\mathrm{II}}_1 = m^{\mathrm{II}}_2 = m^{\mathrm{II}}_3$. To that end, we consider the families of graphs $B^{\mathrm{I}}_p(m, \alpha, \varepsilon', \varepsilon, \mu)$ and $B^{\mathrm{II}}_p(m, \alpha, \varepsilon', \varepsilon, \mu)$ for $m \in \mathbb{N}$ and α, ε', ε, $\mu > 0$ defined as

$$B^{\mathrm{I}}_p(m, \alpha, \varepsilon', \varepsilon, \mu) = B^{\mathrm{I}}_p(pm, m, pm, \alpha, \varepsilon', \varepsilon, \mu)$$

and

$$B^{\mathrm{II}}_p(m, \alpha, \varepsilon', \varepsilon, \mu) = B^{\mathrm{II}}_p(m, m, m, \alpha, \varepsilon', \varepsilon, \mu)\,.$$

Similarly, for integers N and Δ and positive reals α, γ, ε', ε, $\mu > 0$ and $0 < p \leq 1$, we say that a graph $G = (V, E)$ with $V = [N]$ has property $\hat{D}^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$ if G contains no member from $B^{\mathrm{I}}_p(m, \alpha, \varepsilon', \varepsilon, \mu) \cup B^{\mathrm{II}}_p(m, \alpha, \varepsilon', \varepsilon, \mu)$ with $m = \gamma p^{\Delta-2}N$ as a (not necessarily induced) subgraph. Next we prove that $G(N, p)$ has property $\hat{D}^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$ with high probability.

Proposition 6.12. *For an integer* $\Delta \geq 2$ *and positive reals* α, ε' *and* $\mu > 0$ *there exists* $\varepsilon > 0$ *such that for every* $\gamma > 0$ *there exists* $C > 1$ *such that if* $p > C(\log N/N)^{1/\Delta}$*, then* $\mathbb{P}(G(N,p) \in \hat{D}^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)) = 1 - o(1)$*.*

Proof. Below we shall only show that a.a.s. $G(N, p)$ contains no subgraphs from B^{II}_p. The proof for graphs from B^{I}_p is analogous.

Let Δ, α, ε', and μ be given. We set

$$\beta = \left(\frac{1}{4}\right)^{4/\mu} \frac{\alpha^2}{4\mathrm{e}^2} \left(\frac{1}{\mathrm{e}}\right)^{4/(\alpha\mu)}$$

and let ε_0 and L be given by Corollary 6.4 applied with α, β, and ε'. We fix

$$\varepsilon = \min\{\alpha/2\,, \mu/4\,, \varepsilon_0\}\,,$$

and for every $\gamma > 0$ we let $C = 1$. (In fact, any choice of $C > 0$ would suffice for the proof presented here, which concerns only subgraphs from $B_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu)$. For B_p^{I} a more careful choice of C is required.)

Suppose $T = (X \cup Y \cup Z, E_T)$ is a tripartite graph from $B_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu)$. We shall find a subgraph of T that, as we shall show, is unlikely to appear in $G(N, p)$. Because of the assumption on T, the bipartite subgraphs $T[X, Y]$, $T[X, Z]$, and $T[Y, Z]$ of T contain at least $(\alpha - \varepsilon)pm^2$ edges each. Furthermore, there is a set $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N_T(x) \cap Y, N_T(x) \cap Z)$ is not $(\varepsilon', \alpha, p)$-dense. Set

$$X'' = \{x \in X' \colon\ |N_T(x) \cap Y| \geq \alpha pm/2 \text{ and } |N_T(x) \cap Z| \geq \alpha pm/2\}\,.$$

From the (ε, α, p)-denseness of $T[X, Y]$ and $T[X, Z]$ we infer that

$$|X''| \geq (1 - 2\varepsilon/\mu)|X'| \geq |X'|/2 \geq \mu m/2\,.$$

Fix $x \in X''$. An easy averaging argument shows that there are sets $Y'_x \subseteq N_T(x) \cap Y$ and $Z'_x \subseteq N_T(x) \cap Z$ of size $\varepsilon' \alpha pm/2$ each such that $d_{T,p}(Y'_x, Z'_x) < \alpha - \varepsilon'$. Now let Y_x and Z_x be such that $Y'_x \subset Y_x \subseteq N_T(x) \cap Y$ and $Z'_x \subset Z_x \subseteq N_T(x) \cap Z$ and $|Y_x| = |Z_x| = \alpha pm/2$. Then, clearly, $T[Y_x, Z_x]$ is not $(\varepsilon', \alpha, p)$-dense. We may thus find a family of pairs $\{(Y_x, Z_x) \colon\ x \in X''\}$ that are not $(\varepsilon', \alpha, p)$-dense. We shall show that such a configuration is unlikely to occur in $G(N, p)$.

Indeed we can fix the sets X'', Y, Z and the edges of the bipartite graph $T[Y, Z]$ in at most

$$\sum_{t \geq (\alpha - \varepsilon)pm^2} \binom{N}{m}^3 \binom{m^2}{t}$$

ways. Since $m = \gamma p^{\Delta - 2} N$ (see the definition of $\hat{D}^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$) and $p^{\Delta} N > C \log n > 2L/(\alpha\gamma)$ for sufficiently large N we have $\alpha pm/2 \geq L/p$ and, hence, we can apply Corollary 6.4 and infer that there are at most

$$\left(\beta^{\alpha pm/2} \binom{m}{\alpha pm/2}^2\right)^{\mu m/2}$$

possibilities for choosing all pairs (Y_x, Z_x) for $x \in X''$. Combining the two estimates above we infer that the probability that such a configuration appears in $G(N, p)$ is bounded from above by

$$\sum_{t\geq(\alpha-\varepsilon)pm^2}\binom{N}{m}^3\binom{m^2}{t}p^t\times\left(\beta^{\alpha pm/2}\binom{m}{\alpha pm/2}^2\right)^{\mu m/2}p^{\mu\alpha pm^2/2}$$
$$\leq\sum_{t\geq(\alpha-\varepsilon)pm^2}\left(\frac{N\mathrm{e}}{m}\right)^{3m}\left(\frac{pm^2\mathrm{e}}{t}\right)^t\times\left(\sqrt{\beta}\frac{2\mathrm{e}}{\alpha}\right)^{\mu\alpha pm^2/2}$$
$$\leq m^2\left(\frac{N\mathrm{e}}{m}\right)^{3m}\left(\mathrm{e}^{1/\alpha}\left(\frac{2\mathrm{e}}{\alpha}\right)^{\mu/2}\beta^{\mu/4}\right)^{\alpha pm^2},$$

where, for the last inequality, we used the fact that the function $f(t)=(pm^2\mathrm{e}/t)^t$ is maximized for $t=pm^2$. Finally, we note that the right-hand side of the last inequality tends to 0 as $N\to\infty$, since $\mathrm{e}^{1/\alpha}(2\mathrm{e}/\alpha)^{\mu/2}\beta^{\mu/4}=1/4$ (owing to the choice of β) and $pm^2\gg m\log N$ (owing to the choice of p and m). □

Now we deduce Proposition 6.11 from Proposition 6.12.

Proof of Proposition 6.11. In order to prove Proposition 6.11 we will strengthen Proposition 6.12 and consider the families B_p^{I} and B_p^{II} with more general parameters m_1, m_2, and m_3. We shall show that, perhaps surprisingly, this more general statement follows from the "weaker" Proposition 6.12. Indeed, roughly speaking, we show that each "bad" tripartite graph $T\in B_p^{\mathrm{II}}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ with arbitrary m_1, m_2, $m_3\geq m$ contains a subgraph $\hat{T}\in B_p^{\mathrm{II}}(m,\alpha,\varepsilon'/2,\hat{\varepsilon},\mu/4)$ for some appropriate $\hat{\varepsilon}$. The following deterministic statement makes this precise.

Claim 6.13. *For every integer $\Delta\geq 2$ and all positive reals α, ε', μ, and $\hat{\varepsilon}$ there exists $\varepsilon>0$ such that for every $\gamma>0$ there exist $C>1$ and N_0 such that if $N\geq N_0$ and $p>C(\log N/N)^{1/\Delta}$, then every tripartite graph $T=(X\cup Y\cup Z,E_T)\in B_p^{II}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ with*

$$\min\{m_1,m_2,m_3\}\geq m=\gamma p^{\Delta-2}N$$

contains a subgraph $\hat{T}\in B_p^{II}(m,\alpha,\varepsilon'/2,\hat{\varepsilon},\mu/4)$.

The same claim holds for B_p^{I} (and, in fact, the proof is a little simpler), but we only focus on B_p^{II} here. Before we prove Claim 6.13, we note that that claim, combined with Proposition 6.12, yields Proposition 6.11, as Proposition 6.12 guarantees that with probability $1-o(1)$ the random graph $G(N,p)$ contains no such $\hat{T}$ from

$$B_p^{\mathrm{I}}(m,\alpha,\varepsilon'/2,\hat{\varepsilon},\mu/4)\cup B_p^{\mathrm{II}}(m,\alpha,\varepsilon'/2,\hat{\varepsilon},\mu/4)\,.$$

□

Proof of Claim 6.13. Let $\Delta\geq 2$ and α, ε', μ, and $\hat{\varepsilon}$ be given. We let ε_1, ε_2, and ε be as given by Theorem 6.3, so that for $\beta=1/2$ we have $\varepsilon_2=\varepsilon_0(\alpha,\beta,\hat{\varepsilon})$, $\varepsilon_1=\varepsilon_0(\alpha,\beta,\varepsilon_2)$, and $\varepsilon = \min\{\varepsilon_0(\alpha, \beta, \varepsilon_1), \alpha/2, \mu/4\}$. Now for any given γ let $C \geq L/\gamma$, where $L = \max\{L_1, L_2, L_3\}$ and L_1, L_2, and L_3 are given by the three applications of Theorem 6.3 referred to above. Moreover, let C be sufficiently large, so that the asymptotic estimates in the calculations below become valid. Finally, let $\delta = \varepsilon'/8$.

Let $T = (X \cup Y \cup Z, E_T) \in B_p^{\mathrm{II}}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ be given. Hence, there exists a set $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N_T(x) \cap Y, N_T(x) \cap Z)$ is not $(\varepsilon', \alpha, p)$-dense. We consider the set

$$X'' = \{x \in X' \colon\ |N_T(x) \cap Y| \geq \alpha p m_2/2 \text{ and } |N_T(x) \cap Z| \geq \alpha p m_3/2\}\,.$$

Owing to the choice of $\varepsilon \leq \mu/4$, we infer from the (ε, α, p)-denseness of $T[X, Y]$ and $T[X, Z]$ that $|X''| \geq \mu m_1/2$.

Let each of $\hat{X} \in \binom{X}{m}$, $\hat{Y} \in \binom{Y}{m}$, and $\hat{Z} \in \binom{Z}{m}$ be chosen uniformly at random and let $\hat{T} = T[\hat{X}, \hat{Y}, \hat{Z}]$. We shall show that with positive probability $\hat{T} \in B_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \hat{\varepsilon}, \mu/4)$.

By Theorem 6.3, with probability at least $1 - 2\beta^m$ the pairs $(\hat{X}, Y)$ and $(\hat{X}, Z)$ are $(\varepsilon_1, \alpha, p)$-dense. Applying Theorem 6.3 again, we infer that with probability at least $1 - 4\beta^m$ the pairs $(\hat{X}, \hat{Y})$, $(\hat{X}, Z)$, and $(\hat{Y}, Z)$ are $(\varepsilon_2, \alpha, p)$-dense and yet another application finally yields that with probability at least $1 - 6\beta^m$ the pairs

$$(\hat{X}, \hat{Y}),\ (\hat{X}, \hat{Z}),\ \text{and}\ (\hat{Y}, \hat{Z}) \text{ are } (\hat{\varepsilon}, \alpha, p)\text{-dense}, \tag{6.4}$$

which is property (a) of part (II) in Definition 6.9. Below we shall verify that property (b) also holds with high probability.

The concentration of the hypergeometric distribution tells us that, with probability at least $1 - \exp(-\Omega(m))$, if we set $\hat{X}'' = \hat{X} \cap X''$, we have

$$|\hat{X}''| \geq \frac{1}{4}\mu m. \tag{6.5}$$

Similarly, with probability at least $1 - m\exp(-\Omega(pm))$, we have, for every $x \in \hat{X}''$, that

$$|N_{\hat{T}}(x) \cap \hat{Y}| = (1 \pm \delta)\frac{|N_T(x) \cap Y|}{m_2}m = \Omega(pm) \tag{6.6}$$

and

$$|N_{\hat{T}}(x) \cap \hat{Z}| = (1 \pm \delta)\frac{|N_T(x) \cap Z|}{m_3}m = \Omega(pm). \tag{6.7}$$

Recall that for every $x \in \hat{X}'' \subseteq X'$ there exist sets $Y_x \subseteq N_T(x) \cap Y$ and $Z_x \subseteq N_T(x) \cap Z$ of size at least $\varepsilon'|N_T(x) \cap Y| \geq \varepsilon' \alpha p m_2/2$ and $\varepsilon'|N_T(x) \cap Z| \geq \varepsilon' \alpha p m_3/2$, respectively, such that

$$d_{T,p}(Y_x, Z_x) < \alpha - \varepsilon'. \tag{6.8}$$

As before, applying the concentration of the hypergeometric distribution, we obtain that, with probability at least $1 - m\exp(-\Omega(pm))$, we have, for every $x \in \hat{X}''$, that

$$|Y_x \cap \hat{Y}| = (1 \pm \delta)\frac{|Y_x|}{m_2}m = \Omega(pm) \tag{6.9}$$

and

$$|Z_x \cap \hat{Z}| = (1 \pm \delta)\frac{|Z_x|}{m_3}m = \Omega(pm). \tag{6.10}$$

Below we shall show that, with probability $1 - o(1/m)$, for any given $x \in \hat{X}''$, the pair $(N_{\hat{T}}(x) \cap \hat{Y}, N_{\hat{T}}(x) \cap \hat{Z})$ is not $(\varepsilon'/2, \alpha, p)$-dense. Summing the failure probability $o(1/m)$ over all choices of x, we deduce that $\hat{T} = T[\hat{X}, \hat{Y}, \hat{Z}] \in B_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \hat{\varepsilon}, \mu/4)$ with probability $1 - o(1)$ (recall (6.4)).

Fix $x \in \hat{X}''$. Below, we may and shall assume that (6.6), (6.7), (6.9), and (6.10) hold. Let $\zeta = (1 \pm \delta)|Z_x|m/m_3 = \Omega(pm)$. In what follows, we shall consider the conditional space in which $|Z_x \cap \hat{Z}| = \zeta$. To remind ourselves of this conditioning, we shall write $\mathbb{P}_\zeta$ and $\mathbb{E}_\zeta$ to denote the probability and the expectation in this space.

For all $y \in Y_x$, let $\Gamma(y) = N_T(y) \cap Z_x$ and set $d_y = |\Gamma(y)|$. We have

$$\mathbb{E}_\zeta(|\Gamma(y) \cap \hat{Z}|) = \frac{d_y \zeta}{|Z_x|}.$$

Suppose now that $d_y \geq (\varepsilon'/20\mathrm{e})p|Z_x|$. Then

$$\begin{aligned}\mathbb{P}_\zeta\left(|\Gamma(y) \cap \hat{Z}| \geq (1+\delta)\frac{d_y}{|Z_x|}\zeta\right) &\leq \exp\left(-\frac{1}{3}\delta^2\frac{d_y}{|Z_x|}\zeta\right)\\ &\leq \exp\left(-\frac{1}{3}\delta^2\frac{\varepsilon'}{20\mathrm{e}}p\zeta\right) = \exp\left(-\Omega(p^2 m)\right).\end{aligned} \tag{6.11}$$

Consider now the case in which $d_y < (\varepsilon'/20\mathrm{e})p|Z_x|$. Then

$$\begin{aligned}\mathbb{P}_\zeta\left(|\Gamma(y) \cap \hat{Z}| \geq \frac{d_y}{|Z_x|}\zeta + \frac{\varepsilon'}{10}p\zeta\right) &\leq \mathbb{P}_\zeta\left(|\Gamma(y) \cap \hat{Z}| \geq \frac{\varepsilon'}{10}p\zeta\right)\\ &\leq \left(\frac{\mathrm{e}}{(\varepsilon'/10)p\zeta}\frac{d_y}{|Z_x|}\zeta\right)^{(\varepsilon'/10)p\zeta}\\ &\leq \left(\frac{\mathrm{e}}{(\varepsilon'/10)p}(\varepsilon'/20\mathrm{e})p\right)^{(\varepsilon'/10)p\zeta}\\ &= \left(\frac{1}{2}\right)^{(\varepsilon'/10)p\zeta}\\ &= \exp\left(-\Omega(p^2 m)\right).\end{aligned} \tag{6.12}$$

Let us note that, if $d_y < (\varepsilon'/20\mathrm{e})p|Z_x|$, then

$$\frac{d_y}{|Z_x|}\zeta + \frac{\varepsilon'}{10}p\zeta \leq \frac{\varepsilon'}{20\mathrm{e}}p\zeta + \frac{\varepsilon'}{10}p\zeta \leq \frac{1}{8}\varepsilon' p\zeta. \tag{6.13}$$

Because of (6.8), (6.11), (6.12), and (6.13), we have, with probability $1 - o(1/m)$, that

$$\begin{aligned} e(Y_x, Z_x \cap \hat{Z}) &\le \sum_{y \in Y_x} (1+\delta)\frac{d_y}{|Z_x|}\zeta + \sum_{y \in Y_x} \frac{1}{8}\varepsilon' p \zeta \\ &= (1+\delta)\frac{\zeta}{|Z_x|}\sum_{y \in Y_x} d_y + \frac{1}{8}\varepsilon' p \zeta |Y_x| \\ &\le (1+\delta)\frac{\zeta}{|Z_x|}(\alpha - \varepsilon')p|Y_x||Z_x| + \frac{1}{8}\varepsilon' p \zeta |Y_x|, \end{aligned}$$

whence, recalling that $|Z_x \cap \hat{Z}| = \zeta$,

$$d_{T,p}(Y_x, Z_x \cap \hat{Z}) \le (1+\delta)(\alpha - \varepsilon') + \frac{1}{8}\varepsilon' \le \alpha - \frac{1}{4}\varepsilon'. \tag{6.14}$$

Repeating the same argument with Y_x replaced with $Z_x \cap \hat{Z}$ and with Z_x replaced with Y_x, we obtain that, with probability $1 - o(1/m)$,

$$d_{T,p}(Y_x \cap \hat{Y}, Z_x \cap \hat{Z}) \le \alpha - \frac{1}{2}\varepsilon'.$$

This concludes the proof of Claim 6.13. □

6.4 Ramsey universal graphs

6.4.1 Proof of Theorem 1.23

In this section we prove Theorem 1.23, namely, we show that for

$$p = p(N) \ge C(\log N/N)^{1/\Delta}$$

the random graph $G(N,p)$ is partition universal for $\mathcal{H}_{\Delta,n}$ for n of the form $\lfloor cN \rfloor$ for some $c > 0$. In view of the results from Section 6.3 this follows directly from the following deterministic statement.

Lemma 6.14. *For every $\Delta \ge 2$ there exist $\widetilde{\Delta} \ge 2$ and positive constants μ, α, $\varepsilon_0, \dots, \varepsilon_{\widetilde{\Delta}}$, ξ, γ, B, and n_0 such that for every $n \ge n_0$ the following holds. If $G = (V,E)$ is a graph on $V = [N]$, where $N \ge Bn$, such that for some $0 < p \le 1$ we have*

(i) $G \in U_{N,p}$,

(ii) $G \in E^k_{N,p}(\xi)$ *for every* $k = 1, \dots, \Delta$, *and*

(iii) $G \in D^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon_k, \varepsilon_{k-1}, \mu)$ *for every* $k = 1, \dots, \widetilde{\Delta}$,

then G is partition universal for $\mathcal{H}_{\Delta,n}$.

Before we prove Lemma 6.14, we deduce Corollary 6.15 below, which implies Theorem 1.23 immediately.

Corollary 6.15. *For every $\Delta \geq 2$ there exist B and $C > 0$ such that, if $p = p(N) \geq C(\log N/N)^{1/\Delta}$ and $n = n(N) = \lfloor N/B \rfloor$, then*

$$\mathbb{P}(G(N,p) \text{ is partition universal for } \mathcal{H}_{\Delta,n}) = 1 - o(1)\,. \tag{6.15}$$

Proof. For a given $\Delta \geq 2$, let $\widetilde{\Delta}$, μ, α, $\varepsilon_0, \dots, \varepsilon_{\widetilde{\Delta}}$, ξ, γ, and B be given by Lemma 6.14. Then let C be large enough so that Proposition 6.8 holds for every $k = 1, \dots, \Delta$, and ξ and so that Proposition 6.11 holds for every $k = \widetilde{\Delta}, \dots, 1$ with μ, α, $\varepsilon'_k = \varepsilon_k$, and $\varepsilon_{k-1} = \varepsilon(\Delta - 1, \alpha, \varepsilon' = \varepsilon_k, \mu)$ from Proposition 6.11. Consequently, with probability $1 - o(1)$, the random graph $G(N,p)$ satisfies properties (ii)–(iii) of Lemma 6.14 due to Propositions 6.8 and 6.11. Finally, property (i) holds with probability $1 - o(1)$ by Proposition 6.6 as $\Delta \geq 2$. Thus (6.15) follows. □

6.4.2 Proof of Lemma 6.14

In this section we prove the main technical lemma, Lemma 6.14. The proof follows the strategy in the proof of Chvátal et al. in [CRST83], but includes ideas from [AF92] and [RR99], and is based on the sparse regularity lemma.

Proof of Lemma 6.14. The proof consists of four parts. In the first part we fix all constants needed in the proof. In the second part we consider the given graph G along with a fixed 2-coloring of its edges. We have to show that G contains a monochromatic $\mathcal{H}_{\Delta,n}$-universal graph. In other words, we have to embed every graph $H \in \mathcal{H}_{\Delta,n}$ into one of the two monochromatic subgraphs of G. To that end, we first prepare the graph G and here the sparse regularity lemma will be the key tool. In the third part we shall prepare a given graph $H \in \mathcal{H}_{\Delta,n}$ for the embedding. In the last part we then embed H into a monochromatic subgraph of G.

Constants. Let $\Delta \geq 2$ be an integer. We first fix

$$\widetilde{\Delta} = \Delta^4 + 2\Delta + 1$$

and we set

$$r = R(\widetilde{\Delta}, \widetilde{\Delta})\,,$$

where $R(\widetilde{\Delta}, \widetilde{\Delta})$ is the Ramsey number that guarantees that every 2-coloring of the edges of the complete graph K_r yields a monochromatic copy of $K_{\widetilde{\Delta}}$. Next we define the constants μ, α, $\varepsilon_0, \dots, \varepsilon_{\widetilde{\Delta}}$, ξ, γ, B, and n_0 of Lemma 6.14. First we set

$$\mu = \frac{1}{4\Delta^2} \quad \text{and} \quad \alpha = \frac{1}{4}\,, \tag{6.16}$$

and we fix ε_k for $k = \widetilde{\Delta}, \widetilde{\Delta} - 1, \dots, 0$ by setting

$$\varepsilon_{\widetilde{\Delta}} = \frac{1}{12\widetilde{\Delta}} \quad \text{and} \quad \varepsilon_{k-1} = \min\left\{\varepsilon(\Delta - 1, \alpha, \varepsilon' = \varepsilon_k, \mu),\, \varepsilon_k\right\} \text{ for } k = \widetilde{\Delta}, \dots, 1\,, \tag{6.17}$$

where $\varepsilon(\Delta-1,\alpha,\varepsilon'=\varepsilon_k,\mu)$ is given by Proposition 6.11.

Next we set

$$\varepsilon=\min\left\{\frac{\varepsilon_0}{2},\frac{1}{2(r-1)}\right\}\,,\quad K=2\,,\quad \text{and}\quad t_0=2r \tag{6.18}$$

and let T_0, η, and N_0 be the constants guaranteed by the sparse regularity lemma, Theorem 6.1, for ε, K, and t_0 given above. Finally, we set

$$\gamma=\frac{1-\varepsilon}{4^{\Delta-1}T_0}\,,\qquad \xi=\frac{1}{7\cdot 4^{\Delta+1}\cdot T_0}\,,\qquad B=\frac{1}{\xi}\,, \tag{6.19}$$

and

$$n_0=\max\left\{\frac{N_0}{B},\frac{1}{\eta^2},T_0^2,2^{4/\varepsilon_0},e^{1/\eta}\right\}\,. \tag{6.20}$$

This concludes the definition of the constants involved in the proof of Lemma 6.14.

Preparing G. Now let $n\geq n_0$ be given and let $G=(V,E)$ be a graph on $V=[N]$, where $N\geq Bn\geq N_0$, satisfies assumptions (i)–(iii) of Lemma 6.14 for some $0<p\leq 1$. We fix an arbitrary coloring of the edges $E=E_R\cup E_B$ of G with two colors, say red and blue, and let $G_R=(V,E_R)$ and $G_B=(V,E_B)$ be the corresponding monochromatic subgraphs. We have to show that one of G_R or G_B will contain every H in $\mathcal{H}_{\Delta,n}$. To that end, first use the sparse regularity lemma to "locate" an appropriate "regular" subgraph in either G_R or G_B.

More precisely, we apply the sparse regularity lemma, Theorem 6.1, to G_R with

$$\varepsilon=\min\left\{\frac{\varepsilon_0}{2},\frac{1}{r-1}\right\}\,,\quad K=2\,,\quad t_0=2r\,,\quad \text{and}\quad p\,.$$

Note that, owing to property (i) of Lemma 6.14 (see Definition 6.5), the graph G is $(1/\log N,1+1/\log N)$-bounded. Since $G_R\subseteq G$, $1/\log N\leq 1$, and $N/\log N\leq\eta N$ (because of the choice of n_0 in (6.20)) we infer that indeed G_R is (η,K)-bounded (see (6.18)). Consequently, Theorem 6.1 yields an (ε,t)-equitable (ε,G_R,p)-regular partition $V_0\cup V_1\cup\ldots\cup V_t$ of V with $t_0\leq t\leq T_0$.

We consider an auxiliary graph A with vertex set $[t]=\{1,\ldots,t\}$ and $\{i,j\}$ being an edge if and only if the pair (V_i,V_j) is (ε,p)-regular for G_R. Since the partition $V_0\cup V_1\cup\ldots V_t$ is (ε,G_R,p)-regular, at most $\varepsilon\binom{t}{2}\leq\frac{1}{2(r-1)}\binom{t}{2}<(r-1)\binom{t/(r-1)}{2}$ of the pairs of the auxiliary graph are missing and hence, by Turán's theorem, A contains a clique K_r with r vertices. In other words, there exists an index set $I_r=\{i_1,\ldots,i_r\}\subseteq[t]$ such that (V_i,V_j) is (ε,p)-regular for G_R for all $\{i,j\}\in\binom{I_r}{2}$. Moreover, since $G\in U_{N,p}$ and since $1/\log N\leq N/T_0$ (see (6.20)) it follows directly from the definition of (ε,p)-regularity that (V_i,V_j) is $(\varepsilon+2/\log N,p)$-regular for the graph G_B. Because of (6.18) and (6.20), we have $\varepsilon+2/\log N\leq\varepsilon_0/2+\varepsilon_0/2$ and, hence, (V_i,V_j) is (ε_0,p)-regular for G_R and for G_B for all $\{i,j\}\in\binom{I_r}{2}$.

Next we color the edges of the clique $K_r\subseteq A$ red and blue. We color an edge $\{i,j\}\in\binom{I_r}{2}$ red if $d_{G_R,p}(V_i,V_j)\geq d_{G_B,p}(V_i,V_j)$ and blue otherwise. Note that, again from the fact that $G\in U_{N,p}$ and $1/\log N\leq N/T_0$ we infer that $d_{G_R,p}(V_i,V_j)+d_{G_B,p}(V_i,V_j)\geq$

$1 - 1/\log N$ and, therefore,

$$\max\left\{d_{G_R,p}(V_i,V_j), d_{G_B,p}(V_i,V_j)\right\} \geq \frac{1}{2} - \frac{1}{2\log N} \geq \frac{1}{3}$$

for every $\{i,j\} \in \binom{[t]}{2}$.

Because of the choice of $r \geq R(\widetilde{\Delta},\widetilde{\Delta})$ there exists a monochromatic clique $K_{\widetilde{\Delta}} \subseteq K_r \subseteq A$ on $\widetilde{\Delta}$ vertices. Let $J \subseteq I$ be the vertex set of the monochromatic clique $K_{\widetilde{\Delta}}$. Summarizing, the above ensures the existence of a set $J \subseteq I$ of cardinality $\widetilde{\Delta}$ such that either

$$(V_i,V_j) \text{ is } (\varepsilon_0,p)\text{-regular for } G_R \text{ and } d_{G_R,p}(V_i,V_j) \geq 1/3 \text{ for all } \{i,j\} \in \binom{J}{2} \tag{6.21}$$

or the same statement holds for G_B. Without loss of generality we assume that (6.21) holds and we shall show that G_R induced on $\bigcup_{i\in J} V_i$ will contain any H from $\mathcal{H}_{\Delta,n}$.

Preparing H. Fix some $H = (W,F) \in \mathcal{H}_{\Delta,n}$. We consider the third power $H^3 = (W,F^3)$ of H, i.e., $\{w,w'\} \in F^3$ if and only if $w \neq w'$ and there exists a w–w'-path with at most three edges in H. Since $\Delta(H) \leq \Delta$ we have

$$\Delta(H^3) \leq \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 = \Delta^3 - \Delta^2 + \Delta$$

and consequently $\chi(H^3) \leq \Delta^3 - \Delta^2 + \Delta + 1$. Fix a $(\Delta^3 - \Delta^2 + \Delta + 1)$-vertex coloring f of H^3 with colors $1,\ldots,\Delta^3 - \Delta^2 + \Delta + 1$. This way we obtain a partition of W into $\Delta^3 - \Delta^2 + \Delta + 1$ classes such that if two vertices w and w' are elements of the same class, then their distance in H is at least four; in particular, there are no edges between $N_H(w)$ and $N_H(w')$, since otherwise $\{w,w'\}$ would be an edge in H^3. We now refine the partition induced by the color classes of f according to the "left-degrees" of the vertices. More precisely, we say two vertices w and w' are equivalent if $f(w) = f(w')$ and

$$\left|N_H(w) \cap \{x \in W\colon\ f(x) < f(w)\}\right| = \left|N_H(w') \cap \{x \in W\colon\ f(x) < f(w')\}\right|,$$

i.e., w and w' are equivalent if they have the same color in f and the same number of neighbours with colors of smaller number. Clearly, this equivalence relation partitions W into at most $(\Delta^3 - \Delta^2 + \Delta + 1)(\Delta + 1) = \widetilde{\Delta}$ classes. Denote the partition classes by $W_1,\ldots,W_{\widetilde{\Delta}}$ (allowing empty classes if necessary) and let $g\colon W \to [\widetilde{\Delta}]$ be the corresponding partition function, i.e.,

$$g(w) = j \quad \text{if and only if} \quad w \in W_j.$$

Thus, if $g(w) = g(w')$, then $\left|N_H(w) \cap \{x \in W\colon g(x) < g(w)\}\right| = \left|N_H(w') \cap \{x \in W\colon g(x) < g(w')\}\right|$. For an integer $\ell \leq g(w)$ we denote by

$$\mathrm{ldeg}_g^{\ell}(w) := \left|N_H(w) \cap \{x \in W\colon\ g(x) \leq \ell\}\right|$$

the *left-degree of w with respect to g and ℓ.*

Embedding of H into G. After the preparation of G and H we are able to embed H into G_R. We may relabel the vertex classes V_i of G_R with $i \in J$ and assume $J = [\widetilde{\Delta}]$. We proceed inductively and embed the vertex class W_ℓ into V_ℓ one at a time, for $\ell = 1, \dots, \widetilde{\Delta}$. To this end, we verify the following statement (S_ℓ) for $\ell = 0, \dots, \widetilde{\Delta}$.

(S_ℓ) There exists a partial embedding φ_ℓ of $H[\bigcup_{j=1}^{\ell} W_j]$ into $G_R[\bigcup_{j=1}^{\ell} V_j]$ such that for every $z \in \bigcup_{j=\ell+1}^{\widetilde{\Delta}} W_j$ there exists a *candidate set* $C_\ell(z) \subseteq V(G)$ given by

- *(a)* $C_\ell(z) = \bigcap\{N_{G_R}(\varphi_\ell(x))\colon\ x \in N_H(z) \text{ and } g(x) \le \ell\} \cap V_{g(z)}$,

satisfying

- *(b)* $|C_\ell(z)| \ge (p/4)^{\mathrm{ldeg}_g^\ell(z)} m$, where $m = |V_{g(z)}| \ge (1-\varepsilon)N/t$, and
- *(c)* for every $\{z, z'\} \in F = E(H)$ with $g(z), g(z') > \ell$ the pair $(C_\ell(z), C_\ell(z'))$ is $(\varepsilon_\ell, 1/3 - \ell\varepsilon_{\widetilde{\Delta}}, p)$-dense in G_R.

Remark 6.16. *In what follows we shall use the following convention. Vertices from G_R will be denoted by v and vertices from H will be usually named w. However, since the embedding of H into G will be divided into $\widetilde{\Delta}$ rounds, we shall find it convenient to distinguish among the vertices of H. We shall use the letter x for vertices that have already been embedded, the letter y for vertices that will be embedded in the current round, while z will denote vertices that we shall embed at a later step.*

Statement (S_ℓ) ensures the existence of a partial embedding of the first ℓ vertex classes $W_1, \dots, W_\ell$ of H such that for every unembedded vertex z there exists a candidate set $C_\ell(z)$ that is not too small (see part *(b)*). Moreover, if we embed z into its candidate set, then its image will be adjacent to all vertices $\varphi_\ell(x)$ with $x \in (W_1 \cup \dots \cup W_\ell) \cap N_H(z)$ (see part *(a)*). The last property, part *(c)*, says that edges of H for which none of the endvertices are embedded already the respective candidate sets induce (ε, α, p)-dense pairs. This property will be crucial for the inductive proof.

Before we verify (S_ℓ) for $\ell = 0, \dots, \widetilde{\Delta}$ by induction on ℓ we note that $(S_{\widetilde{\Delta}})$ implies that H can be embedded into G_R. Since H was an arbitrary graph from $\mathcal{H}_{\Delta,n}$ and we fixed an arbitrary coloring of the edges of G, this implies $G \longrightarrow H$ for every $H \in \mathcal{H}_{\Delta,n}$. Consequently, verifying (S_ℓ) yields the proof of Lemma 6.14.

Basis of the induction: $\ell = 0$

We first verify (S_0). In this case φ_0 is the empty mapping and for every $z \in W$ we have, according to *(a)*, $C_0(z) = V_{g(z)}$, as there is no vertex $x \in N_H(z)$ with $g(x) \le 0$. Also, property *(b)* holds by definition of $C_0(z)$ for every $z \in W$. Finally, property *(c)* follows from the property that (V_i, V_j) is (ε_0, p)-regular for G_R and, consequently, $(C_0(z), C_0(z'))$ is $(\varepsilon_0, 1/3, p)$-dense in G_R for every edge $\{z, z'\}$ of H (see (6.21)).

Induction step: $\ell \to \ell + 1$

For the inductive step, we suppose that $\ell < \widetilde{\Delta}$ and assume that statement (S_ℓ) holds; we have to construct $\varphi_{\ell+1}$ with the required properties. Our strategy is as follows. In the first step, we find for every $y \in W_{\ell+1}$ an appropriate subset $C(y) \subseteq C_\ell(y)$ of the candidate set such that if $\varphi_{\ell+1}(y)$ is chosen from $C(y)$, then (*) the new candidate set $C_{\ell+1}(z) := C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))$ of every "right-neighbour" z of y will not shrink too much and (**) property (c) will continue to hold.

Note, however, that in general $|C(y)| \leq |C_\ell(y)| = o(N) \ll |W_{\ell+1}|$ (if $\mathrm{ldeg}_g^\ell \geq 1$) and, hence, we cannot "blindly" select $\varphi_{\ell+1}(y)$ from $C(y)$. Instead, in the second step, we shall verify Hall's condition to find a system of distinct representatives for the family $\{C(y)\colon\ y \in W_{\ell+1}\}$ and we let $\varphi_{\ell+1}(y)$ be the representative of $C(y)$. (A similar idea was used in [AF92, RR99].) We now give the details of those two steps.

For the first step, fix $y \in W_{\ell+1}$ and set

$$N_H^{\ell+1}(y) := \{z \in N_H(y)\colon\ g(z) > \ell+1\}\,.$$

A vertex $v \in C_\ell(y)$ will be "bad" (i.e., we shall not select v for $C(y)$) if there exists a vertex $z \in N_H^{\ell+1}(y)$ for which $N_{G_R}(v) \cap C_\ell(z)$, in view of (b) and (c) of $(S_{\ell+1})$, cannot play the rôle of $C_{\ell+1}(z)$.

We first prepare for (b) of $(S_{\ell+1})$. Fix a vertex $z \in N_H^{\ell+1}(y)$. Since $(C_\ell(y), C_\ell(z))$ is an $(\varepsilon_\ell, 1/3 - \ell\varepsilon_{\widetilde{\Delta}}, p)$-dense pair, there exist at most

$$\varepsilon_\ell |C_\ell(y)| \leq \varepsilon_{\widetilde{\Delta}} |C_\ell(y)|$$

vertices v in $C_\ell(y)$ such that

$$|N_{G_R}(v) \cap C_\ell(z)| < (d_{G_R,p}(C_\ell(y), C_\ell(z)) - \varepsilon_{\widetilde{\Delta}})p|C_\ell(y)|\,.$$

Repeating the above for all $z \in N_H^{\ell+1}(y)$, we infer from (a) and (b) of (S_ℓ), that there are at most $\Delta\varepsilon_{\widetilde{\Delta}}|C_\ell(y)|$ vertices $v \in C_\ell(y)$ such that the following fails to be true for some $z \in N_H^{\ell+1}(y)$:

$$\begin{aligned}|N_{G_R}(v) \cap C_\ell(z)| &\geq (d_{G_R,p}(C_\ell(y), C_\ell(z)) - \varepsilon_{\widetilde{\Delta}})p|C_\ell(z)| \\ &\overset{(a),(b)}{\geq} \left(\frac{1}{3} - (\ell+1)\varepsilon_{\widetilde{\Delta}}\right) p \left(\frac{p}{4}\right)^{\mathrm{ldeg}_g^\ell(z)} |V_{g(z)}| \overset{(6.17)}{\geq} \left(\frac{p}{4}\right)^{\mathrm{ldeg}_g^{\ell+1}(z)} |V_{g(z)}|\,. \qquad (6.22)\end{aligned}$$

For property (c) of $(S_{\ell+1})$, we fix an edge $e = \{z, z'\}$ with $g(z)$, $g(z') > \ell+1$ and with at least one end vertex in $N_H^{\ell+1}(y)$. There are at most $\Delta(\Delta-1) < \Delta^2$ such edges. Note that if both vertices z and z' are neighbours of y, i.e., z, $z' \in N_H^{\ell+1}(y)$, then

$$\max\{\,\mathrm{ldeg}_g^\ell(y), \mathrm{ldeg}_g^\ell(z), \mathrm{ldeg}_g^\ell(z')\} \leq \Delta - 2\,,$$

since all three vertices y, z, and z' have at least two neighbours in $W_{\ell+1} \cup \cdots \cup W_{\widetilde{\Delta}}$.

From property (b) of (S_ℓ) we infer

$$\min\left\{|C_\ell(y)|\,,|C_\ell(z)|\,,|C_\ell(z')|\right\} \geq \left(\frac{p}{4}\right)^{\max\{\mathrm{ldeg}_g^\ell(y),\mathrm{ldeg}_g^\ell(z),\mathrm{ldeg}_g^\ell(z')\}}(1-\varepsilon)\frac{N}{T_0} \overset{(6.19)}{\geq} \gamma p^{\Delta-2}N.$$

Furthermore,

$$\frac{1}{3}-\ell\varepsilon_{\widetilde{\Delta}} \geq \alpha = \frac{1}{4}$$

(see (6.17)). Hence $G_R \subseteq G$ and $G \in D^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon_{\ell+1},\varepsilon_\ell,\mu)$ imply that there are at most $\mu|C_\ell(y)|$ vertices $v \in C_\ell(y)$ such that the pair $(N_{G_R}(v)\cap C_\ell(z), N_{G_R}(v)\cap C_\ell(z'))$ fails to be $(\varepsilon_{\ell+1}, 1/3-(\ell+1)\varepsilon_{\widetilde{\Delta}}, p)$-dense.

If, on the other hand, say, only $z \in N_H^{\ell+1}(y)$ and $z' \notin N_H^{\ell+1}(y)$, then

$$\max\{\mathrm{ldeg}_g^\ell(y), \mathrm{ldeg}_g^\ell(z')\} \leq \Delta - 1 \quad \text{and} \quad \mathrm{ldeg}_g^\ell(z) \leq \Delta-2.$$

Consequently, (similarly as above)

$$\min\left\{|C_\ell(y)|\,,|C_\ell(z')|\right\} \geq \gamma p^{\Delta-1}N$$

and

$$|C_\ell(z)| \geq \gamma p^{\Delta-2}N$$

and we can appeal to $G \in D^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon_{\ell+1},\varepsilon_\ell,\mu)$ to infer that there are at most $\mu|C_\ell(y)|$ vertices $v \in C_\ell(y)$ with the property that the pair $(N_{G_R}(v)\cap C_\ell(z), C_\ell(z'))$ fails to be $(\varepsilon_{\ell+1}, 1/3-(\ell+1)\varepsilon_{\widetilde{\Delta}}, p)$-dense. For a given $v \in C_\ell(y)$, let $\hat{C}_\ell(z) = C_\ell(z)\cap N_{G_R}(v)$ if $z \in N_H^{\ell+1}(y)$ and $\hat{C}_\ell(z) = C_\ell(z)$ if $z \notin N_H^{\ell+1}(w)$, and define $\hat{C}_\ell(z')$ analogously.

Summarizing the above we infer that there are at least

$$(1-\Delta\varepsilon_{\widetilde{\Delta}}-\Delta^2\mu)|C_\ell(y)| \tag{6.23}$$

vertices $v \in C_\ell(y)$ such that

(b') $|N_{G_R}(v)\cap C_\ell(z)| \geq (p/4)^{\mathrm{ldeg}_g^{\ell+1}(z)}|V_{g(z)}|$ for every $z \in N_H^{\ell+1}(y)$ (see (6.22) and

(c') $(\hat{C}_\ell(z), \hat{C}_\ell(z'))$ is $(\varepsilon_{\ell+1}, 1/3-(\ell+1)\varepsilon_{\widetilde{\Delta}}, p)$-dense for all edges $\{z,z'\}$ of H with $g(z)$, $g(z') > \ell+1$ and $\{z,z'\}\cap N_H^{\ell+1}(y) \neq \varnothing$.

Let $C(y)$ be the set of those vertices v from $C_\ell(y)$ satisfying properties (b') and (c') above. Recall that $\mathrm{ldeg}_g^\ell(y) = \mathrm{ldeg}_g^\ell(y')$ for all y, $y' \in W_{\ell+1}$ and set

$$k = \mathrm{ldeg}_g^\ell(y) \text{ for some } y \in W_{\ell+1}.$$

Since $y \in W_{\ell+1}$ was arbitrary, we infer from (6.23), the choices of μ and $\varepsilon_{\widetilde{\Delta}}$ in (6.16) and (6.17) and property (b) of (S_ℓ) that

$$\begin{aligned}|C(y)| &\geq (1-\Delta\varepsilon_{\widetilde{\Delta}}-\Delta^2\mu)|C_\ell(y)| \\ &\geq (1-\Delta\varepsilon_{\widetilde{\Delta}}-\Delta^2\mu)\left(\frac{p}{4}\right)^k(1-\varepsilon)\frac{N}{T_0} \geq \frac{1}{4^{k+1}}p^k\frac{N}{T_0}\,. \end{aligned} \tag{6.24}$$

We now turn to the aforementioned second part of the inductive step. Here we ensure the existence of a system of distinct representatives for the set system

$$C_{\ell+1} = \{C(y)\colon\; y \in W_{\ell+1}\}\,.$$

We shall appeal to Hall's condition and show that for every subfamily $C' \subseteq C_{\ell+1}$ we have

$$|C'| \leq \left|\bigcup_{C\in C'} C\right|. \tag{6.25}$$

Because of (6.24), assertion (6.25) holds for all families C' with $1 \leq |C'| \leq 4^{-k-1}p^k N/T_0$.

Thus, consider a family $C' \subseteq C_{\ell+1}$ with $|C'| > 4^{-k-1}p^k N/T_0$. For every $y \in W_{\ell+1}$ we have $\mathrm{ldeg}_g^\ell(y) = k$. Hence, we have a k-tuple $K(y) = \{u_1(y), \dots, u_k(y)\}$ of already embedded vertices of H such that $K(y) = N_H(y) \setminus N_H^{\ell+1}(y)$. Note that for two distinct vertices y, $y' \in W_{\ell+1}$ the sets $K(y)$ and $K(y')$ are disjoint. This follows from the fact that the distance in H between y and y' is at least four and if $K(y) \cap K(y') \neq \varnothing$, then this distance would be at most two. Consequently, the sets $\varphi(K(y))$ and $\varphi(K(y'))$ are disjoint as well and, therefore, $F_k = \{\varphi(K(y))\colon\; y \in W_{\ell+1}\} \subseteq \binom{V}{k}$ is a family of pairwise disjoint k-sets in V. Moreover,

$$C(y) \subseteq \bigcap_{v\in\varphi(K(y))} N_{G_R}(v) \subseteq \bigcap_{v\in\varphi(K(y))} N_G(v)\,.$$

Let

$$U = \bigcup_{C(y)\in C'} C(y) \subseteq V_{\ell+1}\,,$$

and suppose for a contradiction that

$$|U| < |C'| = |F_k|. \tag{6.26}$$

We now use property (ii) of Lemma 6.14, namely, $G \in E^k_{N,p}(\xi)$ applied for F_k and U. We deduce that

$$e_{\Gamma(k,G)}(F_k, U) \leq p^k|F_k||U| + 6\xi N p^k |F_k|\,.$$

On the other hand, because of (6.24), we have

$$e_{\Gamma(k,G)}(F_k, U) \geq \frac{1}{4^{k+1}}p^k\frac{N}{T_0}|F_k|.$$

Combining the last two inequalities we infer

$$\Big|\bigcup_{C(y)\in C'} C(y)\Big| = |U| \geq \left(\frac{1}{4^{k+1}}\frac{1}{T_0} - 6\xi\right) N \overset{(6.19)}{\geq} \xi N \geq \xi B n \overset{(6.19)}{=} n \geq |W_{\ell+1}| \geq |C'|,$$

which contradicts (6.26). This contradiction shows that (6.26) does not hold, that is, Hall's condition (6.25) does hold. Hence, there exists a system of representatives for $\mathcal{C}_{\ell+1}$, i.e., an injective mapping $\psi\colon W_{\ell+1} \to \bigcup_{y\in W_{\ell+1}} C(y)$ such that $\psi(y) \in C(y)$ for every $y \in W_{\ell+1}$.

Finally, we extend φ_ℓ and define $C_{\ell+1}(z)$ for $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$. For that we set

$$\varphi_{\ell+1}(w) = \begin{cases} \varphi_\ell(w)\,, & \text{if } w \in \bigcup_{j=1}^{\ell} W_j,\\ \psi(w)\,, & \text{if } w \in W_{\ell+1}\,. \end{cases}$$

Note that every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ has at most one neighbour in $W_{\ell+1}$, as otherwise there would be two vertices y and $y' \in W_{\ell+1}$ with distance at most 2 in H, which contradicts the fact that g and f are valid vertex colorings of H^3. Consequently, for every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ we can set

$$C_{\ell+1}(z) = \begin{cases} C_\ell(z)\,, & \text{if } N_H(z)\cap W_{\ell+1} = \varnothing,\\ C_\ell(z)\cap N_{G_R}(\varphi_{\ell+1}(y))\,, & \text{if } N_H(z)\cap W_{\ell+1} = \{y\}. \end{cases}$$

In what follows we show that $\varphi_{\ell+1}$ and $C_{\ell+1}(z)$ for every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ have the desired properties and validate $(S_{\ell+1})$.

First of all, from (a) of (S_ℓ), combined with $\varphi_{\ell+1}(y) \in C(y) \subseteq C_\ell(y)$ for every $y \in W_{\ell+1}$ and the property that $\{\varphi_{\ell+1}(y)\colon\, y \in W_{\ell+1}\}$ is a system of distinct representatives, we infer that $\varphi_{\ell+1}$ is indeed a partial embedding of $H[\bigcup_{j=1}^{\ell+1} W_j]$.

Next we shall verify properties (a) and (b) of $(S_{\ell+1})$. So let $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ be fixed. If $N_H(z)\cap W_{\ell+1} = \varnothing$, then $C_{\ell+1}(z) = C_\ell(z)$, $\mathrm{ldeg}_g^{\ell+1}(z) = \mathrm{ldeg}_g^{\ell}(z)$, which yields (a) and (b) of $(S_{\ell+1})$ for that case. If, on the other hand, $N_H(z)\cap W_{\ell+1} \neq \varnothing$, then there exists a unique neighbour $y \in W_{\ell+1}$ of H (owing to the fact that g is a refinement of a valid vertex coloring of H^3). Because of the definition of $C_{\ell+1}(z) = C_\ell(z)\cap N_{G_R}(\varphi_{\ell+1}(y))$ part (a) of $(S_{\ell+1})$ follows in this case. Moreover, since $\varphi_{\ell+1}(y) \in C(y)$, we infer directly from (b') that (b) of $(S_{\ell+1})$ is satisfied in this case.

Finally, we verify property (c) of $(S_{\ell+1})$. Let $\{z, z'\}$ be an edge of H with $z, z' \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$. We consider three cases, depending on the size of $N_H(z)\cap W_{\ell+1}$ and of $N_H(z')\cap W_{\ell+1}$. If $N_H(z)\cap W_{\ell+1} = \varnothing$ and $N_H(z')\cap W_{\ell+1} = \varnothing$, then part (c) of $(S_{\ell+1})$ follows directly from part (c) of (S_ℓ) and $\varepsilon_{\ell+1} \geq \varepsilon_\ell$, combined with $C_{\ell+1}(z) = C_\ell(z)$, $C_{\ell+1}(z') = C_\ell(z')$. If $N_H(z)\cap W_{\ell+1} = \{y\}$ and $N_H(z')\cap W_{\ell+1} = \varnothing$, then (c) of $(S_{\ell+1})$ follows from (c') and the definition of $C_{\ell+1}(z)$ and $C_{\ell+1}(z')$. If $N_H(z)\cap W_{\ell+1} = \{y\}$ and $N_H(z')\cap W_{\ell+1} = \{y'\}$, then $y = y'$, as otherwise there would be a y-y'-path in H with three edges, i.e., $\{y, y'\}$ would be an edge in H^3, which would imply that $g(y) \neq g(y')$. Consequently, (c) of $(S_{\ell+1})$ follows from (c') and the definition of $C_{\ell+1}(z)$ and $C_{\ell+1}(z')$.

We have therefore verified (a)–(c) of $(S_{\ell+1})$, thus concluding the induction step. The proof of Lemma 6.14 follows by induction. □

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Erklärungen

Hiermit erkläre ich, dass

- ich die vorliegende Habilitationsschrift selbstständig ohne fremde Hilfe verfasst und nur die angegebene Literatur und angegebenen Hilfsmittel verwendet habe,
- mein Anteil (an der Konzeption, Durchführung und Berichtabfassung) an den für die vorliegende Habilitationsschrift verwendeten Forschungsarbeiten dem prozentualen Anteil entspricht, der sich durch die Anzahl der mitwirkenden Wissenschaftler ergibt,
- für mich weder ein früheres noch ein schwebendes Habilitationsverfahren existiert, und
- mir die am 17.01.2005 im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin (Nr. 23/2005) veröffentlichte Habilitationsordnung der Mathematisch-Naturwissenschaftlichen Fakultät II bekannt ist.

Berlin, den 25. Juni 2009