How to build a tree of tangles by local refinements

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Abstract

Consider a set of tangles of some set S of separations, e.g. of a graph or matroid, and a nested set T of separations such that all these tangles agree on $T \cap S$. Then we find a nested set $T' \supseteq T$ with $T' \subseteq T \cup S$ that distinguishes all these tangles. We use this local refinement theorem to provide a new inductive way of constructing trees of tangles in general universes of separations. This approach is of particular relevance in cluster analysis applications where the order of separations in $S \setminus T$ should be allowed to depend on the choice of the tangle of T that identifies the location of the dataset which the elements of S are deemed to separate.

Contents

1 Introduction

Tangles were introduced by Robertson and Seymour [12] as a tool to describe clusters in graphs. Tangles do not list the elements of clusters explicitly, but make use of the idea that a cluster cannot be divided equally by separations of 'low order'. Hence, these low-order separations can be oriented towards the side containing most of the cluster; this orientation forms the tangle *pointing to the cluster*. This new paradigm has the advantage that a cluster can be captured even though it might be fuzzy, as happens in most real-world scenarios.

One of the central theorems in the context of tangles in graphs is the following tree-of-tangles theorem. Roughly speaking, it says that the clusters to which the maximal tangles of a graph point can be ordered in a tree-like way (see also [1, Theorem 12.5.4]).

Theorem 1.1 ([12], cited as [10, Theorem 1.1])**.** *Every graph has a tree-decomposition displaying its maximal tangles.*

The paradigm of describing clusters by orienting separations towards them leads to the concept of *profiles* in *abstract separation systems*: tangles of graphs are a special case of (robust) profiles in abstract separation systems. But abstract separation systems do not only arise from graphs but also, for example, from matroids or other discrete structures. Diestel, Hundertmark, and Lemanczyk proved a tree-of-tangles theorem in this more general setting, Theorem 1.2 below. Here the structure tree, i.e. the tree-like way of ordering the profiles, is described by a collection T of nested separations. Their theorem implies Theorem 1.1, but also strengthens it in that its structure tree is invariant under automorphisms of the underlying separation system:

Theorem 1.2 ([6, Theorem 3.6])**.** *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge, |\ |)$ be a submodular universe of separations. Then for every robust set $\mathcal{P}$ of profiles in $\vec{U}$ there is a nested set $T = T(\mathcal{P}) \subseteq U$ of separations such that:*

(i) *every two profiles in $\mathcal{P}$ are efficiently distinguished by some separation in T;*

(ii) *every separation in T efficiently distinguishes a pair of profiles in $\mathcal{P}$;*

(iii) *for every automorphism α of $\vec{U}$ we have $T(\mathcal{P}^\alpha) = T(\mathcal{P})^\alpha$;*

(iv) *if all the profiles in $\mathcal{P}$ are regular, then T is a regular tree set.*

The proof of this theorem given in [6] builds the nested set T inductively, making sure for $k = 1, 2, \dots$ that the sets $T_k \subseteq T$ of separations included in T by step k distinguish any two profiles in $\mathcal{P}$ that can be distinguished by a separation of order $< k$. At each step, the nested set T_k constructed so far consists only of separations of order $< k$. It partitions the set of all the profiles of order $\geq k$ in $\mathcal{P}$ by putting those profiles in the same partition class that cannot be distinguished by a separation of order $< k$ and which therefore 'live in the same location of T_k' in that they induce the same profile on T_k. For each of these locations, local nested sets of separations of order precisely k are then built to distinguish as many of the profiles in $\mathcal{P}$ that live at this location as possible. These local nested sets are all nested with T_k and with each other. Combined with T_k, they form the new nested set T_{k+1}.

Since the profiles in $\mathcal{P}$ can be assumed to be regular in most applications [6], we restrict ourselves to this case for the course of the introduction. Then all the above T_k are tree sets [3]. Every tree set T can be represented by the edge set E of a graph-theoretic tree $\hat{T}$ in an essentially unique way such that the profiles of T correspond to nodes of $\hat{T}$. More precisely, every profile of T defines an orientation of E towards some node of $\hat{T}$, the node corresponding to that profile of T (see [3]).

In the proof of Theorem 1.2 as sketched above, a profile $P \in \mathcal{P}$ of order $\geq k$ induces a profile of the tree set T_k, but it is itself a profile in some larger universe of separations. The location of P then corresponded to a node $t = t(P)$ of $\hat{T}_k$. In the case of a separation system arising from a graph, the node $t(P)$ represents those clusters of the graph that orient T_k as P does. We say that these clusters *live in the location* $t(P)$.

In order to avoid having to refer, rather indirectly, to the graph-theoretic tree $\hat{T}_k$ in our definition of the location of a profile $P \in \mathcal{P}$ with respect to T, let us define the *location of* P as the set $\vec{L}(P) \subseteq \vec{T}_k$ of the maximal separations in the profile that P induces on T_k. Then $\vec{L}(P)$ consists of the separations that correspond to the oriented edges of $\hat{T}_k$ which are incident with $t(P)$ and point towards it. We say that P *lives in* the location $\vec{L}(P)$ of T_k. (These sets $\vec{L}(P)$ are *splitting stars of* T_k [3].)

Analysing the structure of the clusters living in a given location of T_k requires the construction of a local tree set 'inside' this location, using separations of order exactly k. With increasing k, the graph-theoretic tree $\hat{T}_k$ representing T_k gets refined more and more by little trees replacing the nodes of $\hat{T}_k$. Therefore, the order of a separation in $T = \bigcup_k T_k$ reflects its importance for the overall structure of our graph in the sense that low-order separations describe its tree-like structure in terms of its rough clusters, whereas high-order separations reveal more specific local clusters inside a broader one. From this point of view, the inductive approach in the proof of Theorem 1.2 reveals the cluster structure of a graph (say) step-by-step in unfolding detail.

The overall structure of our graph or data set that emerges in this way depends heavily on what order the individual separations in $T = \bigcup_k T_k$ have. Indeed, even the clusters themselves, defined here as tangles (or profiles) of some given order k, depend on the orders that we assigned to the separations we are considering.

In Theorem 1.2, it is assumed that these orders are given by some fixed global order function $|\cdot| : U \to \mathbb{N}$ (so each separation has some fixed order independent of the location in which this separation plays a role in finding a local tree set), which moreover has to be submodular. (A universe of separations, as assumed in Theorem 1.2, is called *submodular* if the order function satisfies $|\vec{s}\wedge\vec{t}|+|\vec{s}\vee\vec{t}| \leq |\vec{s}|+|\vec{t}|$ for any two separations $\vec{s}, \vec{t} \in \vec{U}$.) But neither of these properties can be assumed in all the important applications of abstract tangle theory to clustering. Let us illustrate this with an example from [4].

In the course of a study, a group V of people filled in a questionnaire Q. Each question could be answered yes or no, so it defines a bipartition of V. All these bipartitions together form the separation system induced by Q. Its clusters captured as profiles of Q correspond to those parts of V where most people gave the same answers to many of the questions in our questionnaire. In other words, the profiles in our separation system identify the 'mindsets' found amongst the participants of our study. In order to determine these mindsets and to tell them apart, we would ideally like to find a nested set $T \subseteq Q$ that distinguishes these profiles, as the questions in T alone will then allow us to distinguish those mindsets from one another. Usually, our set Q of questions itself will not contain such a tree set T. But we can construct one that consists of Boolean expressions of questions from Q.

One strategy for algorithmically finding such a tree T of tangles is to build it iteratively, starting with the most important questions and adding less important ones as T grows. As noted earlier, the importance of a separation should be reflected by its order: important questions should be assigned low orders as separations, so that they are considered first. In our previous example, however, this cannot be achieved by any fixed order function on all the separations.

Indeed, suppose we have started building our tree of tangles considering some of the bipartitions arising from the questionnaire Q. We may find that one profile P of these separations points to a place where people like classical music. In order to refine P by more detailed profiles, we now wish to consider some further questions and determine those of their profiles that extend P. Given the nature of this particular P, it seems more promising to look at the 'Do you like Bach?' bipartition than at the 'Do you like football?' one. At the same time, we may have another location of the tree set built so far where people like sports. Here, the 'Do you like football?' bipartition is more relevant than the 'Do you like Bach?' one in order to investigate the structure of this location further. This exemplifies that the importance of a bipartition of V given by a question in Q can depend on the location that we wish to explore in more detail using that bipartition.

Back to the abstract setting, the practical problem of defining the order of separations locally translates into the following mathematical problem: suppose we have a tree set T. In the next iteration step we wish to choose for each splitting star $\vec{L}$ of T individually a set $S_{\vec{L}}$ of further separations. (For tree sets $T = T_k$ as earlier, this choice can be seen as assigning local order k to the separations in $S_{\vec{L}}$ because we believe them to be the next most important separations for the purpose of analysing the location $\vec{L}$.) Note that we may consider the same separation s for different, but perhaps not all, locations of T. Then we are looking for profiles of $S_{\vec{L}} \cup L$ that extend $\vec{L}$ in that they induce the orientation $\vec{L}$ on L, and we seek to extend T at each of its locations $\vec{L}$ by an additional local tree set $T_{\vec{L}} \subseteq S_{\vec{L}}$ of separations inside $\vec{L}$ which distinguishes all the profiles of $S_{\vec{L}} \cup L$ that extend $\vec{L}$. This corresponds to replacing the nodes of $\hat{T}$ with local structure trees, independently for each splitting star $\vec{L}$ of T.

In order for the sets $S_{\vec{L}}$ to contain such local tree sets $T_{\vec{L}}$, they need to satisfy a structural condition which, in Theorem 1.2, used to come with the submodular global order function that we no longer have. This condition, *structural submodularity*, was identified by Diestel, Erde, and Weißauer [5], who showed that it guarantees the existence of structure trees. Alternatively, if there are no refining profiles for some location $\vec{L}$, then we can use the concept of *tangle-tree duality* to find a local tree set $T_{\vec{L}}$ that witnesses the non-existence of certain refinements of $\vec{L}$.

In this thesis, we first give a brief introduction into abstract separation systems with the relevant definitions in Section 2. Then we follow the above sketched iterative approach to cluster analysis on bipartitions of finite sets as exemplified with the questionnaire scenario: in Section 3 we go through this approach in mathematical detail and precisely formulate the theorems needed to make it work. The proofs of these local refinement theorems form Section 4. Inspired by the cluster analysis setup, we investigate how to build trees of tangles using local refinements in general universes of separations in Section 5, and reformulate the 'algorithmic' approach from Section 3 into a more structural framework. In Section 6 we apply our results from Section 5 to reobtain a non-canonical version of Theorem 1.2, and deduce a sequential tree-of-tangles theorem. Finally, in Section 7, we elaborate the idea of using tangle-tree duality in the context of local refinements in general universes of separations, extending the work of Erde [11].

2 Separation Systems, Profiles, and Tangles

In this thesis, we work within the framework of [2], [3], [5], [6], and [7]. This section provides a brief introduction into abstract separation systems and the relevant definitions; its structure is partially inspired by [10, Section 2] and [11, Section 2.2]. For any discussion and context to the definitions made here, we refer the reader to the papers cited above. For the definitions of graphs and their properties, see [1].

Section 2 does not contain any new definitions or results; so this section may be skipped and just used for future reference if the reader is familiar with the framework of abstract separation systems. All further concepts which are not defined in this section will be defined below.

A *separation system* is a triple $(\vec{S}, \leq, ^*)$ which consists of a poset $(\vec{S}, \leq)$ and an order-reversing involution $^* : \vec{S} \to \vec{S}$; we shall write simply $\vec{S}$ as a shortened form. For an element $\vec{s} \in \vec{S}$, its *inverse* under the involution is denoted by $\overleftarrow{s} = (\vec{s})^*$. The involution * is order-reversing in that $\overleftarrow{s} \geq \overleftarrow{t}$ if and only if $\vec{s} \leq \vec{t}$ for $\vec{s}, \vec{t} \in \vec{S}$. The elements of $\vec{S}$ are called *(oriented) separations.*

The underlying *unoriented separation* of $\vec{s} \in \vec{S}$ is the set $\{\vec{s}, \overleftarrow{s}\}$ which we denote by s. We call $\vec{s}$ and $\overleftarrow{s}$ the *orientations* of an unoriented separation s. If $\vec{s} = \overleftarrow{s}$, then s is called *degenerate*; otherwise s is called *non-degenerate.* For a separation system $\vec{S}$, we denote by S the set of all the unoriented separations s with orientations in $\vec{S}$. We will informally use terms defined for unoriented separations also for oriented separations, and vice-versa. For example, we shall also call s a separation and S a separation system.

We say that two unoriented separations $s, t \in S$ are *nested* if there are orientations $\vec{s}$ of s and $\vec{t}$ of t with $\vec{s} \leq \vec{t}$. If s and t are not nested, we say that they *cross.* A *nested set (of separations)* is a set of separations whose elements are pairwise nested. Two sets of separations are said to be *nested* if every element of the first set is nested with every element of the second set.

Given an oriented separation $\vec{s}$ and an unoriented separation t, we say that $\vec{s}$ *points towards* t if there is an orientation $\vec{t}$ of t such that $\vec{s} \leq \vec{t}$. Conversely, $\vec{s}$ *points away from* t if there is an orientation $\vec{t}$ of t such that $\vec{t} \leq \vec{s}$. Note that if two separations s and t are nested, then each orientation of s points towards or away from t; however, a separation $\vec{s}$ may point both towards and away from t.

Such a separation $\vec{s}$ is *trivial in* $\vec{S}$ in that there exists $t \in S$ with $\vec{s} < \vec{t}$ and $\vec{s} < \overleftarrow{t}$; we say that t *witnesses* the triviality of $\vec{s}$ in $\vec{S}$. The inverse $\overleftarrow{s}$ of a trivial separation $\vec{s}$ is called *co-trivial.* If a separation $s \in S$ has no trivial orientation, we shall call it *non-trivial.* If T is a nested set of non-trivial and non-degenerate separations, then we say that T is a *tree set.*

A trivial separation $\vec{s}$ clearly satisfies $\vec{s} \leq \overleftarrow{s}$, but there can be other separations with this property, too. Such a separation $\vec{s}$ with $\vec{s} \leq \overleftarrow{s}$ is called *small*; its inverse is called *co-small.* Note that degenerate separations are non-trivial, but small. A separation system or even a set of separations without small elements is called *regular.*

A *star* is a set $\vec{L}$ of non-degenerate nested oriented separations with $\vec{s} \leq \overleftarrow{t}$ for every two distinct $\vec{s}, \vec{t} \in \vec{L}$. In particular, an element $\vec{s}$ of a star $\vec{L}$ points towards each separation t which has an orientation in $\vec{L}$. Note that by denoting a star with $\vec{L}$, we slightly abuse notation here: $\vec{L}$ does not need to contain both orientations of an unoriented separation s with some orientation $\vec{s} \in \vec{L}$, but will usually only contain one such orientation (otherwise $\vec{s}$ is small).

A *universe of separations* is a quintuple $(\vec{U}, \leq, ^*, \vee, \wedge)$ which consists of a separation system $(\vec{U}, \leq, ^*)$ in which the join operator $\vee$ and the meet operator $\wedge$ are well-defined, and make the poset $(\vec{U}, \leq)$ a lattice. Here, for two oriented separations $\vec{s}, \vec{t} \in \vec{U}$, the separation $\vec{s} \vee \vec{t}$ is defined as *supremum* of $\vec{s}$ and $\vec{t}$ in the partial order $\leq$, and $\vec{s} \wedge \vec{t}$ is defined as their *infimum*. Note that DeMorgan's law holds in universes of separations in that $(\vec{s} \vee \vec{t})^* = \overleftarrow{s} \wedge \overleftarrow{t}$ for $\vec{s}, \vec{t} \in \vec{U}$. Given two unoriented separations $s, t \in U$, a *corner* of s and t is a separation of the form $\vec{s} \vee \vec{t}$ where $\vec{s}$ and $\vec{t}$ are orientations of s and t, respectively.

For a universe $\vec{U}$, an *order function* is a map $|\cdot| : \vec{U} \to \mathbb{N}_{\geq 0}$ with $|\vec{s}| = |\overleftarrow{s}|$; so $|\vec{s}|$ is called the *order of* $\vec{s}$. For an unoriented separation $s \in U$, we let $|s| := |\vec{s}|$ be the *order of* s. Given $\ell \in \mathbb{N}$, let S_ℓ denote the set of all those separations in U that have order $< \ell$. Note that if $\vec{U}$ is finite (as in this thesis), a real-valued map $|\cdot| : \vec{U} \to \mathbb{R}_{\geq 0}$ with $|\vec{s}| = |\overleftarrow{s}|$ for all $s \in U$ can be scaled into an 'essentially integer-valued' order function.

An order function is called *submodular* if

$$|\vec{s} \wedge \vec{t}| + |\vec{s} \vee \vec{t}| \leq |\vec{s}| + |\vec{t}| \quad \forall\, \vec{s}, \vec{t} \in \vec{U}.$$

If a universe $\vec{U}$ is equipped with a submodular order function, we shall call $\vec{U}$ a *submodular universe (of separations)*.

A purely structural analogue of the submodularity of an order function is the following: a separation system $\vec{S}$ in an arbitrary universe $\vec{U}$ of separations is called *structurally submodular* if for any two separations $\vec{s}, \vec{t} \in \vec{S}$, at least one of $\vec{s} \vee \vec{t}$ and $\vec{s} \wedge \vec{t}$ is also in $\vec{S}$. In particular, if $\vec{U}$ is a submodular universe of separations, then the submodularity of its order function implies that the separation systems $S_\ell \subseteq U$ with $\ell \in \mathbb{N}$ are structurally submodular.

Given two separation systems $\vec{S}$ and $\vec{S'}$, a map $\alpha : \vec{S} \to \vec{S'}$ is an *isomorphism* of $\vec{S}$ and $\vec{S'}$ if it is a bijection which commutes with their respective involutions and respects the partial orderings. An *isomorphism* of two universes $\vec{U}$ and $\vec{U}'$ is an isomorphism of their underlying separation systems which also respects their $\vee$ and $\wedge$ operations and, if existing, their order functions.

For a set S of unoriented separations, an *orientation* O of S is a set $O \subseteq \vec{S}$ which contains precisely one orientation of each $s \in S$. A *partial orientation* of S is an orientation of some subset of S. We say that an orientation O of S *extends* a partial orientation O' of S if $O' \subseteq O$.

A non-degenerate separation $s \in U$ is said to *distinguish* two orientations O and O' (not necessarily of the same set of separations) if there are orientations $\vec{s}$ and $\overleftarrow{s}$ of s with $\vec{s} \in O$ and $\overleftarrow{s} \in O'$. In the context of an order function, a separation s *efficiently distinguishes* two orientations O and O' if it has minimal order amongst those separations which distinguish O and O'. We call two orientations O and O' *distinguishable* if there exists a separation which distinguishes O and O'; similarly, a set $\mathcal{O}$ of orientations is called *distinguishable* if every two orientations in $\mathcal{O}$ are distinguishable. For a set $\mathcal{O}$ of orientations, we say that a separation s is *$\mathcal{O}$-relevant* if it distinguishes some two orientations in $\mathcal{O}$.

The above notions extend to sets of separations as follows: We say that a set $S \subseteq U$ of separations *distinguishes* a set $\mathcal{O}$ of orientations if every pair of orientations in $\mathcal{O}$ is distinguished by some separation in S. Similarly, we say that S is *$\mathcal{O}$-relevant* if every separation in S is $\mathcal{O}$-relevant.

An orientation O of a set S of unoriented separations is called *consistent* if we do not have $\overleftarrow{s} < \vec{t}$ for any two distinct $\vec{s}, \vec{t} \in O$. If U is a universe of separations and $S \subseteq U$, then a consistent orientation P of S is a *profile (of S or $\vec{S}$)* if it has the *profile property* in that

$$(\vec{s} \vee \vec{t})^* \notin P \quad \forall\, \vec{s}, \vec{t} \in P.$$

Note that a separation system S which contains a degenerate separation $d \in S$ cannot have a profile by the profile property applied to $\vec{d} \vee \vec{d}$.

A profile P is *regular* if it does not contain any co-small separations; this will often be the case in application scenarios. In particular, two regular profiles cannot be distinguished by separations that have a small orientation. For a universe U of separations equipped with an order function, P is an *ℓ-profile in U* for $\ell \in \mathbb{N}$ if it is a profile of S_ℓ; here, ℓ is called the *order* of P. In general, P is called a *profile in U* if it is an ℓ-profile in U for some $\ell \in \mathbb{N}$.

The following definitions will be needed to formulate the concept of tangle-tree duality. Let S be a set of unoriented separations, and let $\mathcal{F} \subseteq 2^{\vec{S}}$. An orientation O of S is an *$\mathcal{F}$-tangle of S* if O is consistent and there is no $F \subseteq O$ with $F \in \mathcal{F}$. In a universe U with $S \subseteq U$, we shall also call O an $\mathcal{F}$-tangle of S if $\mathcal{F} \subseteq 2^{\vec{U}}$.

In the context of $\mathcal{F}$-tangles, the set $\mathcal{F}$ often consists of stars. Given a set $\mathcal{F} \subseteq 2^{\vec{S}}$ of stars and a separation system $\vec{S}$, we say that $\mathcal{F}$ *forces* $\vec{s} \in \vec{S}$ if $\{\overleftarrow{s}\} \in \mathcal{F}$. The set $\mathcal{F}$ is called *standard for $\vec{S}$* if $\mathcal{F}$ forces every trivial separation $\vec{s} \in \vec{S}$. Moreover, the *essential core* of $\mathcal{F}$ is the set of all $F' \subseteq \vec{S}$ that arise from some $F \in \mathcal{F}$ by deleting those elements of F which are trivial in $\vec{S}$.

An *S-tree* is a pair (G, α) which consists of a (graph-theoretical) tree G and a map $\alpha : \vec{E}(G) \to \vec{S}$ from the oriented edge set $\vec{E}(G)$ of G to $\vec{S}$ such that $\alpha(v, w) = \alpha(w, v)^*$ for all the edges $vw \in \vec{E}(G)$. For a leaf vertex v of G with neighbour w, we call $\alpha(v, w)$ a *leaf separation* of (G, α). If O is an orientation of S and (G, α) an S-tree, then O induces an orientation $E(G)$ via α. This orientation will necessarily have a sink vertex $v \in V(G)$, and we say that O is *contained* in v.

Given a set $\mathcal{F} \subseteq 2^{\vec{S}}$, we call (G, α) an *S-tree over $\mathcal{F}$* if $\alpha(F_v) \in \mathcal{F}$ for all $v \in V(G)$ where

$$F_v := \{(u, v) \mid uv \in E(G)\}.$$

An S-tree (G, α) is called *order-respecting* if α preserves the natural ordering on $\vec{E}(G)$ (see [1] for the definition of this natural ordering). We say that (G, α) is *redundant* if there is a node $v \in V(G)$ with distinct neighbours $v', v'' \in V(G)$ such that $\alpha(v, v') = \alpha(v, v'')$; otherwise, it is called *irredundant*. Moreover, (G, α) is *tight* if each set $\alpha(F_v)$ does not contain both orientations of a non-degenerate separation. Finally, we say that an S-tree (G, α) is *essential* if it is irredundant, tight, and $\alpha(\vec{E}(G))$ does not contain any trivial separations.

Let us briefly collect the most important results regarding these properties of S-trees: In [3, Lemma 6.2], Diestel showed that for each S-tree (G, α) over some $\mathcal{F} \subseteq 2^{\vec{S}}$, there exists an irredundant S-tree (G', α') over $\mathcal{F}$ with $G' \subseteq G$ and $\alpha' = \alpha \restriction \vec{E}(G')$. An irredundant S-tree (G, α) over a set $\mathcal{F} \subseteq 2^{\vec{S}}$ of stars is order-respecting by [3, Lemma 6.3 (i)]; in particular, the set $\alpha(\vec{E}(G))$ of separations is nested. Moreover, for each S-tree (G, α) over some set $\mathcal{F} \subseteq 2^{\vec{S}}$ of stars, there exists an essential S-tree (G', α') over the essential core of $\mathcal{F}$ where G' is a minor of G and $\alpha' = \alpha \restriction \vec{E}(G')$ [3, Corollary 6.7]. For an essential S-tree (G, α) over stars, the map α is even injective [3, Lemma 6.8], and $\alpha(\vec{E}(G))$ is a tree set.

Given a separation $\vec{s} \in \vec{S}$, let $S_{\geq \vec{s}}$ consist of those separations $s' \in S$ where $\vec{s}$ points towards s'. For two non-trivial separations $\vec{s}, \vec{t} \in \vec{S}$ with $\vec{t} \geq \vec{s}$, we define a map $f \downarrow^{\vec{s}}_{\vec{t}}$ on $\vec{S}_{\geq \vec{s}} \setminus \{\overleftarrow{s}\}$ via

$$f \downarrow^{\vec{s}}_{\vec{t}} (\vec{x}) := \vec{x} \vee \vec{t} \text{ and } f \downarrow^{\vec{s}}_{\vec{t}} (\overleftarrow{x}) := (\vec{x} \vee \vec{t})^*,$$

where $\vec{x}$ is the unique orientation of $x \in S_{\geq \vec{s}}$ with $\vec{x} \geq \vec{s}$ (this exists since $\vec{s}$ is non-trivial). If $\alpha' := f \downarrow^{\vec{s}}_{\vec{t}} \circ \alpha$ maps $\vec{E}(G)$ into $\vec{S}$, then (G, α') is again an S-tree, the *shift of* (G, α) *onto* $\vec{t}$.

For two non-trivial and non-degenerate separations $\vec{s}$ and $\vec{t}$ in a separation system $\vec{S}$, we say that $\vec{t}$ *emulates* $\vec{s}$ *in* $\vec{S}$ if $\vec{s} \leq \vec{t}$, and if for any separation $\vec{x} \in \vec{S}_{\geq \vec{s}} \setminus \{\overleftarrow{s}\}$ with $\vec{x} \geq \vec{s}$, we have $\vec{t} \vee \vec{x} \in \vec{S}$. Given a set $\mathcal{F} \subseteq 2^{\vec{S}}$ of stars, we say that $\vec{t}$ *emulates* $\vec{s}$ *in* $\vec{S}$ *for* $\mathcal{F}$ if $\vec{t}$ emulates $\vec{s}$ in $\vec{S}$, and if for every star $\vec{L} \in \mathcal{F}$ with $\vec{L} \subseteq \vec{S}_{\geq \vec{s}} \setminus \{\overleftarrow{s}\}$ that contains a separation $\vec{x} \geq \vec{s}$, we have $f \downarrow^{\vec{s}}_{\vec{t}} (\vec{L}) \in \mathcal{F}$.

A separation system $\vec{S}$ is *separable* if for every two non-trivial and non-degenerate separations $\vec{s}, \overleftarrow{s'} \in \vec{S}$ with $\vec{s} \leq \vec{s'}$, there exists a separation $r \in S$ such that $\vec{r}$ emulates $\vec{s}$ in $\vec{S}$ and $\overleftarrow{r}$ emulates $\overleftarrow{s'}$ in $\vec{S}$. Furthermore, $\vec{S}$ is called $\mathcal{F}$*-separable* if for every two non-trivial and non-degenerate separations $\vec{s}, \overleftarrow{s'} \in \vec{S}$ with $\vec{s} \leq \vec{s'}$ that are not forced by $\mathcal{F}$, there exists a separation $r \in S$ such that $\vec{r}$ emulates $\vec{s}$ in $\vec{S}$ for $\mathcal{F}$ and $\overleftarrow{r}$ emulates $\overleftarrow{s'}$ in $\vec{S}$ for $\mathcal{F}$. Let us say that a set $\mathcal{F} \subseteq 2^{\vec{S}}$ of stars is *closed under shifting in* $\vec{S}$ if $\vec{s}$ emulates $\vec{t}$ in $\vec{S}$ for $\mathcal{F}$ whenever $\vec{s}$ emulates $\vec{t}$ in $\vec{S}$. It is clear from the definitions that a separation system $\vec{S}$ is $\mathcal{F}$-separable if it is separable and $\mathcal{F}$ is closed under shifting in $\vec{S}$.

Let us give two important examples of separation systems. For a graph $G = (V, E)$, a pair (A, B) of subsets of V is an (oriented) separation of G if there is no edge between $A \setminus B$ and $B \setminus A$ in E. We refer to B as the *big side* of (A, B), and call A the *small side* of (A, B). Here, we identify the unoriented separation $\{(A, B), (B, A)\}$ with $\{A, B\}$. The set of all separations of a graph G forms a separation system where $(A, B)^* = (B, A)$ and

$$(A, B) \leq (C, D) :\iff A \subseteq C \text{ and } B \supseteq D$$

for two separations (A, B) and (C, D) of G. This partial order is indeed a lattice in that

$$(A, B) \vee (C, D) = (A \cup C, B \cap D) \text{ and } (A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

The universe of separations of a graph G is usually equipped with the submodular order function $|\{A, B\}| := |A \cap B|$.

Along the same lines, we can define the universe of separations of a finite set V as $\{\{A, B\} \mid A \cup B = V\}$. The involution *, the partial order $\leq$ and the (natural) order function $|\cdot|$ are defined as for separations of graphs. As in Section 3, we often restrict ourselves to the bipartitions of V, i.e. to the universe $\{\{A, B\} \mid A \cap B = \emptyset \text{ and } A \cup B = V\}$ with the same involution and partial order. However, all the bipartitions have order 0 with respect to the natural order function for set separations. There are multiple ways to define more meaningful submodular order functions on bipartition universes; one of them will be sketched in Section 3.

Finally, we end this section with the remark that all the separation systems considered in this thesis are finite without further notice.

3 Algorithmic Cluster Analysis via Local Refinements

In this section we describe the cluster analysis approach sketched in the introduction in more mathematical detail, and state precisely the theorems we need to make it work. We shall prove these theorems in Section 4.

In cluster analysis we usually deal with bipartitions of finite sets. For example, suppose we are given a certain property that elements of some finite ground set V may or may not possess. Then this property naturally induces a bipartition of V which splits V into those elements that have the property and those that do not have it. We refer the reader to [4] for an in-depth discussion of possible setups and application scenarios.

So in this section we work with a universe $\vec{U}$ of separations which consists of all the bipartitions of some finite ground set V, i.e.

$$\vec{U} = \{(A, B) \mid A \cap B = \emptyset \text{ and } A \cup B = V\}.$$

Recall from [6, Section 5.1] that profiles of a separation system $\vec{S} \subseteq \vec{U}$ which consists of set bipartitions and contains a non-trivial separation are regular, i.e. these profiles do not contain any co-small separations: this is because the only small bipartition of V is the trivial $(\emptyset, V)$ whose triviality is witnessed by every non-trivial bipartition. In particular, separations that distinguish two regular profiles are never small. So the structure tree T that we intend to build will not only be a nested set, but even a regular tree set. Therefore, we restrict our attention in this section to regular tree sets T. Note that by the above, almost all tree set T of set bipartitions are regular: the tree set $T = \{\{\emptyset, V\}\}$ is the only exception. (For more general results and discussions, see Sections 5 and 7.)

So let $T \subseteq U$ be a regular tree set in a universe U of set bipartitions. Since every profile of T orients T consistently, it is in particular the down-closure of its maximal elements. The sets of maximal elements of consistent orientations of T are the so-called *splitting stars* of T:

Definition 3.1 ([3])**.** Let $\vec{U}$ be a universe of separations, and let $T \subseteq U$ be a nested set which has no degenerate elements. Then $\vec{L}$ is a *splitting star of* T if there is a consistent orientation O of T such that $\vec{L}$ is the set of maximal elements of O.

As sketched in the introduction and shown by Diestel in [3], we can envision a tree set T as a graph-theoretical tree $\hat{T}$, as follows. Let $\mathcal{L}$ be the set of splitting stars of T. We take $\mathcal{L}$ as the vertex set of $\hat{T}$, and T as its set of edges, where we assign to each edge $t \in T$ as its terminal nodes the two splitting stars in $\mathcal{L}$ that contain $\vec{t}$ or $\overleftarrow{t}$, respectively. In this way, we also have a bijection between the vertices of our graph-theoretical tree $\hat{T}$ and the consistent orientations of T.

In order to analyse algorithmically the cluster structure of V described by a separation system S, we want to build, iteratively, a sequence of regular tree sets $\emptyset = T_0 \subseteq \cdots \subseteq T_n$ that describes this cluster structure in evolving detail. Ideally, we would like to have that $T_k \subseteq S$ for each $0 \leq k \leq n$, but usually T_k will consist of separations in S and corners of separations in S, as we will see below. In the $(k+1)$-th iteration step, T_{k+1} shall arise from T_k by local refinements in that we analyse the clusters in V captured as profiles of T_k. For this purpose we construct, for each cluster captured as a profile of T_k separately, a 'local' regular tree set which describes the structure of the investigated cluster in more detail. These local regular tree sets will be nested with T_k and with each other, and, combined with T_k, they form the refined regular tree set T_{k+1}.

Let us make precise how a regular tree set T describes the cluster structure of V (see also [4, Section 4.1]). For each splitting star $\vec{L}$ of T, let $V_{\vec{L}}$ consist of those elements of V that lie on the big side of all the separations in $\vec{L}$. So $V_{\vec{L}}$ is precisely the big side of the oriented separation which is the supremum of $\vec{L}$, namely $\sup \vec{L} := \bigvee_{\{\vec{t} \in \vec{L}\}} \vec{t} = (V \setminus V_{\vec{L}}, V_{\vec{L}})$. Since $\vec{L}$ is a splitting star of T, the set $V_{\vec{L}}$ is also the set of those elements in V that lie on the big side of all the separations in the consistent orientation of T with maximal elements $\vec{L}$. Each splitting star $\vec{L}$ with non-empty $V_{\vec{L}}$ is induced by a profile of T, but note that $V_{\vec{L}}$ can be empty even if $\vec{L}$ is induced by a profile.

Now the collection of the (non-empty) sets $V_{\vec{L}}$ for all the splitting stars $\vec{L}$ of T form a partition of V: for each $v \in V$, let us orient each bipartition in T towards that side which contains v. This yields a consistent orientation of T with a corresponding splitting star $\vec{L}$ of T such that $v \in V_{\vec{L}}$. Now this consistent orientation is unique: if any bipartition in T is oriented differently, then v is not in the big side of this bipartition, and, in particular, v is not in $V_{\vec{L'}}$ for the respective splitting star $\vec{L'} \neq \vec{L}$.

So from the perspective that the collection of the sets $V_{\vec{L}}$ for all the splitting stars $\vec{L}$ of T partitions V, each splitting star $\vec{L}$ of T captures $V_{\vec{L}}$ as a substructure of V. Now each profile P of T is a consistent orientation of T, and thus corresponds to a certain splitting star $\vec{L}$ of T. This splitting star $\vec{L}$ identifies $V_{\vec{L}}$ as the 'cluster' described by P. Therefore, we loosely refer to $V_{\vec{L}}$ as 'the cluster described/captured by $\vec{L}$'. So in order to investigate the cluster captured by a splitting star $\vec{L}$ in more detail, we have to study how S partitions $V_{\vec{L}}$.

During the first k iteration steps, we may have identified some splitting stars of the already constructed regular tree set T_k that we do not want or need to analyse any further because the cluster that they capture is detailed enough for our purpose. For example, the cluster $V_{\vec{L}}$ described by such an uninteresting splitting star $\vec{L}$ could be small or even empty, or there might not be any separation in S which partitions $V_{\vec{L}}$ into proper subsets. This is why we carry forward, along with T_k, a set $\mathcal{L}_k$ of those splitting stars that we want to investigate further. The splitting stars $\vec{L} \in \mathcal{L}_k$ are called *locations* of T_k. We say that a profile P of T_k *lives* in the location $\vec{L}$ of T_k that it induces.

Suppose that a regular tree set T_k and a set $\mathcal{L}_k$ of splitting stars of T_k have been constructed. For the $(k+1)$-th iteration step we wish to choose, for each location $\vec{L} \in \mathcal{L}_k$ individually, a set of separations $S_{\vec{L}}$ that we believe to be of high relevance for the further investigation of the cluster captured by $\vec{L}$. In particular, the choice of $S_{\vec{L}}$ may depend on the location $\vec{L}$. All the separations in $S_{\vec{L}}$ will be separations in our fixed separation system S; however, for notational simplicity, we will often not mention this explicitly in what follows, and just require $S_{\vec{L}} \subseteq U$.

Now which separations from S are reasonable candidates for $S_{\vec{L}}$ for some given location $\vec{L} \in \mathcal{L}_k$? In order to investigate the cluster $V_{\vec{L}}$ captured by the location $\vec{L}$, we have to look at how the separations in S partition $V_{\vec{L}}$. In particular, we do not care about how $V \setminus V_{\vec{L}}$ is partitioned by S, and any information about this will be irrelevant for the purpose of analysing $\vec{L}$. So suppose we believe that a separation $s \in S$ is of high relevance for the further investigation of the cluster described by $\vec{L}$. Remember that we want to use the separations in $S_{\vec{L}}$ to investigate the cluster captured by $\vec{L}$ in that we construct a 'local' regular tree set $T_{\vec{L}}$ which describes this cluster in more detail. Moreover, the local regular tree set $T_{\vec{L}}$ should be nested with T and with all the local regular tree sets that we construct for other locations in $\mathcal{L}_k$. Now s might cross T_k arbitrarily which greatly complicates our aim to construct such a local regular tree set $T_{\vec{L}}$. But for the purpose of investigating $\vec{L}$, we do only need the information about how s partitions $V_{\vec{L}}$. So instead of including s itself into $S_{\vec{L}}$, we include the 'part' of s into $S_{\vec{L}}$ which contains only the information about how it partitions $V_{\vec{L}}$ (in addition to what we already know from $\vec{L}$ and T_k).

More precisely, instead of $s = \{A, B\}$, we can include one of the separations $\{A \cup (V \setminus V_{\vec{L}}), B \cap V_{\vec{L}}\}$ and $\{B \cup (V \setminus V_{\vec{L}}), A \cap V_{\vec{L}}\}$ into $S_{\vec{L}}$. Let $\vec{s} = (A, B)$. Then these two separations are precisely the unoriented separations which underlie the two corner separations $\vec{s} \vee \sup \vec{L} = (A \cup (V \setminus V_{\vec{L}}), B \cap V_{\vec{L}})$ and $\overleftarrow{s} \vee \sup \vec{L} = (B \cup (V \setminus V_{\vec{L}}), A \cap V_{\vec{L}})$ of s and $\sup \vec{L}$. This is why we shall informally say that we replace s with one of the corner separations $\vec{s} \vee \sup \vec{L}$ and $\overleftarrow{s} \vee \sup \vec{L}$.

These two corner separations capture precisely the information that we are interested in, namely how s partitions $V_{\vec{L}}$: they both have $V \setminus V_{\vec{L}}$ completely on one side, but partition $V_{\vec{L}}$ in the same way as s did. Particularly, note that both corners $\vec{s} \vee \sup \vec{L}$ and $\overleftarrow{s} \vee \sup \vec{L}$ induce the same partition on $V_{\vec{L}}$; so it is sufficient to include one of them into $S_{\vec{L}}$ as we do not gain any information about the structure of $V_{\vec{L}}$ by including them both.

The corners of s with $\sup \vec{L}$ have the advantage over s that they are already 'compatible' with $\vec{L}$ and T_k in a way that helps us to find a local regular tree set $T_{\vec{L}}$ which describes the structure of the cluster captured by $\vec{L}$. These corners are, for example, nested with T_k, but they are compatible with T_k and $\vec{L}$ in an even stronger sense that we are going to make precise below.

However, the above replacement step has one caveat: $S_{\vec{L}}$ was supposed to be a subset of S, but the corners $\vec{s} \vee \sup \vec{L}$ and $\overleftarrow{s} \vee \sup \vec{L}$ will in general not be in S. By definition the corners of separations in S contain the information about how S partitions V that we get by considering combinations of separations in S. In this sense, we do not change the amount of information that S contains about V if we consider such corners. This justifies that we add, for the purpose of analysing a location $\vec{L}$, corners of separations in S to $S_{\vec{L}}$, and thus to S. But we make this change to S locally, for the purpose of analysing $\vec{L}$ only: for each location $\vec{L'} \neq \vec{L}$, the added corners have $V_{\vec{L'}} \subseteq V \setminus V_{\vec{L}}$ completely on one side, so they do not provide any information about how S partitions $V_{\vec{L'}}$.

Let us illustrate this replacement step with the example from the introduction where the separation system $\vec{S}$ was induced by the answers of study participants to questions from a questionnaire Q. Suppose that we so far constructed a regular tree set T_k, and let $\vec{L} \in \mathcal{L}_k$ be a location of T_k that we want to investigate further, e.g. the location of T_k where $V_{\vec{L}}$ contains the lovers of classical music in our study group. Then $V_{\vec{L}}$ consists of those participants of our study that answered all the questions which induce the separations in L in the way that induces the orientation $\vec{L}$ of L, e.g. they answered the question whether they like classical music in the affirmative.

Suppose that we want to analyse $\vec{L}$ further using a separation $s \in S$ that corresponds to the question 'Do you like Bach?'. Now even though the participants in $V \setminus V_{\vec{L}}$ do not like classical music in general, some of them might still enjoy the music of Johann Sebastian Bach while others do not like any classical music at all. Consequently, the separation s might partition the small side of the 'Do you like classical music?'-separation in $\vec{L}$. In this case, s itself carries more information about V than we need for the further analysis of $\vec{L}$: we are not interested in what the study participants in $V \setminus V_{\vec{L}}$ think about Bach; we are interested in whether the study participants in $V_{\vec{L}}$, those that like classical music, do also like Bach or not. So instead of including s into $S_{\vec{L}}$, we include the separation into $S_{\vec{L}}$ that corresponds to the combined question 'Do you like classical music *and* do you like Bach?'. On the formal level, considering such Boolean expressions of questions from Q corresponds to taking corners of the corresponding induced separations in S. In particular, if we want to include the separation into $S_{\vec{L}}$ that corresponds to logical conjunction of the question inducing s with all the questions inducing the separations in $\vec{L}$, then we include precisely the corners $\vec{s} \vee \sup \vec{L}$ and $\overleftarrow{s} \vee \sup \vec{L}$.

However, these corners do not have to be in $\vec{S}$ since the corresponding combined questions may not appear on the questionnaire Q. But for any Boolean expression of questions from Q, we can infer how the participants of the study would have answered the combined question based on their answers to the questions from Q (assuming that the participants of our study answered the questions in Q consistently). In this sense, adding the corresponding corner separation to S for the purpose of analysing a location $\vec{L}$ only does not change the amount of information that S provides about $V_{\vec{L}}$.

By looking at the questionnaire example, we explained how to construct the sets $S_{\vec{L}}$ of separations 'by hand' in that we determine, for each location $\vec{L}$ separately, the most relevant separations for the analysis of $\vec{L}$ by content-related considerations. But from an algorithmic perspective it is not desirable to stop the algorithm after each iteration step to manually select a suitable set $S_{\vec{L}}$ for every location $\vec{L}$; instead, we want to choose the sets $S_{\vec{L}}$ mechanically in each iteration step. In the next paragraph we will describe an approach to such a mechanical construction by adapting a construction of Diestel in [4, Section 3.4].

In the introduction we explained using the questionnaire example that we cannot assume the existence of a global submodular order function which reasonably describes the next important separations simultaneously at each location in $\mathcal{L}_k$. Therefore, we suggested to use local order functions whose definition depends on the respective location instead. Formally we can rephrase the choice of the sets $S_{\vec{L}}$ in the following way: For each location $\vec{L}$, we define a 'local' order function $|\cdot|_{\vec{L}}$ on U depending on $\vec{L}$. Then the separations of highest relevance for the analysis of $\vec{L}$ are those separations in S that have low order with respect to $|\cdot|_{\vec{L}}$, namely order $< \ell$ for some $\ell \in \mathbb{N}$. From this set S_ℓ of separations, we can construct $S_{\vec{L}}$ by the replacement process as described above. But with a suitably constructed local order function $|\cdot|_{\vec{L}}$, we can avoid this replacement step since all the separations constructed in the replacement process are already in S_ℓ, and we can just pick them therefrom (see below).

One typical construction of a global submodular order function on bipartition universes is as follows (for any details, see in particular [4, Section 3.4]): For each two elements u, v of the finite ground set V, we choose a similarity measure $\sigma(u, v) \in \mathbb{N}$ where u and v are seen as similar if the pair $\{u, v\}$ has a high similarity measure $\sigma(u, v)$. (As Diestel explains, it is possible to define a reasonable similarity measure only referencing S, but without a concrete interpretation. [4]) Now the order of a bipartition $\{A, B\}$ of V is defined as

$$|\{A, B\}| = \sum_{a \in A} \sum_{b \in B} \sigma(a, b). \tag{$\star$}$$

So $|\cdot|$ assigns low order to those bipartitions of V that do not separate many similar pairs.

This construction can be adapted to define a local order function $|\cdot|_{\vec{L}}$ as follows: When analysing the location $\vec{L}$, we are interested in how the separations in S partition the elements of $V_{\vec{L}}$. Therefore, if we calculate the local order of a bipartition $\{A, B\}$ at $\vec{L}$, we do only need to consider those pairs (a, b) of elements $a \in A$ and $b \in B$ in the double sum in $(\star)$ where both a and b are in $V_{\vec{L}}$. The resulting local order function $|\cdot|_{\vec{L}}$ on U assigns to a bipartition $\{A, B\}$ of V an order that is based solely on the similarity of elements of $V_{\vec{L}}$. To put it in another way, the order assigned by $|\cdot|_{\vec{L}}$ to $\{A, B\}$ equals the order that the order function $|\cdot|$ constructed in $(\star)$ on the ground set $V_{\vec{L}}$ would assign to the bipartition $\{A \cap V_{\vec{L}}, B \cap V_{\vec{L}}\}$ of $V_{\vec{L}}$.

From this perspective, it is immediate that $|\cdot|_{\vec{L}}$ assigns the same order to a separation s and its corner separations $\vec{s} \vee \sup \vec{L}$ and $\overleftarrow{s} \vee \sup \vec{L}$ with $\sup \vec{L}$. So we can choose $S_{\vec{L}}$ as the set of those separations in S_ℓ which have the form of such a corner separation without loosing any information about how the separations of low order partition $V_{\vec{L}}$.

This inconvenient and somehow artificial restriction step can be avoided by directly choosing $S_{\vec{L}}$ with respect to the order function defined by $(\star)$ on the ground set $V_{\vec{L}}$: We first apply this order function to the set of bipartitions of the ground set $V_{\vec{L}}$ that are induced by S, and choose $S'_{\vec{L}}$ to be the set of all those bipartitions of $V_{\vec{L}}$ induced by S that have order $< \ell$ for some fixed $\ell \in \mathbb{N}$. Then we define $S_{\vec{L}} = \{(A \cup (V \setminus V_{\vec{L}}), B) \mid \{A, B\} \in S'_{\vec{L}}\}$ which contains precisely the separations in $\vec{U}$ that have the form $\vec{s} \vee \sup L$ for some $\vec{s} \in \vec{S}$ of order $< \ell$ with respect to the above constructed $|\cdot|_{\vec{L}}$. Note that by construction, this $\vec{S}_{\vec{L}}$ contains precisely one of $\vec{s} \vee \sup \vec{L}$ and $\overleftarrow{s} \vee \sup \vec{L}$ for each $s \in S'_{\vec{L}}$.

With the replacement process described above, we can assume that each separation s which we include into $S_{\vec{L}}$ does not partition $V \setminus V_{\vec{L}}$, otherwise we include a corner separation of s with $\sup \vec{L}$ instead of s itself without loosing any information about $\vec{L}$. Hence, every separation $s \in S_{\vec{L}}$ has an orientation $\vec{s}$ such that $V \setminus V_{\vec{L}}$ is a subset of the small side of $\vec{s}$. In other words this orientation $\vec{s}$ satisfies $\sup \vec{L} \leq \vec{s}$ by the definition of $V_{\vec{L}}$, so $\sup \vec{L}$ points towards any separation $s \in S_{\vec{L}}$.

However, as we shall see in the course of this thesis, this property is stronger than what we actually need to build the desired sequence of regular tree sets through local refinements. Indeed, we do not need the separations in $S_{\vec{L}}$ to have $V \setminus V_{\vec{L}}$ completely on their small side; it is sufficient for our purposes if they do not partition the small side of any single $\vec{t} \in \vec{L}$. We say that such separations are 'inside' $\vec{L}$. To define this formally, we first extend the notion of 'pointing towards' to sets of separations as follows:

Definition 3.2. Let $\vec{X}$ be a set of oriented separations in a universe $\vec{U}$ of separations, and let $Y \subseteq U$ be a set of unoriented separations. Then $\vec{X}$ *points towards* Y if every $\vec{x} \in \vec{X}$ points towards every $y \in Y$, i.e. if for every $\vec{x} \in \vec{X}$ and every $y \in Y$, there is an orientation $\vec{y}$ of y such that $\vec{x} \leq \vec{y}$. Conversely, $\vec{X}$ *points away* from Y if every $\vec{x} \in \vec{X}$ points away from every $y \in Y$. If $\vec{X} = \{\vec{x}\}$, then we say that $\vec{x}$ *points towards* or *points away* from Y, and similarly for $Y = \{y\}$.

Now we are ready to make precise what we mean by separations inside a location $\vec{L}$:

Definition 3.3. Let $\vec{L}$ be a star of separations in a universe $\vec{U}$ of separations. We say that an unoriented separation $s \in U$ is *inside* $\vec{L}$ if $\vec{L}$ points towards s. Conversely, we call s *outside* $\vec{L}$ if some $\vec{t} \in \vec{L}$ points away from s, i.e. if there is an orientation $\vec{s}$ of s and some $\vec{t} \in \vec{L}$ with $\vec{s} \leq \vec{t}$. A set $X \subseteq U$ of unoriented separations is *inside* or *outside* $\vec{L}$ if every $x \in X$ is inside or outside $\vec{L}$, respectively.

By this definition, a separation is nested with L if it is inside or outside $\vec{L}$. Conversely, a separation is inside or outside $\vec{L}$ if it is nested with L. If $\vec{L} = \emptyset$, then Definition 3.3 says that every separation in U is inside $\vec{L}$, but not outside $\vec{L}$.

Note that every separation in L is inside and outside $\vec{L}$, we can view them intuitively as the *borders* of the star $\vec{L}$. But there can also be other separations which are inside and outside $\vec{L}$, namely those separations that have a trivial orientation with a witness in L: suppose a separation $s \notin L$ is inside and outside $\vec{L}$. Then there are orientations $\vec{s}$ and $\overleftarrow{s}$ of s and some $\vec{t} \in \vec{L}$ with $\vec{s} < \vec{t} < \overleftarrow{s}$, so t witnesses the triviality of $\vec{s}$. Particularly, in a universe U of set bipartitions, the trivial separation $(\emptyset, V)$ is inside every splitting star $\vec{L}$ of a (regular) tree set T.

However, for non-empty nested sets T in arbitrary universes of separations, every separation in T is outside a splitting star $\vec{L}$ of T, since $\vec{L}$ is the set of maximal elements of some consistent orientation O of T. Furthermore, if T is a (regular) tree set, then there is no separation from $T \setminus L$ inside $\vec{L}$ since all the separations in T are non-trivial in $\vec{T}$.

Now the property 's is inside $\vec{L}$' is strictly weaker than the property '$\sup \vec{L}$ points towards s': If $\sup \vec{L}$ points towards a non-trivial separation s, then s is clearly inside $\vec{L}$. But the converse does not hold: the supremum $\sup \vec{L}$ does not need to point towards a separation s inside $\vec{L}$. Indeed, s does not even have to be nested with $\sup \vec{L}$ as can be seen in the following example:

Example 3.4. Let $\vec{U}$ be a universe of bipartitions, and let $T = \{t, t'\} \subseteq U$ be a regular tree set where there are orientations $\vec{t}$ of t and $\vec{t'}$ of t' such that $\vec{t} < \vec{t'}$. Then $\vec{L} = \{\vec{t}, \overleftarrow{t'}\}$ is a splitting star of T. Given $\vec{s} \in \vec{U}$ with $\vec{t} < \vec{s} < \vec{t'}$, the underlying unoriented separation s is inside $\vec{L}$, but it crosses $\sup \vec{L} = \vec{t} \vee \overleftarrow{t'}$ because s partitions both the small and the big side of $\vec{t} \vee \overleftarrow{t'}$.

In the light of this, let us briefly revisit the above replacement step. Suppose a separation $s \in U$ is of high relevance for the further investigation of a location $\vec{L}$. Above, we included either $\vec{s} \vee \sup \vec{L}$ or $\overleftarrow{s} \vee \sup \vec{L}$ into $S_{\vec{L}}$ instead of including s into $S_{\vec{L}}$ directly. Thereby, we ensured that $V \setminus V_{\vec{L}}$ is completely on one side of each separation in $S_{\vec{L}}$, i.e. $\sup \vec{L}$ points towards $S_{\vec{L}}$. If we now relax this requirement in that we only want to ensure that each separation in $S_{\vec{L}}$ is inside $\vec{L}$, we could modify the replacement step as follows: Given $s = \{A, B\}$, we go through all the separations in $\vec{L}$ which do not point towards s one by one. For each such $\vec{t} = (C, D) \in \vec{L}$, we replace s with either $\vec{s} \vee \vec{t} = (A \cup C, B \cap D)$ or $\overleftarrow{s} \vee \vec{t} = (B \cup C, A \cap D)$, and then continue with the chosen corner as our new s. The choice between $\vec{s} \vee \vec{t}$ and $\overleftarrow{s} \vee \vec{t}$ can be made individually for each considered $\vec{t} \in \vec{L}$. In this way we ensure that each $\vec{t} \in \vec{L}$ points towards every separation in $S_{\vec{L}}$, so that $\vec{L}$ points towards $S_{\vec{L}}$. However, in practice, it is usually more convenient to avoid making these choices and determining those $t \in L$ not pointing towards s in that we directly replace s by $\vec{s} \vee \sup \vec{L}$ or $\overleftarrow{s} \vee \sup \vec{L}$.

To put it in a nutshell: Suppose we are given the separations that we believe to be of high relevance for the purpose of investigating the cluster $V_{\vec{L}}$ captured by a location $\vec{L}$ of T_k. Then we can modify them into a set $S_{\vec{L}}$ of separations inside $\vec{L}$ which contains the same amount of information about $V_{\vec{L}}$. This set $S_{\vec{L}}$ is the one that we use to find a local regular tree set $T_{\vec{L}}$ describing the structure of the cluster captured by $\vec{L}$.

Suppose now that we have chosen a set $S_{\vec{L}} \subseteq U$ of separations inside $\vec{L}$ for each location $\vec{L} \in \mathcal{L}_k$. Given such a location $\vec{L} \in \mathcal{L}_k$, the structure of the cluster $V_{\vec{L}}$ captured by $\vec{L}$ is described by how $V_{\vec{L}}$ is partitioned by separations in S. We now investigate the structure of $V_{\vec{L}}$ using the separations in $S_{\vec{L}}$. The 'sub-clusters' of $V_{\vec{L}}$ that we can identify using $S_{\vec{L}}$ are captured by those profiles of $T_k \cup S_{\vec{L}}$ that extend the profile of T_k which corresponds to $V_{\vec{L}}$ itself. Since the splitting star $\vec{L}$ of T_k already determines the full profile of T_k which corresponds to $V_{\vec{L}}$, we hence consider profiles of $S_{\vec{L}} \cup L$ extending $\vec{L}$. We shall call such profiles *refinements* of $\vec{L}$ into $S_{\vec{L}}$, and make the following definition:

Definition 3.5. Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a regular tree set, and $\vec{L}$ a splitting star of T. Let $S_{\vec{L}} \subseteq U$ be a set of separations. Then we say that P *refines* $\vec{L}$ *into* $S_{\vec{L}}$ if P is a profile of $S_{\vec{L}} \cup L$ extending $\vec{L}$. We call P a *refinement of* $\vec{L}$ *into* $S_{\vec{L}}$. The set of all refinements of $\vec{L}$ into $S_{\vec{L}}$ is denoted by $\mathcal{P}_{\vec{L}}(S_{\vec{L}})$. If $S_{\vec{L}}$ is clear from the context, we usually write just $\mathcal{P}_{\vec{L}}$.

Note that our definition of a refinement includes that P is a profile. So if $\vec{L}$ itself is not a profile of L, then there are no refinements of $\vec{L}$ into $S_{\vec{L}}$. But since each (non-empty) cluster of V is captured by a profile of T_k, the locations $\vec{L}$ that we investigate will arise from profiles of T_k, and, in particular, $\vec{L}$ will be a profile of L. Some of our later results also work for so-called $\mathcal{F}$*-tangles* of $S_{\vec{L}} \cup L$ that orient L as $\vec{L}$, but which do not need to be profiles themselves; we will make clear when this is the case.

In practice, we are often not interested in whether *any* refinements of $\vec{L}$ into some set $S_{\vec{L}}$ of separations exist (i.e. whether $\mathcal{P}_{\vec{L}} \neq \emptyset$), but we look for certain *special* refinements $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$. The main reason for this is the following (see also [4, Section 3.1]): Recall (e.g. from [5]) that each element v of our ground set V defines a *principal profile* of U which is 'focused' on v, namely

$$P_v = \{(A, B) \in \vec{U} \mid v \in B\}.$$

In particular, if $v \in V_{\vec{L}}$ for some splitting star $\vec{L}$, then for any set $S_{\vec{L}} \subseteq U$ of separations, the profile $P_v \cap (\vec{S}_{\vec{L}} \cup \vec{L})$ refines $\vec{L}$ into $S_{\vec{L}}$. Throughout our iterative cluster analysis, all the $\vec{L} \in \mathcal{L}_k$ will usually have $V_{\vec{L}} \neq \emptyset$. So in this case, it cannot happen that there do not exist any refinements of $\vec{L}$ into $S_{\vec{L}}$.

However, if $\vec{L}$ itself is not a profile of L, then there are clearly no refinements of $\vec{L}$ into any set $S_{\vec{L}} \subseteq U$ of separations. Furthermore, even if $\vec{L}$ itself is induced by a profile of T, then it can occur that there exist no refinements of $\vec{L}$ into a set $S_{\vec{L}} \subseteq U$ of separations if $V_{\vec{L}}$ is empty:

Example 3.6. Let $V = \{1, 2, 3, 4\}$ and consider the bipartitions $\vec{s}_i = (\{i\}, V \setminus \{i\})$ for $i \in V$. Then $P = \{\vec{s}_i \mid i \in V\}$ is a profile of the regular tree set $T = \{s_i \mid i \in V\}$, and P lives in the location $\vec{L} = P$ of T. Clearly, we have $V_{\vec{L}} = \emptyset$ here.

If we now consider the bipartition $\vec{r} = (\{1, 2\}, \{3, 4\})$ of V which is inside $\vec{L}$, then there exists no refinement P' of $\vec{L}$ into $\{r\}$: by the profile property for $\vec{s}_1$ and $\vec{s}_2$, we would need $\vec{r} \in P'$, whereas the profile property for $\vec{s}_3$ and $\vec{s}_4$ required P' to orient r as $\overleftarrow{r}$.

Now if we assume in our algorithmic setup that $V_{\vec{L}}$ is non-empty for every $\vec{L} \in \mathcal{L}_k$, then there exist refinements of $\vec{L}$ into $S_{\vec{L}}$ exist for every set $S_{\vec{L}} \subseteq U$ of separations, namely those which are induced by the principal profile of some $v \in V_{\vec{L}}$. From this perspective, the sheer existence of such refinements does not directly tell us anything about the cluster structure of $V_{\vec{L}}$ as described by $S_{\vec{L}}$; so the existence of such refinements becomes interesting if they have additional properties (see below).

Therefore, we often do not consider the set $\mathcal{P}_{\vec{L}}$ of all the refinements of $\vec{L}$ into $S_{\vec{L}}$; instead, we focus on the existence of a set $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$ of certain 'special' refinements with additional properties. A typical class of such special refinements consists of those refinements which are also $\mathcal{F}_{\vec{L}}$-*tangles* for some set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars. In particular, the additional tangle-structure of such refinements has the advantage that their non-existence is witnessed by a certain structure tree; we will make this precise below (see Theorem 3.11 and the preceding discussion).

Let us briefly discuss possible choices for the set $\mathcal{F}_{\vec{L}}$ of stars. In general there are many suitable choices for $\mathcal{F}_{\vec{L}}$ depending on the specific setup. Note that for an arbitrary set $\mathcal{F}$ of stars, the $\mathcal{F}$-tangles of a separation system $\vec{S} \subseteq \vec{U}$ do not have to be profiles of $\vec{S}$. But here we want to consider $\mathcal{F}_{\vec{L}}$-tangles that refine a location $\vec{L}$ into some set $S_{\vec{L}} \subseteq U$ of separations. So in order to make sure that all the $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ that orient L as $\vec{L}$ are also refinements of $\vec{L}$ into $S_{\vec{L}}$, we should choose $\mathcal{F}_{\vec{L}}$ in such a way that all the $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ are indeed profiles. A typical choice for such a set $\mathcal{F}_{\vec{L}}$ of stars is as follows (see also [4, Section 3.1] and [5, Section 5.3]).

For an integer $n \geq 1$, consider the set $\mathcal{F}_n \subseteq 2^{\vec{U}}$ of stars defined as

$$\mathcal{F}_n = \{\vec{L} \subseteq \vec{U} \mid \vec{L} = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \text{ is a star in } \vec{U} \text{ with } |B_1 \cap B_2 \cap B_3| < n\}.$$

In the definition of $\mathcal{F}_n$, we do not require the (A_i, B_i) to be distinct. Particularly, an $\mathcal{F}_n$-tangle of a separation system $\vec{S} \subseteq \vec{U}$ orients every separation $\{A, B\} \in S$ with $|A| < n$ as (A, B). So an $\mathcal{F}_n$-tangle P of $\vec{S}$ has the property that for every star $\vec{L} \subseteq P$ of size $|\vec{L}| \leq 3$, the set $V_{\vec{L}}$ has size at least n. In [5, Lemma 25], it is shown that if $\vec{S}$ is structurally submodular, then the $\mathcal{F}_n$-tangles of $\vec{S}$ are precisely those orientations P of S where for every set $\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq P$ (and not just every such set which is also a star), we have $|B_1 \cap B_2 \cap B_3| < n$.

Note that $\mathcal{F}_n$ is standard for every non-empty separation system $\vec{S} \subseteq \vec{U}$, since it clearly contains the singleton star $\{(V, \emptyset)\}$, and hence forces even every small separation in $\vec{U}$. By definition an $\mathcal{F}_n$-tangle P of some separation system $S \subseteq U$ is also a profile of S: Suppose this does not hold for some $\mathcal{F}_n$-tangle P of S and let $\vec{r}, \vec{s}, (\vec{r} \vee \vec{s})^* \in P$ contradict the profile property. Then these three separations form a star $\vec{L} \subseteq \vec{U}$ of size 3 which has $V_{\vec{L}} = \emptyset$. In particular, $\vec{L} \in \mathcal{F}_n$ for all $n \geq 1$ which contradicts that P is an $\mathcal{F}_n$-tangle.

Now the profile of some separation system $\vec{S} \subseteq \vec{U}$ which is induced by a principal profile of $\vec{U}$ is always an $\mathcal{F}_1$-tangle of $\vec{S}$, too; so the $\mathcal{F}_n$-tangles of $\vec{S}$ become interesting for $n \geq 2$. Note that the profiles of S induced by principal profiles of $\vec{U}$ can also be $\mathcal{F}_n$-tangles of S for some $n > 1$, e.g. for a tree set T and a splitting star $\vec{L}$ of T, the profiles of T induced by principal profiles for $v \in V_{\vec{L}}$ are at least $\mathcal{F}_{|V_{\vec{L}}|}$-tangles of T. But this is not true for every separation system $\vec{S} \subseteq \vec{U}$, and hence it does already reveal something about the cluster structure of V as described by S.

Moreover, since $\mathcal{F}_n \supseteq \mathcal{F}_{n'}$ for $n \geq n'$, every $\mathcal{F}_n$-tangle is also an $\mathcal{F}_{n'}$-tangle for $n \geq n'$. This means that the number of $\mathcal{F}_n$-tangles decreases with increasing n. From this point of view, the $\mathcal{F}_n$-tangles for large n capture rather broad substructures of V. So if we choose $\mathcal{F}_{\vec{L}} = \mathcal{F}_n$ for some $n \geq 1$, then the choice of n is a parameter that we can use to specify the degree of fineness of the refinements that we want to consider.

Back to our algorithmic analysis, we aim to extend T_k to a 'refined' regular tree set $T_{k+1} \supseteq T_k$ which also describes the local cluster structure at each location $\vec{L}$ as far as possible using the separations in $S_{\vec{L}}$. More precisely, we construct T_{k+1} such that T_{k+1}, for each location $\vec{L} \in \mathcal{L}_k$, contains a local regular tree set $T_{\vec{L}} \subseteq S_{\vec{L}} \cup L$ that describes the structure of $V_{\vec{L}}$ as captured by a set of 'interesting' refinements $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$ of $\vec{L}$ into $S_{\vec{L}}$. In what follows, let us always keep in mind as a key example that $\mathcal{P}'_{\vec{L}}$ could be chosen as the set of all $\mathcal{F}_{\vec{L}}$-tangles in $\mathcal{P}_{\vec{L}}$ for some suitable locally chosen set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars.

We will construct the refined regular tree set $T_{k+1} \supseteq T_k$ in two steps: First, we show for each location $\vec{L}$ separately that if $S_{\vec{L}}$ admits certain structural properties, then there exists such a local regular tree set $T_{\vec{L}} \subseteq S_{\vec{L}} \cup L$ describing $\mathcal{P}'_{\vec{L}}$ in that it either distinguishes all the refinements in $\mathcal{P}'_{\vec{L}}$ or it witnesses the absence of certain refinements of $\vec{L}$ into $S_{\vec{L}}$. Secondly, we combine all the local regular tree sets $T_{\vec{L}}$ with T_k and with each other to T_{k+1}.

Before we describe structural properties of the sets $S_{\vec{L}} \subseteq U$ of separations that guarantee the existence of the desired local regular tree sets $T_{\vec{L}} \subseteq S_{\vec{L}} \cup L$, let us address the question how to deal with sets $S_{\vec{L}}$ that do not satisfy these structural conditions. As we will see, all the structural conditions that we need to require on the $S_{\vec{L}}$ have one crucial property in common: they demand the set $S_{\vec{L}}$ to contain specific corners of its separations.

Hence, if $S_{\vec{L}}$ does not satisfy the required structural conditions, we 'pre-process' it in that we repeatedly add corners to $S_{\vec{L}}$ until we get a superset $S'_{\vec{L}} \supseteq S_{\vec{L}}$ which satisfies these structural conditions (see also [4, Section 4.3]). Note that this process of repeatedly adding corners will terminate since V, and hence $\vec{U}$ and $\vec{S}$, are finite. After this pre-processing step, we replace $S_{\vec{L}}$ by the newly constructed set $S'_{\vec{L}}$. Since $S_{\vec{L}}$ is inside $\vec{L}$, the set $S'_{\vec{L}}$ is again inside $\vec{L}$: if a separation $\vec{t}$ points towards two separations s and s', then it is immediate from the definition that $\vec{t}$ points towards all four corners of s and s'.

As explained above, the amount of information that $S_{\vec{L}}$ provides about the cluster captured by $\vec{L}$ does not change by adding these corners; this justifies that we change S locally for the purpose of analysing $\vec{L}$, and add the demanded corners to S. But note that the size of $S_{\vec{L}}$ (and S) may increase significantly if many corners have to be added.

Unfortunately, the modification of $S_{\vec{L}}$ by adding corners can change the set of refinements of $\vec{L}$ into $S_{\vec{L}}$. More precisely, the set of refinements of $\vec{L}$ into the original $S_{\vec{L}}$ does not have to be in one-to-one correspondence to the set of refinements of $\vec{L}$ into the newly constructed $S'_{\vec{L}}$. Clearly, every refinement of $\vec{L}$ into $S'_{\vec{L}}$ induces a refinement of $\vec{L}$ into $S_{\vec{L}}$ since $S_{\vec{L}} \subseteq S'_{\vec{L}}$; but the converse does not need to be true as in the following example:

Example 3.7. Let $\vec{U}$ be a universe of bipartitions, and let $S = \{s_1, s_2, s_3\} \subseteq U$ consist of three distinct separations which cross pairwise. Suppose that there exists an orientation $P = \{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ of $\vec{S}$ such that $\overleftarrow{s}_1 < \vec{s}_2 \vee \vec{s}_3 =: \vec{r}$ (see Figure 3.1). Now P is surely a profile of $\vec{S}$ since neither consistency nor the profile property can be violated if no two separations in S are even nested.

But if we consider a separation system $\vec{S'} \supseteq \vec{S}$ which contains the corner $\vec{r}$, then there is no profile P' of $\vec{S'}$ that extends the profile P of $\vec{S}$: by the profile property for $\vec{s}_2$ and $\vec{s}_3$, we must have $\vec{r} \in P'$ whereas we need $\overleftarrow{r} \in P'$ by consistency with $\vec{s}_1$ - a contradiction.

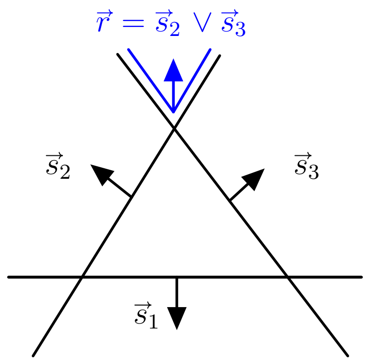

Figure 3.1: The situation in Example 3.7: the profile $P = \{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ cannot be extended to a profile which contains an orientation of the corner r of s_2 and s_3.

This means that we cannot consider the refinements of $\vec{L}$ into $S_{\vec{L}}$ and then fully describe them by finding a local regular tree set in $S'_{\vec{L}} \cup L$, since the refinements of $\vec{L}$ into $S_{\vec{L}}$ may have no counterpart in $S'_{\vec{L}}$. For example, a regular tree set $T_{\vec{L}}$ distinguishing the refinements of $\vec{L}$ into $S'_{\vec{L}}$ does not have to distinguish all the refinements of $\vec{L}$ into $S_{\vec{L}}$. Moreover, if we consider special refinements of $\vec{L}$ into $S_{\vec{L}}$ (e.g. $\mathcal{F}_{\vec{L}}$-tangles), then there does necessarily exist any such refinement of $\vec{L}$ into $S'_{\vec{L}}$. But in practice, these cases do not occur very often, and most refinements of $\vec{L}$ into $S_{\vec{L}}$ will indeed extend to refinements of $\vec{L}$ into $S_{\vec{L}}$ (see also [4, Section 4.3]).

Back to the algorithm, this means that the set of refinements of $\vec{L}$ which we can analyse precisely is the set of refinements of $\vec{L}$ into $S'_{\vec{L}}$. Thereby, we also obtain information about the refinements of $\vec{L}$ into $S_{\vec{L}}$, but as described above, our findings about the refinements of $\vec{L}$ into $S'_{\vec{L}}$ do not need to provide the full picture about the refinements of $\vec{L}$ into $S_{\vec{L}}$. Therefore, we will consider the set of refinements of $\vec{L}$ into $S'_{\vec{L}}$ as the set $\mathcal{P}_{\vec{L}}$ of refinements that we want to investigate in the following. So we first pre-process $S_{\vec{L}}$ suitably by adding corners to get $S'_{\vec{L}}$, and then replace $S_{\vec{L}}$ with $S'_{\vec{L}}$. After this replacement step, we have the set $S_{\vec{L}}$ that we are using for the course of our cluster analysis, and we finally determine the set $\mathcal{P}_{\vec{L}}$ of refinements of $\vec{L}$ into this $S_{\vec{L}}$.

Given a regular tree set T in a universe U of set bipartitions, a splitting star $\vec{L}$ of T, and a set $S_{\vec{L}} \subseteq U$ of separations, we will now study structural properties of $S_{\vec{L}}$ which guarantee the existence of a suitable local regular tree set $T_{\vec{L}} \subseteq S_{\vec{L}} \cup L$ that describes the structure of the cluster captured by $\vec{L}$ as characterized by a set $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$ of refinements of $\vec{L}$ into $S_{\vec{L}}$. In doing so, we distinguish three possible cases: either $\mathcal{P}'_{\vec{L}}$ consists of multiple refinements of $\vec{L}$ into $S_{\vec{L}}$, or there is precisely one such refinement in $\mathcal{P}'_{\vec{L}}$, or $\mathcal{P}'_{\vec{L}}$ is empty, i.e. there does not exist any refinement of $\vec{L}$ into $S_{\vec{L}}$ in $\mathcal{P}'_{\vec{L}}$.

Let us first assume that $\mathcal{P}'_{\vec{L}}$ contains multiple refinements of $\vec{L}$ into $S_{\vec{L}}$. In this case, we want to use a structural version of the tree-of-tangles theorem to refine T in that we add a local regular tree set $T_{\vec{L}} \subseteq S_{\vec{L}}$ to T which distinguishes all the refinements in $\mathcal{P}'_{\vec{L}}$. As we shall see in the following, $S_{\vec{L}}$ even contains such a regular tree set $T_{\vec{L}}$ which distinguishes the set $\mathcal{P}_{\vec{L}}$ of all refinements of $\vec{L}$ into $S_{\vec{L}}$. Such a regular tree set $T_{\vec{L}}$ clearly distinguishes $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$, too.

In general, a separation system $\vec{S}$ in a universe $\vec{U}$ is guaranteed to contain a tree set distinguishing all its profiles if it is *structural submodular.* Let us recall from Section 2 that a separation system $\vec{S} \subseteq \vec{U}$ is structurally submodular if for any two separations $\vec{s}, \vec{t} \in \vec{S}$, at least one of $\vec{s} \vee \vec{t}$ and $\vec{s} \wedge \vec{t}$ is also in $\vec{S}$. This purely structural analogue of the submodularity of a global order function was introduced by Diestel, Erde, and Weißauer in [5]. They showed that structural submodularity implies the existence of structure trees:

Theorem 3.8 ([5, Theorem 6]). *Let $\vec{U}$ be a universe of separations, $\vec{S} \subseteq \vec{U}$ a structurally submodular separation system, and let $\mathcal{P}$ be a set of profiles of $\vec{S}$. Then $\vec{S}$ contains a tree set that distinguishes $\mathcal{P}$.*

We want to use Theorem 3.8 to refine a given regular tree set T into a regular tree set $T' \supseteq T$ by adding a local regular tree set $T_{\vec{L}} \subseteq S_{\vec{L}}$ which distinguishes all the refinements in $\mathcal{P}_{\vec{L}}$. It turns out that it is enough to require $S_{\vec{L}}$ to be structurally submodular if $S_{\vec{L}}$ is inside $\vec{L}$. But by the previous construction, we can clearly assume $S_{\vec{L}}$ to be inside $\vec{L}$ in our cluster analysis scenario.

So our first local refinement theorem reads as follows:

Theorem 3.9. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, and let $T \subseteq U$ be a regular tree set. Let $\vec{L}$ be a splitting star of T, and let $S_{\vec{L}}$ be a set of separations in U which contains a non-trivial separation. Moreover, assume that*

(i) *$\vec{S}_{\vec{L}}$ is a structurally submodular separation system;*

(ii) *$S_{\vec{L}}$ is inside $\vec{L}$.*

Let $\mathcal{P}_{\vec{L}}$ be the set of refinements of $\vec{L}$ into $S_{\vec{L}}$. Then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup S_{\vec{L}}$ such that every two refinements in $\mathcal{P}_{\vec{L}}$ are distinguished by some separation in $T_{\vec{L}} = (T' \setminus T) \cap S_{\vec{L}}$.

We will prove Theorem 3.9 in Section 4. Note that the condition that $S_{\vec{L}}$ contains a non-trivial separation is included in Theorem 3.9 to rule out the pathological case that $\vec{T} = \vec{L} = \emptyset$ and $S_{\vec{L}} = \{\{\emptyset, V\}\}$. In this case, the refinements of $\vec{L}$ into $S_{\vec{L}}$ are not regular, and they cannot be distinguished by any regular tree set in U.

If $S_{\vec{L}}$ is structurally submodular, then Theorem 3.9 provides the regular tree set T' that we are looking for. Recall that by definition, a separation system has to contain specific corners to be structurally submodular. So if $S_{\vec{L}}$ is not structurally submodular, we can pre-process $S_{\vec{L}}$ as described above to make it structurally submodular before we determine the set $\mathcal{P}_{\vec{L}}$ of refinements of $\vec{L}$ into $S_{\vec{L}}$ and apply Theorem 3.9. This pre-processing step will not always be needed: if $S_{\vec{L}}$ comes as the set of those separations in some universe of set bipartitions which have low order with respect to some local submodular order function (e.g. a local order function based on pairwise similarity as in ($\star$)), then $S_{\vec{L}}$ is already structurally submodular.

For the second case, assume that there is precisely one refinement in $\mathcal{P}'_{\vec{L}}$. There are two possible reasons for this: either there is only one cluster inside $\vec{L}$, and we have thus completed our analysis of the cluster captured by $\vec{L}$. Or there are multiple clusters inside $\vec{L}$, but $S_{\vec{L}}$ does not distinguish them. It is up to us to decide which of these reasons applies to $\vec{L}$ and $S_{\vec{L}}$.

The decision which of the two reasons applies can be done mechanically in different ways: For example, we could try to investigate the cluster described by $\vec{L}$ in a fixed number n of iteration steps by repeated choices of $S_{\vec{L}}$. If we still do not find more than one refinement in n iteration steps, then we stop investigating $\vec{L}$ and decide that $\vec{L}$ is the most local structure that we wish to find. In this case, we drop $\vec{L}$ and do not carry it forward in $\mathcal{L}_{k+1}$. On the other hand, if we investigate $\vec{L}$ further and keep it in $\mathcal{L}_{k+1}$, then it might be sensible to choose the new $S_{\vec{L}}$ as a superset of the set of separations that we considered in the preceding iteration step. Thereby, we ensure that the refinements found in the next iteration step extend the currently found unique refinement.

In the third case, there does not exist any refinement of $\vec{L}$ into $S_{\vec{L}}$ in $\mathcal{P}'_{\vec{L}}$, i.e. $\mathcal{P}'_{\vec{L}}$ is empty. Then we can make use of another central result in tangle theory, the so-called *tangle-tree duality theorem*, Theorem 3.10 below. We will use the tangle-tree duality theorem to find a local regular tree set $T_{\vec{L}} \subseteq S_{\vec{L}} \cup L$ that witnesses the absence of certain refinements, namely such refinements that are also $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$ for some suitably chosen set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars. In particular, if $\mathcal{P}'_{\vec{L}}$ is empty, then the set $\mathcal{F}_{\vec{L}}$ should be chosen such that there is also no $\mathcal{F}_{\vec{L}}$-tangle refining $\vec{L}$ into $S_{\vec{L}}$. Especially, this is true if we consider as $\mathcal{P}'_{\vec{L}}$ precisely the set of $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$.

The tangle-tree duality theorem says that a separation system $\vec{S}$ either has an $\mathcal{F}$-tangle or there exists an S*-tree over* $\mathcal{F}$, i.e. a tree-like object that witnesses, in some easily checkable way, the non-existence of $\mathcal{F}$-tangles of S. It reads as follows:

Theorem 3.10 ([7, Theorem 4.3])**.** *Let $\vec{U}$ be a universe of separations containing a finite separation system $\vec{S}$. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for $\vec{S}$. If $\vec{S}$ is $\mathcal{F}$-separable, exactly one of the following holds:*

(i) *There exists an $\mathcal{F}$-tangle of S.*

(ii) *There exists an S-tree over $\mathcal{F}$.*

Let us first make precise how an S-tree over a set $\mathcal{F}$ of stars helps us find a regular tree set which witnessing the non-existence of $\mathcal{F}$-tangles of S. In [3, Section 6], Diestel showed that for an essential S-tree (G, α) over the essential core of $\mathcal{F}$, the separation system $\alpha(\vec{E}(G))$ is a tree set that does not contain any separations which are trivial in $\vec{S}$. Since we work in a universe of set bipartitions, this implies that the separation system $\alpha(\vec{E}(G))$ is even a regular tree set. In this case, we call $\alpha(\vec{E}(G))$ the *regular tree set associated with* (G, α). We shall informally say that the regular tree set associated with an essential S-tree (G, α) over $\mathcal{F}$ witnesses the non-existence of $\mathcal{F}$-tangles of S in the sense of tangle-tree duality.

However, for general S-trees over $\mathcal{F}$, the set $\alpha(\vec{E}(G))$ does not even have to be nested. But if $\mathcal{F}$ contains its essential core, then we can make a general S-tree (G, α) over $\mathcal{F}$ essential: by [3, Corollary 6.7], there exists an essential S-tree (G', α') over the essential core of $\mathcal{F}$ with G' a minor of G and $\alpha' = \alpha \restriction \vec{E}(G')$. In the light of this result we assume for the rest of this section without further notice that every S-tree over a set $\mathcal{F}$ of stars containing its essential core is essential and over the essential core of $\mathcal{F}$. In particular, speaking of the regular tree set associated with an S-tree over $\mathcal{F}$ is always well-defined. In the setup of cluster analysis as in this section, the set $\mathcal{F}$ of stars often contains its essential core (e.g. each of the sets $\mathcal{F}_n$ of stars defined above).

As described above, we aim to use Theorem 3.10 to refine T by adding a local regular tree set $T_{\vec{L}}$ to T which witnesses in the sense of tangle-tree duality that there are no $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$ for some locally chosen set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars containing its essential core.

Note that $T_{\vec{L}}$ cannot be associated with an $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}_{\vec{L}}$: such a tree would witness the non-existence of any $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$. But the absence of $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$ does not necessarily prevent the existence of $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ that do not orient L as $\vec{L}$. The additional restriction 'L is oriented as $\vec{L}$' can be translated into the language of tangles by considering the set $\mathcal{F}'_{\vec{L}} = \mathcal{F}_{\vec{L}} \cup \{\{\overleftarrow{t}\} | \vec{t} \in \vec{L}\}$ of stars instead of $\mathcal{F}_{\vec{L}}$ itself. Then the $\mathcal{F}'_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ are precisely the $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$. Therefore, $T_{\vec{L}}$ will be a regular tree set associated with an $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}}$.

If $S_{\vec{L}}$ is inside $\vec{L}$ (as it is in our cluster analysis setup), and if both $S_{\vec{L}}$ and $\mathcal{F}_{\vec{L}}$ satisfy certain structural conditions which we will discuss below, then we can proceed as desired, and use Theorem 3.10 to get a refined regular tree set $T' \supseteq T$.

So our second local refinement theorem reads as follows:

Theorem 3.11. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, and let $T \subseteq U$ be a regular tree set. Let $\vec{L}$ be a splitting star of T, and let $S_{\vec{L}}$ be a set of separations in U. Suppose that $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ is a set of stars, standard for $S_{\vec{L}} \cup L$, which contains its essential core. Moreover, assume that*

(i) *$S_{\vec{L}} \cup L$ is separable;*

(ii) *$S_{\vec{L}}$ is inside $\vec{L}$;*

(iii) *$\mathcal{F}_{\vec{L}}$ is closed under shifting in $S_{\vec{L}} \cup L$.*

If there are no $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$, then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup S_{\vec{L}}$ such that T' contains a regular tree set $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ which is associated with an essential $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}} = \mathcal{F}_{\vec{L}} \cup \{\{\overleftarrow{t}\} | \vec{t} \in \vec{L}\}$.

For the proof, we again defer the reader to Section 4. The proof will indeed show that Theorem 3.11 still holds if we do not require the considered $\mathcal{F}_{\vec{L}}$-tangles to be profiles of $S_{\vec{L}} \cup L$. Note that we have the inclusion $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ instead of an equality as in Theorem 3.11 since Theorem 3.10 does not tell us which separations from L are in $T_{\vec{L}}$; however, we still have $(T' \setminus T) \cap S_{\vec{L}} \subseteq T_{\vec{L}}$.

Let us discuss the structural conditions on $S_{\vec{L}}$ and $\mathcal{F}_{\vec{L}}$ in Theorem 3.11. There we require $S_{\vec{L}} \cup L$ to be separable and $\mathcal{F}_{\vec{L}}$ to be closed under shifting in $S_{\vec{L}} \cup L$. In the proof of Theorem 3.11, we use these assumptions to show that $S_{\vec{L}} \cup L$ is $\mathcal{F}'_{\vec{L}}$-separable as required for the application of Theorem 3.10. In our setup, separation systems are often separable, and many suitable choices of $\mathcal{F}_{\vec{L}}$ are closed under shifting and standard for $S_{\vec{L}} \cup L$. For example, the above defined sets $\mathcal{F}_n$ of stars are standard for $S_{\vec{L}} \cup L$ and also closed under shifting by [5, Lemma 26].

Note that by the definition of separability, we can again make a separation system separable using the previously described pre-processing routine in that we repeatedly add corners if needed. However, we will not need to do this if $S_{\vec{L}}$ is a structurally submodular separation system:

Lemma 3.12 ([5, Lemma 13]). *Let $\vec{U}$ be a universe of separations and $\vec{S} \subseteq \vec{U}$ a structurally submodular separation system. Then $\vec{S}$ is separable.*

If $S_{\vec{L}}$ is structurally submodular and inside a splitting star $\vec{L}$ of T, then $S_{\vec{L}} \cup L$ is structurally submodular, too. So if $S_{\vec{L}}$ is inside $\vec{L}$, then $S_{\vec{L}} \cup L$ is separable if $S_{\vec{L}}$ is structurally submodular.

Until now, we investigated how to refine a regular tree set T at a single splitting star $\vec{L}$ of T using the separations in $S_{\vec{L}}$. The next theorem says that we can do these local refinements simultaneously for an arbitrary set $\mathcal{L}$ of splitting stars of T: we can extend T to a regular tree set $T' \supseteq T$ such that for each $\vec{L} \in \mathcal{L}$, there exists a local regular tree set $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ which describes the structure of a set $\mathcal{P}'_{\vec{L}}$ of refinements of $\vec{L}$ into $S_{\vec{L}}$. Let us keep in mind that an exemplary choice for each $\mathcal{P}'_{\vec{L}}$ would be the set of $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$ for a locally chosen set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars (see also Corollary 3.14).

The *complete local refinement theorem for regular tree sets* combines the two previous single location refinement theorems, Theorem 3.9 and Theorem 3.11, as follows:

Theorem 3.13. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, let $T \subseteq U$ be a regular tree set, and let $\mathcal{L}$ be a set of splitting stars of T. For each splitting star $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}} \subseteq U$ be a set of separations inside $\vec{L}$, and let $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$ be a set of refinements of $\vec{L}$ into $S_{\vec{L}}$. For every $\vec{L} \in \mathcal{L}$, let us assume that*

(i) *if $\mathcal{P}'_{\vec{L}}$ is non-empty, then $S_{\vec{L}}$ is a structurally submodular separation system which contains a non-trivial separation;*

(ii) *if $\mathcal{P}'_{\vec{L}}$ is empty, then $S_{\vec{L}} \cup L$ is a separable separation system, and we have a set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars containing its essential core, which is standard for $S_{\vec{L}} \cup L$, closed under shifting in $S_{\vec{L}} \cup L$, and such that there is no $\mathcal{F}_{\vec{L}}$-tangle refining $\vec{L}$ into $S_{\vec{L}}$.*

Then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup \bigcup_{\vec{L} \in \mathcal{L}} S_{\vec{L}}$ such that for every $\vec{L} \in \mathcal{L}$,

(a) *if $\mathcal{P}'_{\vec{L}}$ is non-empty, then every pair of refinements in $\mathcal{P}'_{\vec{L}}$ is distinguished by some separation in $T_{\vec{L}} = (T' \setminus T) \cap S_{\vec{L}}$;*

(b) *if $\mathcal{P}'_{\vec{L}}$ is empty, then there exists a regular tree set $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ which is associated with an essential $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}} = \mathcal{F}_{\vec{L}} \cup \{\{\overleftarrow{t}\} | \vec{t} \in \vec{L}\}$.*

For the formal proof, see Section 4. Essentially, the assumptions (i) and (ii) enable us to apply our previous two local refinement theorems iteratively to the splitting stars in $\mathcal{L}$ one by one. Since the refined regular tree sets obtained from one of the previous two local refinement theorems were constructed by adding 'local' regular tree sets inside the investigated splitting star, the already constructed local regular tree sets do not interfere with the new local regular tree set.

Note that condition (i) in Theorem 3.13 forbids $S_{\vec{L}} = \emptyset$ if $\vec{L}$ is a profile of L; but if we set $S_{\vec{L}} = \emptyset$, then this means that we do not analyse $\vec{L}$ any further. So we can also just remove $\vec{L}$ from $\mathcal{L}$ before applying Theorem 3.13. Similarly, if condition (ii) is not met in that we have $\mathcal{P}'_{\vec{L}} = \emptyset$, but do not find a suitable set $\mathcal{F}_{\vec{L}}$ of stars in $\vec{U}$, then we cannot use tangle-tree duality to witness the absence of refinements in $\mathcal{P}'_{\vec{L}}$. In this case, we can again remove $\vec{L}$ from $\mathcal{L}$, and then apply Theorem 3.13 to the remaining splitting stars.

Theorem 3.13 is formulated in a very general form. In order to illustrate its practical use, let us give the subsequent corollary which follows directly from Theorem 3.13 combined with Lemma 3.12:

Corollary 3.14. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, let $T \subseteq U$ be a regular tree set, and let $\mathcal{L}$ be a set of splitting stars of T. For each splitting star $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}} \subseteq U$ be a structurally submodular set of separations inside $\vec{L}$ which is either empty or contains a non-trivial separation, and let $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ be a set of stars containing its essential core which is standard for $S_{\vec{L}} \cup L$ and closed under shifting in $S_{\vec{L}} \cup L$.*
Let $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$ be the set of $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$. Then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup \bigcup_{\vec{L} \in \mathcal{L}} S_{\vec{L}}$ such that for every $\vec{L} \in \mathcal{L}$,

(a) *if $\mathcal{P}'_{\vec{L}}$ is non-empty, then every pair of refinements in $\mathcal{P}'_{\vec{L}}$ is distinguished by some separation in $T_{\vec{L}} = (T' \setminus T) \cap S_{\vec{L}}$;*

(b) *if $\mathcal{P}'_{\vec{L}}$ is empty, then there exists a regular tree set $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ which is associated with an essential $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}} = \mathcal{F}_{\vec{L}} \cup \{\{\overleftarrow{t}\} | \vec{t} \in \vec{L}\}$.* $\square$

Let us return to our inductive construction of building a sequence $\emptyset = T_0 \subseteq \cdots \subseteq T_n$ of regular tree sets by local refinements that describes the cluster structure of V as given by $\vec{S}$ in evolving detail. Suppose $T_0, \ldots, T_k$ (and the corresponding sets $\mathcal{L}_0, \ldots, \mathcal{L}_k$ of 'interesting' splitting stars) were already constructed.

In the $(k+1)$-th iteration step, we determine, for each location $\vec{L} \in \mathcal{L}_k$ separately, a set $S_{\vec{L}}$ of separations that we believe to be of highest relevance for the further investigation of $\vec{L}$, e.g. mechanically using the local order function approach. By the replacement step presented at the beginning of this section, we may assume that each $S_{\vec{L}}$ is inside the respective location $\vec{L}$. Based on what the set $\mathcal{P}'_{\vec{L}}$ of 'interesting' refinements of $\vec{L}$ into $S_{\vec{L}}$ looks like, we can use the pre-processing step described above to ensure that $S_{\vec{L}}$ has the structural properties that guarantee the existence of a suitable local regular tree set $T_{\vec{L}}$ inside $\vec{L}$ (see conditions (i) and (ii) of Theorem 3.13 above). By neither of these modification steps, the amount of information that $S_{\vec{L}}$ provides about the cluster captured by $\vec{L}$ changes.

However, recall that some refinements into the original set of separations do not extend to refinements into this pre-processed $S_{\vec{L}}$. Hence, the number of refinements of $\vec{L}$ into the pre-processed $S_{\vec{L}}$ may be lower than the one before the pre-processing step. If this results in that there are no interesting refinements any more, then note that if $S_{\vec{L}}$ satisfies condition (i), it also satisfies condition (ii): remember from Lemma 3.12 and the comment thereafter that if $S_{\vec{L}}$ is structurally submodular, then $S_{\vec{L}} \cup L$ is also separable.

Now for those locations $\vec{L} \in \mathcal{L}_k$ without interesting refinements into the pre-processed $S_{\vec{L}}$, we choose a local set $\mathcal{F}_{\vec{L}}$ of stars satisfying condition (ii). Then we apply the local refinement theorem for regular tree sets, Theorem 3.13 to T_k, $\mathcal{L}_k$, and the collection of the corresponding $S_{\vec{L}}$ for $\vec{L} \in \mathcal{L}_k$, and obtain the next and more detailed regular tree set T_{k+1}.

How do we choose the set $\mathcal{L}_{k+1}$ of interesting splitting stars of T_{k+1}? Every refinement P from some $\mathcal{P}'_{\vec{L}}$ is a partial consistent orientation of the regular tree set T_{k+1}. Therefore, we can extend it to a complete consistent orientation O of T_{k+1} [2, Lemma 4.1 (i)]. Since $\vec{L}$ is a splitting star of T_k, all the local regular tree sets $T_{\vec{L'}}$ for $\vec{L'} \neq \vec{L}$ are outside $\vec{L}$ (see Lemma 4.1 below). Thus, the orientation O has the same maximal elements as P has because O is consistent and P orients $L \cup T_{\vec{L}}$. So for $\mathcal{L}_{k+1}$, we choose those splitting stars $\vec{L'}$ of T_{k+1} that are induced by such an extension of a refinement from some $\mathcal{P}'_{\vec{L}}$. Furthermore, we may again restrict ourselves to such splitting stars with $V_{\vec{L'}} \neq \emptyset$.

Then we begin the next iteration step dealing with T_{k+1} and $\mathcal{L}_{k+1}$. This iterative procedure will lead to a sequence of regular tree sets that displays the structure of our separation system in more and more detail.

The algorithmic approach as presented in this section is rather a framework and a proof of concept; questions of implementation are not within the scope of this thesis. However, let us end this section with a brief discussion about the key factors determining the computational complexity of a single local refinement step as described in Theorem 3.13. In each such local refinement step, we need to analyse $|\mathcal{L}_k| \leq |T_k| + 1$ locations. First, we choose a suitable set $S_{\vec{L}}$ of separations for each location $\vec{L} \in \mathcal{L}_k$ which we potentially need to suitably pre-process by repeatedly adding corners. Secondly, we compute for each location $\vec{L} \in \mathcal{L}_k$ separately either a tree of tangles or a suitable duality tree, depending on the size of the set $\mathcal{P}'_{\vec{L}}$ of 'interesting' refinements of $\vec{L}$ into $S_{\vec{L}}$. Finally, we combine all these local regular tree sets to our refined regular tree set T_{k+1}. So the complexity of each refinement step depends heavily on how the first step and the second step are implemented. For a discussion of current algorithms, we refer the reader to [4, Section 4.3].

4 Proofs of the Local Refinement Theorems

In this section we show the validity of the algorithm described above. Therefore, it merely consists of proofs of the local refinement theorems that we claimed in Section 3. As described in the introduction, the approach of local refinements was first taken in the proof of Theorem 1.2 as [6, Theorem 3.6]. So some arguments and approaches in the following proofs are inspired by arguments thereof.

We begin by proving the first local refinement theorem, Theorem 3.9 above. It says that we can use the structural version of the tree-of-tangles theorem, Theorem 3.8 above, to extend a given regular tree set such that it also distinguishes the refinements of a fixed splitting star:

Theorem 3.9. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, and let $T \subseteq U$ be a regular tree set. Let $\vec{L}$ be a splitting star of T, and let $S_{\vec{L}}$ be a set of separations in U which contains a non-trivial separation. Moreover, assume that*

(i) *$\vec{S}_{\vec{L}}$ is a structurally submodular separation system;*

(ii) *$S_{\vec{L}}$ is inside $\vec{L}$.*

Let $\mathcal{P}_{\vec{L}}$ be the set of refinements of $\vec{L}$ into $S_{\vec{L}}$. Then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup S_{\vec{L}}$ such that every two refinements in $\mathcal{P}_{\vec{L}}$ are distinguished by some separation in $T_{\vec{L}} = (T' \setminus T) \cap S_{\vec{L}}$.

Proof. We consider the set $\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}$ of those profiles of $S_{\vec{L}}$ which are induced by profiles in $\mathcal{P}_{\vec{L}}$. Since $S_{\vec{L}}$ is structurally submodular, we can apply Theorem 3.8 to $(\vec{S}_{\vec{L}}, \leq, *)$ and $\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}$, and obtain a local tree set $T_{\vec{L}} \subseteq S_{\vec{L}}$ distinguishing $\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}$. As all the refinements of $\vec{L}$ into $S_{\vec{L}}$ orient L in the same way, distinct refinements in $\mathcal{P}_{\vec{L}}$ induce distinct profiles of $S_{\vec{L}}$. So $T_{\vec{L}}$ also distinguishes $\mathcal{P}_{\vec{L}}$.

All the refinements in $\mathcal{P}_{\vec{L}}$ are regular profiles as $S_{\vec{L}}$ contains a non-trivial separation. In particular, such refinements cannot be distinguished by a small separation. So after removing separations from $T_{\vec{L}}$ that are not $\mathcal{P}_{\vec{L}}$-relevant, we may assume that $T_{\vec{L}}$ is a regular tree set.

By (ii), $T_{\vec{L}} \subseteq S_{\vec{L}}$ is inside $\vec{L}$. Since $\vec{L}$ is a splitting star of T, every separation in T is outside $\vec{L}$. So $T_{\vec{L}}$ is also nested with T. Hence, $T' = T \cup T_{\vec{L}}$ is the desired refined regular tree set. □

Theorem 3.9 holds with the same proof even in general universes of separations as long as all the refinements in $\mathcal{P}_{\vec{L}}$ are regular. For a discussion of more general tree-of-tangles-type theorems in the context of local refinements in arbitrary universes of separations, see Section 5. In particular, Theorem 3.9 is a special case of Theorem 5.5.

Now we turn to the second local refinement theorem, Theorem 3.11 above. Here we use the concept of tangle-tree duality, described in Theorem 3.10 above, to witness the non-existence of refinements that are $\mathcal{F}_{\vec{L}}$-tangles for some set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars. As we already mentioned in Section 3, the second local refinement theorem holds even if the $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ are not profiles themselves.

Theorem 3.11. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, and let $T \subseteq U$ be a regular tree set. Let $\vec{L}$ be a splitting star of T, and let $S_{\vec{L}}$ be a set of separations in U. Suppose that $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ is a set of stars, standard for $S_{\vec{L}} \cup L$, which contains its essential core. Moreover, assume that*

(i) *$S_{\vec{L}} \cup L$ is separable;*

(ii) *$S_{\vec{L}}$ is inside $\vec{L}$;*

(iii) *$\mathcal{F}_{\vec{L}}$ is closed under shifting in $S_{\vec{L}} \cup L$.*

If there are no $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$, then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup S_{\vec{L}}$ such that T' contains a regular tree set $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ which is associated with an essential $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}} = \mathcal{F}_{\vec{L}} \cup \{\{\overleftarrow{t}\} | \vec{t} \in \vec{L}\}$.

Proof. By construction the $\mathcal{F}'_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ are precisely those $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ that refine $\vec{L}$ into $S_{\vec{L}}$. Therefore, we want to apply Theorem 3.10 to $S_{\vec{L}} \cup L$ and $\mathcal{F}'_{\vec{L}}$. Suppose that we could do so, and obtained an $(S_{\vec{L}} \cup L)$-tree (G, α) over $\mathcal{F}'_{\vec{L}}$.

Since T is a regular tree set in $\vec{U}$, there are no separations in $\vec{L}$ which are co-trivial in $\vec{U}$. Hence, $\mathcal{F}'_{\vec{L}}$ still contains its essential core. So by [3, Corollary 6.7], we can assume without loss of generality that (G, α) is even an essential $(S_{\vec{L}} \cup L)$-tree over the essential core of $\mathcal{F}'_{\vec{L}}$. Since the trivial separation $(\emptyset, V)$ is the only small separation in $\vec{U}$, this implies that the nested set $T_{\vec{L}}$ associated with (G, α) is a regular tree set. By (ii), $T_{\vec{L}} \subseteq S_{\vec{L}} \cup L$ is inside $\vec{L}$. Since $\vec{L}$ is a splitting star of T, every separation in T is outside $\vec{L}$. Therefore, $T_{\vec{L}}$ is also nested with T. So the regular tree set $T' = T \cup T_{\vec{L}}$ is as desired.

It remains to prove that $S_{\vec{L}} \cup L$ and $\mathcal{F}'_{\vec{L}}$ satisfy the assumptions of Theorem 3.10. Since $\mathcal{F}_{\vec{L}}$ is standard, so is $\mathcal{F}'_{\vec{L}}$. Moreover, since $S_{\vec{L}} \cup L$ is separable by (i), it is sufficient to show that $\mathcal{F}'_{\vec{L}}$ is closed under shifting. By (iii), $\mathcal{F}_{\vec{L}}$ is closed under shifting. Now $\mathcal{F}'_{\vec{L}} \setminus \mathcal{F}_{\vec{L}}$ consists only of singleton stars, and every relevant shift maps such a singleton star $\{\overleftarrow{t}\}$ to $\{\overleftarrow{s}\}$ for some separation $\overleftarrow{s} \geq \overleftarrow{t}$. But since $\vec{t}$ points towards $S_{\vec{L}} \cup L$, this yields that either $\overleftarrow{s} = \overleftarrow{t}$, or $\overleftarrow{s}$ is co-trivial in $\vec{U}$. Since $\mathcal{F}'_{\vec{L}}$ is standard, we have $\{\overleftarrow{s}\} \in \mathcal{F}'_{\vec{L}}$ in both cases. Hence, $\mathcal{F}'_{\vec{L}}$ is closed under shifting which completes the proof. □

Theorem 3.11 is also a direct corollary of Proposition 7.2 which shows that we can use the concept of tangle-tree duality in the context of local refinements even in more general universes of separations, and even if $S_{\vec{L}}$ is not necessarily inside $\vec{L}$. Therefore, the above proof is essentially an extract of the proofs in Section 7 which, in turn, follow closely the lines of the proof of [11, Lemma 8].

It remains to show that we can deduce the complete local refinement theorem for regular tree sets, Theorem 3.13 above, from the previous two local refinement theorems, Theorem 3.9 and Theorem 3.11. The complete local refinement theorem for regular tree sets says that given a regular tree set T, a set $\mathcal{L}$ of splitting stars of T, and for each $\vec{L} \in \mathcal{L}$, a suitable set $S_{\vec{L}}$ of separations and a set $\mathcal{P}'_{\vec{L}}$ of refinements of $\vec{L}$ into $S_{\vec{L}}$, we can extend T to a regular tree set $T' \supseteq T$ such that for each $\vec{L}$ in $\mathcal{L}$, there exists a local regular tree set $T_{\vec{L}} \subseteq T' \cap (S_{\vec{L}} \cup L)$ which describes the structure of the refinements in $\mathcal{P}'_{\vec{L}}$.

To prove Theorem 3.13, we are going to apply the suitable local refinement theorem to the splitting stars in $\mathcal{L}$ one by one. Each such local refinement step should leave all the other splitting stars of T 'untouched' in the following sense: if T is a regular tree set with distinct splitting stars $\vec{L}$ and $\vec{L'}$, then $\vec{L'}$ should again be a splitting star of the refined regular tree set $T' \supseteq T$ resulting from either Theorem 3.9 or Theorem 3.11 applied to T, $\vec{L}$, and $S_{\vec{L}}$. Since the refined regular tree set T', as constructed in the proofs of the previous local refinement theorems, differs from T only by a local regular tree set $T_{\vec{L}}$ inside $\vec{L}$, it is sufficient to show that a separation which is inside $\vec{L}$ must be outside every other location $\vec{L'}$ of T. This is precisely the statement of the following lemma:

Lemma 4.1. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, let $T \subseteq U$ be a tree set, and let $\vec{L}$ and $\vec{L'}$ be distinct splitting stars of T. Then if a separation $r \notin L$ is inside $\vec{L}$, it must be outside $\vec{L'}$.*

Proof. Suppose that there exist $\vec{s} \in \vec{L}$ and $\vec{s'} \in \vec{L'}$ with $\overleftarrow{s} \leq \vec{s'}$. Then we were done: since r is inside $\vec{L}$ and $r \notin L$, there is an orientation $\vec{r}$ of r with $\vec{s} < \vec{r}$, and we have $\overleftarrow{r} < \overleftarrow{s} \leq \vec{s'}$. So r is outside $\vec{L'}$ as claimed.

It remains to show that there are such $\vec{s} \in \vec{L}$ and $\vec{s'} \in \vec{L'}$ with $\overleftarrow{s} \leq \vec{s'}$. By definition of a splitting star, there is an orientation O of T with maximal elements $\vec{L}$. Similarly, there is an orientation O' of T with maximal elements $\vec{L'}$. Since $\vec{L} \neq \vec{L'}$, we have $O \neq O'$, i.e. there exists a separation $t \in T$ with orientations $\vec{t}$ and $\overleftarrow{t}$ such that $\vec{t} \in O$ and $\overleftarrow{t} \in O'$. Since $\vec{L}$ is the set of maximal elements of O, there must be some $\vec{s} \in \vec{L}$ with $\vec{t} \leq \vec{s}$. Analogously, we find $\vec{s'} \in \vec{L'}$ with $\overleftarrow{t} \leq \vec{s'}$. The combination of these inequalities yields $\overleftarrow{s} \leq \overleftarrow{t} \leq \vec{s'}$, as desired. □

Lemma 4.1 holds with the same proof in general universes of separations, and even for splitting stars of a nested set T. It also follows from Proposition 5.11 and Proposition 5.4 since for a splitting star $\vec{L}$ of a nested set T, every separation in T is outside $\vec{L}$. From this perspective, Lemma 4.1 uses an argument from the proof of Theorem 1.2 in [6] (see in particular Proposition 5.4 below).

We are now ready to prove the complete local refinement theorem for regular tree sets:

Theorem 3.13. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be the universe of bipartitions of a finite set V, let $T \subseteq U$ be a regular tree set, and let $\mathcal{L}$ be a set of splitting stars of T. For each splitting star $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}} \subseteq U$ be a set of separations inside $\vec{L}$, and let $\mathcal{P}'_{\vec{L}} \subseteq \mathcal{P}_{\vec{L}}$ be a set of refinements of $\vec{L}$ into $S_{\vec{L}}$. For every $\vec{L} \in \mathcal{L}$, let us assume that*

(i) *if $\mathcal{P}'_{\vec{L}}$ is non-empty, then $S_{\vec{L}}$ is a structurally submodular separation system which contains a non-trivial separation;*

(ii) *if $\mathcal{P}'_{\vec{L}}$ is empty, then $S_{\vec{L}} \cup L$ is a separable separation system, and we have a set $\mathcal{F}_{\vec{L}} \subseteq 2^{\vec{U}}$ of stars containing its essential core, which is standard for $S_{\vec{L}} \cup L$, closed under shifting in $S_{\vec{L}} \cup L$, and such that there is no $\mathcal{F}_{\vec{L}}$-tangle refining $\vec{L}$ into $S_{\vec{L}}$.*

Then there exists a regular tree set $T' \supseteq T$ in U with $T' \subseteq T \cup \bigcup_{\vec{L} \in \mathcal{L}} S_{\vec{L}}$ such that for every $\vec{L} \in \mathcal{L}$,

(a) *if $\mathcal{P}'_{\vec{L}}$ is non-empty, then every pair of refinements in $\mathcal{P}'_{\vec{L}}$ is distinguished by some separation in $T_{\vec{L}} = (T' \setminus T) \cap S_{\vec{L}}$;*

(b) *if $\mathcal{P}'_{\vec{L}}$ is empty, then there exists a regular tree set $T_{\vec{L}} \subseteq ((T' \setminus T) \cap S_{\vec{L}}) \cup L$ which is associated with an essential $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}} = \mathcal{F}_{\vec{L}} \cup \{\{\overleftarrow{t}\} | \vec{t} \in \vec{L}\}$.*

Proof. We build T' iteratively going through the splitting stars in $\mathcal{L} = \{\vec{L}_1, \dots, \vec{L}_m\}$ one by one. More precisely, we construct a sequence $T = T_0 \subseteq \cdots \subseteq T_m$ of regular tree sets such that for each $0 \leq i \leq m$, the conclusions (a) and (b) hold with $T' = T_i$ for those splitting stars $\vec{L}_j$ with $j \leq i$, while the $\vec{L}_j$ with $j > i$ are still splitting stars of T_i. Then $T' = T_m$ is clearly as desired.

Suppose $T_0, \dots, T_i$ were already constructed for some $0 \leq i < m$. By the induction hypothesis, $\vec{L} := \vec{L}_{i+1}$ is a splitting star of T_i. Now the assumptions (i) and (ii) guarantee that we can apply the appropriate local refinement theorem to T_i and $\vec{L}$.

First, suppose that $\mathcal{P}'_{\vec{L}}$ is non-empty. If $\mathcal{P}'_{\vec{L}}$ contains only one refinement, then we set $T_{i+1} = T_i$ which trivially satisfies (a) for $\vec{L}$. If $\mathcal{P}'_{\vec{L}}$ has size at least 2, then by assumption (i), we can apply Theorem 3.9 to T_i and $\vec{L}$. This gives us a refined regular tree set $T_{i+1} \supseteq T_i$ that ensures (a) for $\vec{L}$.

Secondly, if $\mathcal{P}'_{\vec{L}}$ is empty, then there are no $\mathcal{F}_{\vec{L}}$-tangles refining $\vec{L}$ into $S_{\vec{L}}$ for a locally chosen set $\mathcal{F}_{\vec{L}}$ of stars in $\vec{U}$ by assumption (ii). So we can apply Theorem 3.11 to T_i and $\vec{L}$ by assumption (ii), and we obtain a refined regular tree set $T_{i+1} \supseteq T_i$ that ensures (b) for $\vec{L}$.

In all these constructions, the refined regular tree set T_{i+1} differs from T_i only by a local regular tree set $T_{\vec{L}}$ inside $\vec{L} = \vec{L}_i$. By Lemma 4.1, all the separations in $T_{\vec{L}}$ are outside each splitting star $\vec{L}_j$ with $j > i + 1$. So by the induction hypothesis, all the $\vec{L}_j$ with $j > i + 1$ are again splitting stars of T_{i+1}. Moreover, for every two distinct splitting stars $\vec{L}_j$ and $\vec{L}_{j'}$, we have $S_{\vec{L}_j} \cap S_{\vec{L}_{j'}} \subseteq (L \cap L') \cup \{\{\emptyset, V\}\}$ by Lemma 4.1 since each $S_{\vec{L}}$ is inside the respective $\vec{L}$. Hence, we also have that the correct inclusions on $T_{\vec{L}_j}$ in (a) and (b) for $j \leq i$. This completes the induction step, and hence the proof. $\square$

5 Distinguishing Profiles by Local Refinements

In Section 3 we investigated how we can use a structural version of the tree-of-tangles theorem in a universe U of set bipartitions to distinguish multiple refinements of a splitting star, and thereby refine a given regular tree set (see Theorem 3.9). Inspired from this result, we now turn to the more theoretical question how we can use local refinement steps to distinguish profiles of separation systems in arbitrary universes. In particular, such profiles do not have to be regular any more; hence, a structure tree can contain trivial and small separations.

As described in the introduction, the approach of using local refinements to distinguish profiles was first taken in the proof of Theorem 1.2 in [6] in the context of universes of separations equipped with a submodular order function; arguments from this proof inspired many ideas in this section. However, we do not assume the existence of an order function here; all the conditions formulated in this section will be of a structural nature in that they are defined purely in terms of the considered separation systems. We shall revisit Theorem 1.2 in Section 6, and reobtain it using results from our structural setup. In order to approach the question how we can use local refinement steps in general universes of separations, we move from the algorithmic perspective to a more structural one as follows.

In the algorithmic setup of Section 3, we start with a regular tree set T and a set $\mathcal{L}$ of splitting stars of T. For each location $\vec{L} \in \mathcal{L}$, we consider the set $\mathcal{P}_{\vec{L}}$ of refinements of $\vec{L}$ into a locally chosen set $S_{\vec{L}}$ of separations inside $\vec{L}$, i.e. the profiles of $S_{\vec{L}} \cup L$ orienting L as $\vec{L}$. In particular, each refinement $P \in \mathcal{P}_{\vec{L}}$ has the same set of maximal elements in the partial orientation $P \cap \vec{T}$ of our tree set T, namely the splitting star $\vec{L}$. We said that the profile P *lives in the location* $\vec{L}$ *of* T. Note that since T is a regular tree set, the consistent partial orientation $P \cap \vec{T}$ can be uniquely extended to a consistent orientation of T which is the down-closure of $\vec{L}$ [3].

Now, instead of constructing the profiles given $S_{\vec{L}}$, we will assume in the following that we begin with a set $\mathcal{P}$ of profiles and a nested set T in some arbitrary universe U of separation. Here, each profile $P \in \mathcal{P}$ is a profile of some individual separation system $S(P) \subseteq U$. We now aim to refine the nested set T into a nested set $T' \supseteq T$ by 'local refinements' such that T' distinguishes more profiles in $\mathcal{P}$ than T does.

Analogously to the algorithmic setup, we will identify the maximal separations of the partial orientation $P \cap \vec{T}$ as the location $\vec{L} = \vec{L}(P)$ of T in which the profile $P \in \mathcal{P}$ *lives* - even though $\vec{L}(P)$ does not need to be the set of maximal elements of a consistent orientation of the whole nested set T. Indeed, the consistent partial orientation $P \cap \vec{T}$ cannot necessarily be extended to a consistent orientation of T as $\vec{T}$ may contain co-trivial separations [2, Lemma 4.1 (i)].

Let $\mathcal{P}_{\vec{L}}$ be the set of those profiles in $\mathcal{P}$ that live in the same location $\vec{L}$ of T. We now aim to refine T locally as follows: For each location $\vec{L}$ of $\mathcal{P}$ with respect to T separately, we consider a local separation system $S_{\vec{L}}$ that is oriented by all the profiles in $\mathcal{P}_{\vec{L}}$. If $S_{\vec{L}}$ satisfies certain structural conditions (e.g. structural submodularity), then we find a local nested set $T_{\vec{L}} \subseteq S_{\vec{L}}$ of separations inside $\vec{L}$ such that $T_{\vec{L}}$ distinguishes the profiles in $\mathcal{P}_{\vec{L}}$ as far as possible by $S_{\vec{L}}$. Then we combine T with all these local nested sets $T_{\vec{L}}$ to our refined nested set T'.

Back to the algorithmic setup of Section 3, we can define the set $\mathcal{P}$ as the set of refinements of some splitting star $\vec{L}$ of T into the corresponding set $S_{\vec{L}}$ of separations. Then all the profiles in $\mathcal{P}$ live in the location $\vec{L}$ of T - which may justify the reuse of the notation $\mathcal{P}_{\vec{L}}$ in our general problem formulation. Clearly, the separation systems $S_{\vec{L}}$ that we used to define the refinements in $\mathcal{P}_{\vec{L}}$ are oriented by all the profiles in $\mathcal{P}_{\vec{L}}$. With this modelling, the algorithmic setup of Section 3 can be seen as a special case of the general problem that we are going to investigate in this section.

Section 5 is organized as follows. We first make the above sketched general setup precise and obtain a general local refinement theorem: Given that all the locally chosen sets $S_{\vec{L}} \subseteq U$ of separations are nested with T, we identify conditions on the $S_{\vec{L}}$ and on the interplay of T and $\mathcal{P}$ which assure that the $S_{\vec{L}}$ contain the local nested sets $T_{\vec{L}}$ which are nested with T and with each other, so that we combine all these nested sets into a refined nested set $T' \supseteq T$. In a second step, we investigate which properties of a set $S_{\vec{L}}$ of separations allow us to reduce to the case where $S_{\vec{L}}$ is nested with T. Moreover, we show how knowledge about the interplay of T and $\mathcal{P}$ can help us to build a tree of tangles for $\mathcal{P}$ by a sequence of local refinements.

Let us start by making precise what profiles we are looking at. Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be an arbitrary universe of separations. Since U is not equipped with an order function, we cannot talk about profiles *in* $\vec{U}$ as they were defined as profiles of the set S_ℓ of separations of order $< \ell$ for some $\ell \in \mathbb{N}$. Instead, we consider profiles of arbitrary separation systems $\vec{S} \subseteq \vec{U}$.

Definition 5.1. Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be a universe of separations. We call P a *profile within* $\vec{U}$ if P is a profile of some non-empty separation system $\vec{S}(P) \subseteq \vec{U}$.

A non-degenerate separation $s \in U$ is said to *distinguish* two profiles P and Q within $\vec{U}$ if there are distinct orientations $\vec{s}$ and $\overleftarrow{s}$ of s with $\vec{s} \in P$ and $\overleftarrow{s} \in Q$. In particular, every $s \in U$ that distinguishes P and Q must be in $S(P) \cap S(Q)$.

We say that a set $T \subseteq U$ *distinguishes* a set $\mathcal{P}$ of profiles within $\vec{U}$ if any pair of profiles in $\mathcal{P}$ is distinguished by some $t \in T$. Moreover, if T is nested and distinguishes $\mathcal{P}$, then we call T a *tree of tangles for* $\mathcal{P}$.

For the remainder of this section, let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be a (finite) universe of separations, $T \subseteq U$ a nested set of separations, and $\mathcal{P}$ a set of profiles within $\vec{U}$. We shall often not mention this explicitly in the following.

As sketched above, a profile P within $\vec{U}$ lives in a location of a nested set T in the following sense: P induces a partial consistent orientation of T. The maximal separations of this partial orientation fully determine the orientation of the other separations in $P \cap \vec{T}$ by consistency. This leads to the following definition:

Definition 5.2. Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a nested set, and $\mathcal{P}$ a set of profiles within $\vec{U}$. For each profile $P \in \mathcal{P}$, its *location (with respect to T)* is a set of oriented separations defined as

$$\vec{L}(P, T) := \max P \cap \vec{T} = \max\{\vec{t} \in P \mid t \in T\},$$

where $\max \vec{X}$ denotes the set of maximal elements of a set $\vec{X} \subseteq \vec{U}$ of oriented separations in the partial order $\leq$. We say that P *lives* in the location $\vec{L}(P, T)$ of T, and write $L(P, T)$ for the separations in $\vec{L}(P, T)$ without orientation.

The set consisting of all the locations of profiles in $\mathcal{P}$ with respect to T is denoted by $\mathcal{L}(\mathcal{P}, T) := \{\vec{L}(P) \mid P \in \mathcal{P}\}$. Moreover, let $\mathcal{P}_{\vec{L}} = \{P \in \mathcal{P} \mid \vec{L}(P) = \vec{L}\}$ be the set of profiles in $\mathcal{P}$ that live in a location $\vec{L} \in \mathcal{L}(\mathcal{P}, T)$.

We usually write $\vec{L}(P)$ and $L(P)$ for $\vec{L}(P,T)$ and $L(P,T)$, respectively, if T is clear from the context. Similarly, we write just $\mathcal{L}$ for $\mathcal{L}(\mathcal{P},T)$ if $\mathcal{P}$ and T are clear from the context.

Since a profile does not contain any degenerate separation, the location $\vec{L}(P)$ of some profile $P \in \mathcal{P}$ with respect to T cannot contain a degenerate separation even if there is one in T. So $\vec{L}(P)$ is a star of separations by definition. With Definition 3.3, we can again speak of a separation s being inside or outside a location $\vec{L}(P)$. Since $\vec{T}$ may contain separations that are small or trivial in $\vec{T}$, note again that we can have separations in $T \setminus L(P)$ that are inside *and* outside $\vec{L}(P)$, but such separations must have an orientation which is trivial with a witness in $L(P)$.

Let us briefly investigate the relation between locations and splitting stars if T is a tree set. It turns out that a location $\vec{L}$ of T is a splitting star precisely if there is no $t \in T \setminus L$ inside $\vec{L}$:

Proposition 5.3. *Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a tree set, and P a profile within $\vec{U}$. If there is no $t \in T \setminus L(P)$ inside $\vec{L}(P)$, then $\vec{L}(P)$ is a splitting star of T. Conversely, there is no $t \in T \setminus L$ inside a splitting star $\vec{L}$ of T.*

Proof. Suppose that there is no $t \in T \setminus L(P)$ inside $\vec{L}(P)$. Since $P \cap \vec{T}$ is a partial consistent orientation of a tree set T, we can extend it to a consistent orientation O of T by [2, Lemma 4.1 (i)]. Now if $\vec{L}(P)$ is not a splitting star, then $\vec{L}(P)$ particularly cannot be the set of maximal elements of O. Hence, there exists a maximal element $\vec{t} \in O \setminus \vec{L}(P)$. Since t is not inside $\vec{L}(P)$ and t is non-trivial in T, t must be outside $\vec{L}(P)$. So there exists $\vec{s} \in \vec{L}(P)$ with $\overleftarrow{t} < \vec{s}$ or $\vec{t} < \vec{s}$. But the first case cannot occur as O is consistent whereas the second case contradicts the maximality of $\vec{t}$ in O.

For the converse, let $\vec{L}$ be a splitting star of T. By the definition of a splitting star, there exists a consistent orientation O of T with $\vec{L}$ as its set of maximal elements. But in this orientation, we have for every $t \in T \setminus L$ an orientation $\vec{t} \in O$ that is not maximal in O, i.e. there exists a separation $\vec{s} \in \vec{L}$ with $\vec{t} < \vec{s}$. Thus, t is outside $\vec{L}$ and hence not inside $\vec{L}$ since every separation t in a tree set T is non-trivial in T. □

Both parts of Proposition 5.3 are false if T is not a tree set. For the first part, we follow [2, Lemma 4.1 (i)]: If $T = \{s,t\}$ is a nested set with $\vec{t}$ trivial in $\vec{T}$ witnessed by s and $\overleftarrow{t}$ maximal in $\vec{T}$, then $\vec{L} = \{\overleftarrow{t}\}$ is the location of the profile $P = \{\overleftarrow{t}\}$ with respect to T. Furthermore, $\vec{L}$ does not point towards any element of $T \setminus L$. But $\vec{L}$ is not a splitting star of T, since the partial orientation P of T cannot be extended to a consistent orientation of T: neither orientation of s is consistent with orienting t as $\overleftarrow{t}$.

For the second part, consider the profile $P' = \{\vec{t}, \vec{s}\}$ of the nested set T above and its location $\vec{L'} = \{\vec{s}\}$ with respect to T. Clearly, $\vec{L'}$ is a splitting star of T since P' orients T consistently. But t is inside $\vec{L'}$ because it has a trivial orientation $\vec{t}$ witnessed by $s \in L'$.

Suppose we have a nested set T in a universe U and a set $\mathcal{P}$ of profiles within $\vec{U}$, and let $\mathcal{L} = \mathcal{L}(\mathcal{P},T)$ be the corresponding set of locations. For each location $\vec{L} \in \mathcal{L}$ separately, we want that the locally chosen set $S_{\vec{L}}$ of separations contains a local tree set $T_{\vec{L}}$ which distinguishes $\mathcal{P}_{\vec{L}}$ at least as well as $S_{\vec{L}}$ does. All these $T_{\vec{L}}$ together with T should form a new nested set T' which refines the location structure given by T. In order to achieve this, the $T_{\vec{L}}$ should be nested with T as well as with each other.

Similar to Section 3, this seems possible under the assumption that $S_{\vec{L}}$ is nested with T. By consistency, any two profiles from $\mathcal{P}_{\vec{L}}$ cannot be distinguished by a separation outside $\vec{L}$. So any local tree set $T_{\vec{L}} \subseteq S_{\vec{L}}$ which consists of $\mathcal{P}_{\vec{L}}$-relevant separations is not only nested with T, but even inside $\vec{L}$. However, we are not guaranteed at all that two local tree sets inside distinct locations are nested with each other.

For example, suppose we have two distinct locations $\vec{L}$ and $\vec{L'}$ such that for every $\vec{t} \in \vec{L}$, there exists $\vec{t'} \in \vec{L}$ with $\vec{t} \leq \vec{t'}$; in particular, $\vec{L}$ points towards L'. In this case, the separate construction of local tree sets $T_{\vec{L}}$ and $T_{\vec{L'}}$ inside $\vec{L}$ and $\vec{L'}$, respectively, can easily result in crossing separations.

Therefore, we require $\mathcal{L}$ to satisfy the following additional condition whose definition is inspired by an argument in the proof of Theorem 1.2 in [6]:

$$\textit{For every two distinct } \vec{L}, \vec{L'} \in \mathcal{L}, \textit{ there are } \vec{s} \in \vec{L} \textit{ and } \vec{s'} \in \vec{L'} \textit{ with } \overleftarrow{s} \leq \vec{s'}. \tag{L}$$

Roughly speaking, (L) makes sure that the difference of two distinct locations is witnessed by two respective elements that point away from each other. Now property (L) guarantees that separations inside distinct locations are nested with each other:

Proposition 5.4. *Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a nested set, and $\mathcal{P}$ a set of profiles within $\vec{U}$. Suppose that the corresponding set $\mathcal{L}$ of locations satisfies (L). For each $\vec{L} \in \mathcal{L}$, let $T_{\vec{L}} \subseteq U$ be a set of separations that is inside $\vec{L}$. Then for every two distinct locations $\vec{L}$ and $\vec{L'}$ in $\mathcal{L}$, every separation $t \in T_{\vec{L}}$ is nested with every separation $t' \in T_{\vec{L'}}$.*

Proof. Let $t \in T_{\vec{L}}$ and $t' \in T_{\vec{L'}}$ for two distinct locations $\vec{L}$ and $\vec{L'}$ in $\mathcal{L}$. By (L), there exist $\vec{s} \in \vec{L}$ and $\vec{s'} \in \vec{L'}$ with $\overleftarrow{s} \leq \vec{s'}$. As t is inside $\vec{L}$, the separation $\vec{s}$ points towards t, i.e. there is an orientation $\vec{t}$ of t with $\vec{s} \leq \vec{t}$. Similarly, there is an orientation $\vec{t'}$ of t' with $\vec{s'} \leq \vec{t'}$. Putting this together, we have $\overleftarrow{t} \leq \overleftarrow{s} \leq \vec{s'} \leq \vec{t'}$. In particular, t and t' are nested. □

Using Proposition 5.4, we can now deduce the following general version of a local refinement theorem in arbitrary universes of separations:

Theorem 5.5. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be a universe of separations, $\mathcal{P}$ a set of profiles within $\vec{U}$, and $T \subseteq U$ a nested set of separations such that the set $\mathcal{L}$ of locations of $\mathcal{P}$ with respect to T satisfies (L). For each location $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}}$ be a set of separations in U such that*

(i) *$\vec{S}_{\vec{L}}$ is a structurally submodular separation system;*

(ii) *$S_{\vec{L}}$ is oriented by every profile $P \in \mathcal{P}_{\vec{L}}$;*

(iii) *$S_{\vec{L}}$ is nested with T.*

Then there exists a nested set $T' = T \cup \bigcup_{\vec{L} \in \mathcal{L}} T_{\vec{L}}$ in U where for each location $\vec{L} \in \mathcal{L}$, the nested set $T_{\vec{L}} \subseteq S_{\vec{L}} \setminus T$ is a set of $\mathcal{P}_{\vec{L}}$-relevant separations inside $\vec{L}$ such that every pair of profiles in $\mathcal{P}_{\vec{L}}$ which is distinguished by $S_{\vec{L}}$ is also distinguished by $T_{\vec{L}}$. Moreover, the set $\mathcal{L}' = \mathcal{L}(\mathcal{P}, T')$ of locations of $\mathcal{P}$ with respect to T' satisfies (L) again.

Proof. For each location $\vec{L} \in \mathcal{L}$, every profile $P \in \mathcal{P}_{\vec{L}}$ induces a profile of $S_{\vec{L}}$ by (ii). Let $\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}$ be the set of profiles of $S_{\vec{L}}$ induced by profiles in $\mathcal{P}_{\vec{L}}$. By (i), the $S_{\vec{L}}$ are structurally submodular. Thus, we can apply Theorem 3.8 separately to $(S_{\vec{L}}, \leq, *)$ and $\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}$ for each $\vec{L} \in \mathcal{L}$ to find 'local' tree sets $T_{\vec{L}} \subseteq S_{\vec{L}}$ distinguishing $\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}$. In particular, $T_{\vec{L}}$ distinguishes every pair of profiles in $\mathcal{P}_{\vec{L}}$ which is distinguished by $S_{\vec{L}}$. Note that the local tree sets $T_{\vec{L}}$ are tree sets in $S_{\vec{L}}$, but they do not need to be tree sets in other separation systems (e.g. $T \cup T_{\vec{L}}$) as $\vec{T}_{\vec{L}}$ may contain separations that are trivial therein, but not in $\vec{S}_{\vec{L}}$.

By (iii), each local tree set $T_{\vec{L}}$ is nested with T, and in particular with L. So by transition to a subset if necessary, we may assume that $T_{\vec{L}}$ is a set of $\mathcal{P}_{\vec{L}}$-relevant separations inside $\vec{L}$ as by consistency, no separation outside $\vec{L}$ can distinguish two profiles in $\mathcal{P}_{\vec{L}}$. In particular, we also have $T_{\vec{L}} \subseteq S_{\vec{L}} \setminus T$. Now we apply Proposition 5.4 to deduce that the $T_{\vec{L}}$ are also nested with each other. Then our desired nested set is given by

$$T' = T \cup \bigcup_{\vec{L} \in \mathcal{L}} T_{\vec{L}}.$$

It is left to show that $\mathcal{L}' = \mathcal{L}(\mathcal{P}, T')$ again satisfies (L). For the rest of the proof, we write $\vec{L'}(P) := \vec{L}(P, T') \in \mathcal{L}'$ for the location of a profile $P \in \mathcal{P}$ with respect to T'. So let $\vec{L'}(P), \vec{L'}(Q) \in \mathcal{L}'$ be distinct locations.

First, if $\vec{L}(P,T) = \vec{L}(Q,T) =: \vec{L}$, then both P and Q are in $\mathcal{P}_{\vec{L}}$. In particular, P and Q orient $T_{\vec{L}}$, and they orient $T_{\vec{L}}$ differently because they are in different locations of T'. So there exists $t \in T_{\vec{L}}$ with orientations $\vec{t}$ and $\overleftarrow{t}$ such that $\vec{t} \in P$ and $\overleftarrow{t} \in Q$. Then there are $\vec{p} \in \vec{L'}(P)$ and $\vec{q} \in \vec{L'}(Q)$ with $\vec{t} \leq \vec{p}$ and $\overleftarrow{t} \leq \vec{q}$. In particular, we have $\overleftarrow{q} \leq \vec{t} \leq \vec{p}$, and thus, $\vec{p}$ and $\vec{q}$ are as desired.

Secondly, suppose $\vec{L}(P) \neq \vec{L}(Q)$. Since $\mathcal{L}$ satisfies (L), there are separations $\vec{p} \in \vec{L}(P)$ and $\vec{q} \in \vec{L}(Q)$ with $\overleftarrow{p} \leq \vec{q}$. By the definition of a location, there exists $\overrightarrow{p'} \in \vec{L'}(P)$ with $\vec{p} \leq \overrightarrow{p'}$, and, similarly, there is $\overrightarrow{q'} \in \vec{L'}(Q)$ with $\vec{q} \leq \overrightarrow{q'}$. But then we are done by transitivity because $\overleftarrow{p'} \leq \overleftarrow{p} \leq \vec{q} \leq \overrightarrow{q'}$. $\square$

Let us briefly see why Theorem 5.5 is a direct generalization of Theorem 3.9: In the latter, we had a regular tree set T in a universe U of set bipartitions, a splitting star $\vec{L}$ of T, and a set $\mathcal{P}_{\vec{L}}$ of refinements of $\vec{L}$ into a set $S_{\vec{L}}$ of separations which is nested with T and structurally submodular. If we set $\mathcal{P} = \mathcal{P}_{\vec{L}}$, then each profile $P \in \mathcal{P}$ lives in the location $\vec{L}$ of T. The set $S_{\vec{L}}$ satisfies conditions (i)-(iii) by construction. Moreover, $\mathcal{L}(\mathcal{P}, T) = \{\vec{L}\}$ trivially has property (L). By Theorem 5.5, we get a nested set $T' \supseteq T$ with the required properties. Since all the profiles in $\mathcal{P}$ are regular, small separations cannot distinguish any two profiles in $\mathcal{P}$. Hence, $T_{\vec{L}}$ is even a regular tree set in $\vec{U}$, and so T' can be assumed to be a regular tree set again.

The assumption that $S_{\vec{L}}$ is nested with T seems to be very strong at first sight. But in the following, we find conditions on the sets $S_{\vec{L}}$ which allow us to pass from $S_{\vec{L}}$ to the set $S'_{\vec{L}} \subseteq S_{\vec{L}}$ of all the separations nested with T without loosing any power in distinguishing $\mathcal{P}_{\vec{L}}$. In particular, we obtain conditions that are satisfied in the induction step of the proof of Theorem 1.2; we will see this in detail when we revisit it in Section 6.

In order to restrict ourselves to the separations from $S_{\vec{L}}$ that are nested with T without loss of power in distinguishing $\mathcal{P}_{\vec{L}}$, we have to make sure that the following holds: if two profiles P and Q in $\mathcal{P}_{\vec{L}}$ are distinguished by a separation s in $S_{\vec{L}}$ which crosses T, then there is a separation s' in $S_{\vec{L}}$ which is nested with T and still distinguishes P and Q. We ensure this by two conditions: the first one deals with the case where s crosses an element of $T \setminus L$ which is inside $\vec{L}$ whereas the second one handles the case where s crosses an element of L. This second condition again inspired by an argument in the proof of Theorem 1.2 in [6]; however, condition (i) is not needed there since in each induction step, the current nested set T_k 'strongly partitions' $\mathcal{P}$ (see Section 6). The next proposition also shows that if the separation system $\vec{S}_{\vec{L}}$ is structurally submodular, then the set of all those separations in $\vec{S}_{\vec{L}}$ which are nested with T does again have this property.

Proposition 5.6. *Let $\vec{U}$ be a universe of separations, and let $\mathcal{P}$ be a set of profiles within $\vec{U}$. Let $T \subseteq U$ be a nested set, and let $\mathcal{L}$ denote the set of locations of $\mathcal{P}$ with respect to T. Let $\vec{L} \in \mathcal{L}$, and let $S_{\vec{L}} \subseteq \vec{U}$ be a set of separations that is oriented by every profile $P \in \mathcal{P}_{\vec{L}}$. Now suppose that*

(i) *if a separation $t \in T \setminus L$ inside $\vec{L}$ and a separation $s \in S_{\vec{L}}$ which distinguishes two profiles P and Q in $\mathcal{P}_{\vec{L}}$ cross, then there exists a corner of s and t in $S_{\vec{L}}$ which also distinguishes P and Q;*

(ii) *if $\vec{t} \in \vec{L}$ and a $\mathcal{P}_{\vec{L}}$-relevant separation $s \in S_{\vec{L}}$ cross, then at least one of the two corners $\vec{t} \vee \vec{s}$ and $\vec{t} \vee \overleftarrow{s}$ is in $\vec{S}_{\vec{L}}$.*

Let $S'_{\vec{L}} \subseteq S_{\vec{L}}$ consist of those separations in $S_{\vec{L}}$ which are nested with T. Then the following holds: if two profiles $P, Q \in \mathcal{P}_{\vec{L}}$ are distinguished by $S_{\vec{L}}$, then they are also distinguished by $S'_{\vec{L}}$.
Moreover, if $\vec{S}_{\vec{L}}$ is a structurally submodular separation system, then $\vec{S'}_{\vec{L}}$ is also structurally submodular.

The proof of this proposition heavily relies on the following basic fact about crossing separations and their corners (this fact is sometimes called the 'fish lemma'):

Lemma 5.7 ([6, Lemma 2.1])**.** *Let $\vec{U}$ be a universe of separations, and let $s, t \in U$ be two crossing separations. Every separation r that is nested with both s and t is also nested with all four corner separations of s and t.*

Proof of Proposition 5.6. Let $P, Q \in \mathcal{P}_{\vec{L}}$ be a pair of profiles that is distinguished by $S_{\vec{L}}$. Choose $s \in S_{\vec{L}}$ to be a separation distinguishing P and Q that crosses the minimal number of separations in T amongst those separations in $S_{\vec{L}}$ that distinguish P and Q. If s is nested with T, then we are done. So suppose for a contradiction that s crosses some $t \in T$. Since t is nested with L, it is inside $\vec{L}$ or outside $\vec{L}$, and we can proceed via case distinction.

First, assume that t is inside $\vec{L}$ and $t \in T \setminus L$. By (i), there exists a corner r of s and t in $S_{\vec{L}}$ that still distinguishes P and Q. By Lemma 5.7, r is nested with all the separations that t and s are nested with. But then r distinguishes P and Q and crosses fewer elements of T than s does which contradicts the minimality of s in $S_{\vec{L}}$.

Secondly, suppose that t is outside $\vec{L}$. We claim that we may assume $t \in L$: If $t \notin L$, then there is an orientation $\vec{t}$ of t and $\vec{t'} \in \vec{L}$ with $\vec{t} < \vec{t'}$ since t is outside $\vec{L}$. We want to show that s crosses t'; so suppose for a contradiction that s is nested with t'. Then $\vec{t'}$ points towards or away from s. In the first case, we have $\vec{t} < \vec{t'} < \vec{s}$ for some orientation $\vec{s}$ of s contradicting that s and t cross. In the second case where $\vec{t'}$ points away from s, we get $\vec{s} < \vec{t'}$ for some orientation $\vec{s}$ of s. But P and Q orient t' and s, and they both orient t' as $\vec{t'}$. So by consistency, they both orient s as $\vec{s}$ contradicting that s distinguishes P and Q. Hence, s and t' cross, and we may assume that $t \in L$, for if not, we replace t with t'.

Since s distinguishes P and Q, one of the corners $\vec{t} \vee \vec{s}$ and $\vec{t} \vee \overleftarrow{s}$ is in $\vec{S}_{\vec{L}}$ by (ii). Without loss of generality, we assume $\vec{r} = \vec{t} \vee \vec{s} \in \vec{S}_{\vec{L}}$; the case of $\vec{t} \vee \overleftarrow{s} \in \vec{S}_{\vec{L}}$ is symmetric. We claim that r distinguishes P and Q: Since s distinguishes P and Q, we have $\vec{s} \in P$ and $\overleftarrow{s} \in Q$ for suitable orientations $\vec{s}$ and $\overleftarrow{s}$ of s. By the profile property of P, we have $\vec{r} \in P$. If r does not distinguish P and Q, then Q orients r as $\vec{r}$, too. But $\vec{s} \leq \vec{r}$ and hence, $\vec{s} \in Q$ by consistency which contradicts $\overleftarrow{s} \in Q$. So $\overleftarrow{r} \in Q$, and r distinguishes P and Q. Since r is a corner of s and t, and r distinguishes P and Q, this again contradicts the minimality s by Lemma 5.7.

Finally, the moreover-part of this proposition follows directly from Lemma 5.7 in that corners of separations which are nested with T are also nested with T. $\square$

Often enough, the first case in Proposition 5.6 does not occur: in the setup of Section 3, for example, where T is a regular tree set in a universe U of set bipartitions and $\vec{L}$ is a splitting star of T, there are no separations of $T \setminus L$ inside $\vec{L}$ by Proposition 5.3.

The next proposition shows that the set $\mathcal{L}$ of locations of $\mathcal{P}$ with respect to T has property (L) if there are no separations of $T \setminus L$ inside any location $\vec{L}$.

Proposition 5.8. *Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a nested set, $\mathcal{P}$ a set of profiles within $\vec{U}$, and $\mathcal{L}$ the set of locations of $\mathcal{P}$ with respect to T. Suppose that for every location $\vec{L} \in \mathcal{L}$, there is no separation $t \in T \setminus L$ inside $\vec{L}$. Then $\mathcal{L}$ satisfies property (L).*

Proof. If $|\mathcal{L}| = 1$, then we are done by the definition of (L). So let $|\mathcal{L}| > 1$, and let $P, Q \in \mathcal{P}$ with distinct locations $\vec{L}(P), \vec{L}(Q) \in \mathcal{L}$. We have to show that there are $\vec{p} \in \vec{L}(P)$ and $\vec{q} \in \vec{L}(Q)$ with $\overleftarrow{p} \leq \vec{q}$. Note that this is clear if any separation in $L(P) \cap L(Q)$ is oriented differently in $\vec{L}(P)$ and $\vec{L}(Q)$. So we may assume that $L(P) \cap L(Q)$ is oriented in the same way by P and Q.

Let us first suppose that $L(P) \subseteq L(Q)$ (the case $L(Q) \subseteq L(P)$ is symmetrical). Then $\vec{L}(P) \subseteq \vec{L}(Q)$ by the above assumption. Since $\vec{L}(Q)$ is a star, $\vec{L}(P)$ points towards every separation in $L(Q) \setminus L(P)$. As $\vec{L}(P)$ does not point towards any separation in the set $T \setminus L(P)$ by assumption, this implies $L(P) = L(Q)$ and hence, $\vec{L}(P) = \vec{L}(Q)$ contradicting that these two locations are distinct.

So suppose that $L(P) \not\subseteq L(Q)$ and $L(Q) \not\subseteq L(P)$, and that $L(P) \cap L(Q)$ is oriented in the same way by both P and Q. Since there is no separation $t \in T \setminus L(P)$ inside $\vec{L}(P)$, the location $\vec{L}(P)$ cannot point towards any separation in $L(Q) \setminus L(P)$. We have $L(Q) \not\subseteq L(P)$, so there exists some $q \in L(Q) \setminus L(P)$. Then there is $\vec{p} \in \vec{L}(P)$ and an orientation $\vec{q}$ of q such that $\vec{q} < \vec{p}$. If $\overleftarrow{q} \in \vec{L}(Q)$, then we are done; so suppose that $\vec{q} \in \vec{L}(Q)$. Then we have $p \in L(P) \setminus L(Q)$, as otherwise $\vec{p} \in \vec{L}(Q)$ and $\vec{q} < \vec{p}$ which contradicts the maximality of $\vec{q}$ in $Q \cap \vec{T}$.

Since T is nested, each separation $\vec{q'} \in \vec{L}(Q)$ points towards or away from p. If some $\vec{q'} \in \vec{L}(Q)$ points away from p, then either $\vec{q'} > \vec{p} > \vec{q}$ which again contradicts the maximality of $\vec{q}$ in $Q \cap \vec{T}$, or $\vec{q'} > \overleftarrow{p}$ in which case $\vec{p}$ and $\vec{q'}$ are as desired. So we may suppose that all the separations in $\vec{L}(Q)$ point towards p. But then p is inside $\vec{L}(Q)$ and $p \notin L(Q)$ which contradicts the assumption that there is no separation $t \in T \setminus L(Q)$ inside $\vec{L}(Q)$. $\square$

By Proposition 5.8, we can apply the general local refinement theorem, Theorem 5.5 above, if condition (ii) of Proposition 5.6 is satisfied for each $\vec{L} \in \mathcal{L}$ and the corresponding $S_{\vec{L}}$. However, the refined nested set $T' \supseteq T$ obtained by Theorem 5.5 does not need to satisfy the assumption of Proposition 5.8 again: We might have added a separation s into T' that now witnesses the triviality of some separation $\vec{t} \in \vec{T}$ in $\vec{T'}$. Then the newly trivial $\vec{t}$ is automatically inside each location $\vec{L}$ of T' with $s \in L$.

A separation in $\vec{T}$ that becomes trivial in $\vec{T'}$ must be small. In particular, if all the profiles in $\mathcal{P}$ are regular, then we can assume T to be regular since no small separation distinguishes any two profiles in $\mathcal{P}$. In this case, no separation in $\vec{T}$ can become trivial in $\vec{T'}$, and the assumption of Proposition 5.8 is again satisfied for T' and $\mathcal{P}$ (after possibly removing separations from T' which are not $\mathcal{P}$-relevant).

In order to enable the iterative application of Theorem 5.5 even if not all the profiles in $\mathcal{P}$ are regular, let us put the assumption of Proposition 5.8 into a more general form. Instead of forbidding the existence of any separations inside a location $\vec{L} \in \mathcal{L}$, we require that every separation in $T \setminus L$ is outside $\vec{L}$.

Definition 5.9. Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a nested set, $\mathcal{P}$ a set of profiles within $\vec{U}$, and $\mathcal{L}$ the set of locations of $\mathcal{P}$ with respect to T. We say that T *properly partitions* $\mathcal{P}$ if for every location $\vec{L} \in \mathcal{L}$, each separation $s \in T \setminus L$ is outside $\vec{L}$, i.e. s has an orientation $\vec{s}$ with $\vec{s} \leq \vec{t}$ for some $\vec{t} \in \vec{L}$.

By the definition of 'inside' and 'outside', T properly partitions $\mathcal{P}$ if and only if for each location $\vec{L} \in \mathcal{L}(\mathcal{P}, T)$, a separation in $T \setminus L$ which is inside $\vec{L}$ must be trivial witnessed by some separation in L. In particular, there is no non-trivial separation from $T \setminus L$ that lies inside $\vec{L}$. Moreover, note that if T properly partitions $\mathcal{P}$ and there is $\emptyset = \vec{L} \in \mathcal{L}(\mathcal{P}, T)$, then T itself must be empty as every separation $t \in T$ needs a separation in $\vec{L}$ which 'witnesses' that t is outside $\vec{L}$.

Now T properly partitions $\mathcal{P}$ in the wide class of examples which satisfy that each profile $P \in \mathcal{P}$ orients the nested set T completely: Here, for each $P \in \mathcal{P}$, the location $\vec{L}(P)$ consists of the maximal separations of the consistent orientation that P induces on T. So each separation in $T \setminus \vec{L}(P)$ has an orientation which is $\leq$ some element of $\vec{L}(P)$, i.e. it is outside $\vec{L}(P)$. Note that in each iteration step of the proof of Theorem 1.2 in [6], every profile P in the considered set $\mathcal{P}_k$ of profiles orients the current nested set T_k since all the separations in T_k have order less than the order of P (see Section 6).

If T properly partitions $\mathcal{P}$, then condition (i) of Proposition 5.6 is sufficient to ensure its conclusion: in the proof of Proposition 5.6, we needed condition (ii) only if $t \in T \setminus L$ is inside $\vec{L}$, but not outside $\vec{L}$; but this cannot happen if T properly partitions $\mathcal{P}$. This yields the following proposition:

Proposition 5.10. *Let $\vec{U}$ be a universe of separations, and let $T \subseteq U$ be a nested set which properly partitions a set $\mathcal{P}$ of profiles within $\vec{U}$. Let $\mathcal{L}$ be the set of locations of $\mathcal{P}$ with respect to T. For some $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}} \subseteq \vec{U}$ be a set of separations that is oriented by every profile $P \in \mathcal{P}_{\vec{L}}$. Suppose that the separation system $S_{\vec{L}}$ satisfies that if $\vec{t} \in \vec{L}$ and a $\mathcal{P}_{\vec{L}}$-relevant separation $s \in S_{\vec{L}}$ cross, then at least one of the two corners $\vec{t} \vee \vec{s}$ and $\vec{t} \vee \overleftarrow{s}$ is in $\vec{S}_{\vec{L}}$.*
Let $S'_{\vec{L}} \subseteq S_{\vec{L}}$ consist of those separations in $S_{\vec{L}}$ which are nested with T. Then the following holds: if two profiles $P, Q \in \mathcal{P}_{\vec{L}}$ are distinguished by $S_{\vec{L}}$, then they are also distinguished by $S'_{\vec{L}}$.
Moreover, if $\vec{S}_{\vec{L}}$ is a structurally submodular separation system, then $\vec{S}'_{\vec{L}}$ is also structurally submodular. □

The definition of 'properly partitions' ensures that a nested set $T_{\vec{L}}$ inside $\vec{L}$ is nested with T: a separation outside $\vec{L}$ is nested with every separation inside $\vec{L}$ by definition. In the next proposition, we show that this fact guarantees that the refined nested set T' obtained from T by the application of Theorem 5.5 properly partitions $\mathcal{P}$ if T properly partitions $\mathcal{P}$. This also provides a formal proof of the claim made above that if T and $\mathcal{P}$ are both regular and there is no separation of $T \setminus L$ inside any $\vec{L} \in \mathcal{L}$, then the same holds for T' (after possibly removing some separations from T' such that all the $T_{\vec{L}}$ are $\mathcal{P}_{\vec{L}}$-relevant, and hence regular themselves).

Proposition 5.11. *Let $\vec{U}$ be a universe of separations, and let $T \subseteq U$ be a nested set which properly partitions a set $\mathcal{P}$ of profiles within $\vec{U}$. Let $\mathcal{L}$ be the set of locations of $\mathcal{P}$ with respect to T, and assume that $\mathcal{L}$ satisfies property (L). For each location $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}} \subseteq U$ be a set of separations which satisfies the conditions (i)-(iii) of Theorem 5.5. Then the nested set $T' \supseteq T$ obtained from Theorem 5.5 again properly partitions $\mathcal{P}$.*

Proof. Let $\vec{L}(P,T) \in \mathcal{L}(\mathcal{P},T)$ for some $P \in \mathcal{P}$. We want to show that every separation in $T \setminus L(P,T')$ is outside $\vec{L}(P,T')$. By definition, $\vec{L}(P,T)$ points towards $L(P,T')$ as $T \subseteq T'$. In particular, each separation inside $\vec{L}(P,T')$ is also inside $\vec{L}(P,T)$, and each separation outside $\vec{L}(P,T)$ is outside $\vec{L}(P,T')$ as well.

Theorem 5.5 yields $T' = T \cup \bigcup T_{\vec{L}}$ where $\vec{L}$ ranges over $\mathcal{L}(\mathcal{P},T)$ and each $T_{\vec{L}}$ is inside the respective location $\vec{L}$. Since T properly partitions $\mathcal{P}$, every $t \in T$ is outside $\vec{L}(P,T)$, and hence outside $\vec{L}(P,T')$. Moreover, if $t \in T_{\vec{L}}$ for some $\vec{L} \in \mathcal{L}(\mathcal{P},T)$ with $\vec{L} \neq \vec{L}(P,T)$, then t is outside $\vec{L}(P,T)$ by property (L), and thus t is also outside $\vec{L}(P,T')$.

So it is enough to consider $t \in T_{\vec{L}(P,T)} \setminus L(P,T')$. Since P orients $T_{\vec{L}(P,T)} \subseteq S_{\vec{L}(P,T)}$, every separation in $T_{\vec{L}(P,T)}$ is outside $\vec{L}(P,T')$ since $\vec{L}(P,T')$ is the set of maximal elements of $P \cap \vec{T'} \supseteq P \cap \vec{T}_{\vec{L}(P,T)}$. This completes the proof. □

By Proposition 5.11, the property of 'T properly partitions $\mathcal{P}$' is maintained through iterative applications of Theorem 5.5. Note that we can always use $T = \emptyset$ as a suitable starting point for such an iterative process since the empty set clearly properly partitions $\mathcal{P}$.

Perhaps surprisingly, the conclusion of Proposition 5.8 still holds if T properly partitions $\mathcal{P}$, in that the corresponding set $\mathcal{L}$ of locations satisfies (L). The following proof of this fact is very similar to the one of Proposition 5.8 given above.

Proposition 5.12. *Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a nested set, $\mathcal{P}$ a set of profiles within $\vec{U}$, and $\mathcal{L}$ the set of locations of $\mathcal{P}$ with respect to T. If T properly partitions $\mathcal{P}$, then $\mathcal{L}$ satisfies property (L).*

Proof. If $|\mathcal{L}| = 1$, then we are done by the definition of (L). So let $|\mathcal{L}| > 1$, and let $P, Q \in \mathcal{P}$ with distinct locations $\vec{L}(P), \vec{L}(Q) \in \mathcal{L}$. We have to show that there are $\vec{p} \in \vec{L}(P)$ and $\vec{q} \in \vec{L}(Q)$ with $\overleftarrow{p} \leq \vec{q}$. As in Proposition 5.8, we may assume that all the separations in $L(P) \cap L(Q)$ are oriented in the same way by both P and Q.

If $L(P) \subseteq L(Q)$ (the case $L(Q) \subseteq L(P)$ is symmetrical), then $\vec{L}(P) \subseteq \vec{L}(Q)$ by the above assumption. Since $\vec{L}(Q)$ is a star, $\vec{L}(P)$ points towards $L(Q) \setminus L(P)$. So as T properly partitions $\mathcal{P}$, each separation $q \in L(Q) \setminus L(P)$ has an orientation $\vec{q}$ which is trivial witnessed by some $p \in L(P)$. Now $p \in L(P) \subseteq L(Q) \subseteq S(Q) \cap T$, so $\vec{q}$ is also trivial in $\vec{S}(Q)$ with witness p. Thus, $\vec{q}$ cannot be maximal in $Q \in \vec{T}$ which contradicts that $\vec{q} \in \vec{L}(Q)$. So we cannot have $L(P) \subseteq L(Q)$.

Therefore, we may suppose that $L(P) \not\subseteq L(Q)$, that $L(Q) \not\subseteq L(P)$, and that all the separations in $L(P) \cap L(Q)$ are oriented in the same way by both P and Q. If no separation in $L(Q) \setminus L(P)$ is inside $\vec{L}(P)$ and no separation in $L(P) \setminus L(Q)$ is inside $\vec{L}(Q)$, then we can proceed as in the proof of Proposition 5.8. So let us assume that there exists $q \in L(Q) \setminus L(P)$ which is inside $\vec{L}(P)$ (the case of $p \in L(P) \setminus L(Q)$ inside $\vec{L}(Q)$ is symmetric).

Since T properly partitions $\mathcal{P}$, the separation q has a trivial orientation $\vec{q}$ with a witness $p \in L(P)$. If $\overleftarrow{q} \in \vec{L}(Q)$, then we are done since $\vec{q} < \vec{p}$ for the orientation $\vec{p} \in \vec{L}(P)$ of p by the triviality of $\vec{q}$. So suppose that $\vec{q} \in \vec{L}(Q)$. Then we must have $p \in L(P) \setminus L(Q)$ since $\vec{q}$ cannot be maximal in $\vec{L}(Q)$ otherwise.

Now by assumption, p is outside $\vec{L}(Q)$. Hence, there exists an orientation of p that is smaller or equal some $\vec{q'} \in \vec{L}(Q)$ with $q' \in L(Q) \setminus L(P)$. But p witnesses the triviality of $\vec{q}$, so we get $\vec{q} < \vec{q'}$ contradicting the maximality of $\vec{q}$ in $Q \cap \vec{T}$. This completes the proof. $\square$

We can now combine the previous propositions with Theorem 5.5 to obtain the following refinement theorem whose conditions are often easier to check than constructing a suitable separation system $S_{\vec{L}}$ nested with T by hand (see e.g. Section 6):

Theorem 5.13. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be a universe of separations, $\mathcal{P}$ a set of profiles within $\vec{U}$ and $T \subseteq U$ a nested set that properly partitions $\mathcal{P}$. Let $\mathcal{L}$ be the set of locations of $\mathcal{P}$ with respect to T. For each location $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}}$ be a set of separations in U such that*

(i) *$\vec{S}_{\vec{L}}$ is a structurally submodular separation system;*

(ii) *$S_{\vec{L}}$ is oriented by every profile $P \in \mathcal{P}_{\vec{L}}$;*

(iii) *if $\vec{t} \in \vec{L}$ and a $\mathcal{P}_{\vec{L}}$-relevant separation $s \in S_{\vec{L}}$ cross, then at least one of the corners $\vec{t} \vee \vec{s}$ and $\vec{t} \vee \overleftarrow{s}$ is in $\vec{S}_{\vec{L}}$.*

Then there exists a nested set $T' = T \cup \bigcup_{\vec{L} \in \mathcal{L}} T_{\vec{L}}$ in U where for each location $\vec{L} \in \mathcal{L}$, the nested set $T_{\vec{L}} \subseteq S_{\vec{L}} \setminus T$ is a set of $\mathcal{P}_{\vec{L}}$-relevant separations inside $\vec{L}$ such that every pair of profiles in $\mathcal{P}_{\vec{L}}$ which is distinguished by $S_{\vec{L}}$ is also distinguished by $T_{\vec{L}}$. Moreover, T' properly partitions $\mathcal{P}$ again.

Proof. For each location $\vec{L} \in \mathcal{L}$, let $S'_{\vec{L}} \subseteq S_{\vec{L}}$ be the set of separations in $S_{\vec{L}}$ which is nested with T. By Proposition 5.10, $S'_{\vec{L}}$ is again structurally submodular and distinguishes $\mathcal{P}_{\vec{L}}$ as far as $S_{\vec{L}}$. Since $\mathcal{L}$ satisfies property (L) by Proposition 5.12, we can apply Theorem 5.5 to T, $\mathcal{P}$, and the collection of $S_{\vec{L}}$ for $\vec{L} \in \mathcal{L}$. Thereby, we obtain a nested set $T' \supseteq T$ with the required properties, and T' properly partitions $\mathcal{P}$ again by Proposition 5.11. $\square$

Now suppose that $\mathcal{P}$ is a distinguishable set of profiles within $\vec{U}$. If we apply Theorem 5.13 repeatedly to build a sequence $T_0 \subseteq \cdots \subseteq T_n$ of nested sets towards a tree of tangles T_n for $\mathcal{P}$, then the empty set can always serve as a starting point $T_0 = \emptyset$ as it trivially partitions $\mathcal{P}$ properly. During the iterative construction, we are guaranteed that each T_k properly partitions $\mathcal{P}$. Therefore, we only have to care about finding appropriate separation systems $S_{\vec{L}}$ satisfying (i)-(iii), but this might not be possible without further assumptions as we shall see below in Example 5.18.

Before we turn to this, let us first address the problem that if we start our iterative construction with some arbitrary nested set $T_0 = T$, then T itself can already prevent us from extending it to a tree of tangles for $\mathcal{P}$ even if T properly partitions $\mathcal{P}$.

Example 5.14. For a universe $\vec{U}$ of separations, let $\vec{r} < \vec{t}$ be nested separations in $\vec{U}$ and let $s \in U$ cross r and t. Consider the three profiles $P = \{\overleftarrow{r}, \vec{s}\}$, $Q = \{\overleftarrow{s}, \vec{t}\}$, and $R = \{\vec{r}, \overleftarrow{t}\}$ on their respective underlying separation systems (see Figure 5.1 on the next page). Suppose our current nested set is $T = \{r, t\}$ which partitions the set $\mathcal{P} = \{P, Q, R\}$ of profiles properly. In order to distinguish P and Q, we would have to add s to T, but this is not possible.

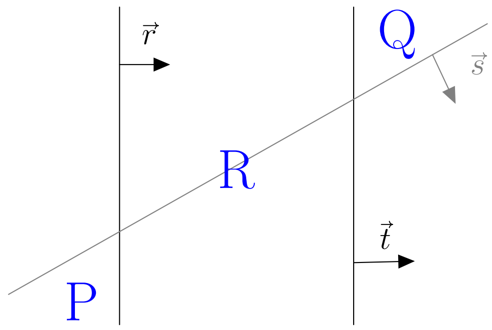

Figure 5.1: The situation in Example 5.14: the nested set $T = \{r, t\}$ properly partitions the set $\mathcal{P}$ of profiles consisting of $P = \{\overleftarrow{r}, \vec{s}\}$, $Q = \{\overleftarrow{s}, \vec{t}\}$, and $R = \{\vec{r}, \overleftarrow{t}\}$, but T cannot be extended to a nested set which distinguishes $\mathcal{P}$.

The key obstruction in this example is that P and Q live in different locations of T while there is no separation in T that distinguishes them. Indeed, the iterative application of Theorem 5.13 with suitably chosen $S_{\vec{L}}$ in every step can give us a nested set such that each two profiles from $\mathcal{P}$ live in different locations of it. In order to ensure that such a nested set also distinguishes $\mathcal{P}$, we need to require $T_0 = T$ to distinguish profiles living in different locations of T.

Definition 5.15. Let $\vec{U}$ be a universe of separations, $T \subseteq U$ a nested set, and $\mathcal{P}$ a set of profiles within $\vec{U}$. We say that T *strongly partitions* $\mathcal{P}$ if T properly partitions $\mathcal{P}$, and if for every pair of profiles $P, Q \in \mathcal{P}$ that live in different locations of T, there exists a separation $t \in T$ distinguishing P and Q.

If all the profiles in $\mathcal{P}$ orient T completely, then T clearly strongly partitions $\mathcal{P}$. Note that if T strongly partitions $\mathcal{P}$, then the corresponding set $\mathcal{L}$ of locations satisfies property (L) directly: If $\vec{L}(P) \neq \vec{L}(Q)$ for some two profiles $P, Q \in \mathcal{P}$, then there exists some $t \in T$ distinguishing them in that (say) $\vec{t} \in P$ and $\overleftarrow{t} \in Q$. Then there exist $\vec{p} \in \vec{L}(P)$ and $\vec{q} \in \vec{L}(Q)$ with $\vec{t} \leq \vec{p}$ and $\overleftarrow{t} \leq \vec{q}$. But this yields $\overleftarrow{p} \leq \overleftarrow{t} \leq \vec{q}$, so property (L) holds.

Now the following proposition shows that the property 'T strongly partitions $\mathcal{P}$' is preserved by Theorem 5.5, too.

Proposition 5.16. *Let $\vec{U}$ be a universe of separations, and let $T \subseteq U$ be a nested set which strongly partitions a set $\mathcal{P}$ of profiles within $\vec{U}$. Let $\mathcal{L}$ be the set of locations of $\mathcal{P}$ with respect to T, and for each location $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}} \subseteq U$ be a set of separations which satisfies the conditions (i)-(iii) of Theorem 5.5.*
Then the nested set $T' \supseteq T$ obtained from Theorem 5.5 strongly partitions $\mathcal{P}$, too.

Proof. Write $\mathcal{L}' = \mathcal{L}(\mathcal{P}, T')$ for the set of locations of $\mathcal{P}$ with respect to T'. By Theorem 5.5, we have that $T' = T \cup \bigcup_{\vec{L} \in \mathcal{L}} T_{\vec{L}}$ where each $T_{\vec{L}}$ is inside the respective $\vec{L}$. Since $\mathcal{L}$ satisfies property (L), the following holds for every location $\vec{L} \in \mathcal{L}$: if $t \in T' \setminus T_{\vec{L}}$, then t is outside $\vec{L}$.

By Proposition 5.11, T' again properly partitions $\mathcal{P}$. It remains to check that every pair of profiles $P, Q \in \mathcal{P}$ with $\vec{L}(P, T') \neq \vec{L}(Q, T')$ is distinguished by a separation $t \in T'$. If P and Q did already live in distinct locations of T, then we are done since $T \subseteq T'$ strongly partitions $\mathcal{P}$.

So suppose $\vec{L}(P, T) = \vec{L}(Q, T) =: \vec{L}$. As observed above, every $t \in T' \setminus T_{\vec{L}}$ is outside $\vec{L}$. So if t is oriented by some profile in $\mathcal{P}_{\vec{L}}$, then it is oriented as $\vec{t}$ by consistency. Therefore, it cannot witness that $\vec{L}(P,T')$ and $\vec{L}(Q,T')$ are distinct. Hence, there exists $t \in T_{\vec{L}}$ which witnesses $\vec{L}(P,T') \neq \vec{L}(Q,T')$. So since both P and Q orient $T_{\vec{L}}$, they must orient $T_{\vec{L}}$ differently. But then there exists some separation in $T_{\vec{L}}$ which is oriented differently by P and Q, and which hence distinguishes P and Q as desired. □

If we now use Proposition 5.16 instead of Proposition 5.11 in the proof of Theorem 5.13, then we obtain the following theorem which we can use to do each single iteration step in the process of building a tree of tangles for a distinguishable set $\mathcal{P}$ of profiles iteratively by local refinements:

Theorem 5.17. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be a universe of separations, $\mathcal{P}$ a set of profiles within $\vec{U}$, and $T \subseteq U$ a nested set that strongly partitions $\mathcal{P}$. Let $\mathcal{L}$ be the set of locations of $\mathcal{P}$ with respect to T. For each location $\vec{L} \in \mathcal{L}$, let $S_{\vec{L}}$ be a set of separations in U such that*

(i) *$\vec{S}_{\vec{L}}$ is a structurally submodular separation system;*

(ii) *$S_{\vec{L}}$ is oriented by every profile $P \in \mathcal{P}_{\vec{L}}$;*

(iii) *if $\vec{t} \in \vec{L}$ and a $\mathcal{P}_{\vec{L}}$-relevant separation $s \in S_{\vec{L}}$ cross, then at least one of the corners $\vec{t} \vee \vec{s}$ and $\vec{t} \vee \overleftarrow{s}$ is in $\vec{S}_{\vec{L}}$.*

Then there exists a nested set $T' = T \cup \bigcup_{\vec{L} \in \mathcal{L}} T_{\vec{L}}$ in U where for each location $\vec{L} \in \mathcal{L}$, the nested set $T_{\vec{L}} \subseteq S_{\vec{L}} \setminus T$ is a set of $\mathcal{P}_{\vec{L}}$-relevant separations inside $\vec{L}$ such that every pair of profiles in $\mathcal{P}_{\vec{L}}$ which is distinguished by $S_{\vec{L}}$ is also distinguished by $T_{\vec{L}}$. Moreover, T' strongly partitions $\mathcal{P}$. □

So given that suitable separation systems $S_{\vec{L}}$ exist throughout the iterative process, we can apply Theorem 5.17 repeatedly to build a sequence $T_0 \subseteq \cdots \subseteq T_n$ of nested sets towards a tree of tangles T_n for a distinguishable set $\mathcal{P}$ of profiles if our starting point T_0 strongly partitions $\mathcal{P}$. Note again that the empty set strongly partitions every set of profiles, so it can always serve as a starting point for this iterative process.

If there is a tree of tangles $T \subseteq U$ for a distinguishable set $\mathcal{P}$ of profiles, then it is not necessarily possible to construct a tree of tangles by an iterative application of Theorem 5.17 starting with $T = \emptyset$ without further assumptions on the separation systems $\vec{S}(P)$ for $P \in \mathcal{P}$.

Example 5.18. Let $\vec{U}$ be a universe of separations. Consider three distinct and nested separations $s_1, s_2, s_3 \in U$ and the separation systems $S_i = \{s_j \mid j \neq i\} \subseteq U$. The profiles $P_1 = \{\vec{s}_2, \vec{s}_3\}$ of $\vec{S}_1$, $P_2 = \{\vec{s}_2, \overleftarrow{s}_3\}$ of $\vec{S}_2$, and $P_3 = \{\overleftarrow{s}_1, \overleftarrow{s}_2\}$ of $\vec{S}_3$ have a tree of tangles $T = \{s_1, s_2, s_3\}$ in U. But $\bigcap S_i = \emptyset$, and so there is no choice of a non-empty set $S_{\vec{L}} \subseteq U$ of separations with respect to the unique location $\vec{L} = \emptyset$ for $T_0 = \emptyset$.

In Example 5.18, we could fix the problem by extending all the P_i to profiles of the separation system $\{s_1, s_2, s_3\}$. But we do not know whether this holds in general.

Question 5.19. *Let $\vec{U}$ be a universe of separations, and let $\mathcal{P}$ be a set of profiles within $\vec{U}$ such that for each $P \in \mathcal{P}$, we cannot extend P to a profile on a proper superset of $S(P)$. If there exists a tree of tangles for $\mathcal{P}$, can we construct one by the iterative local refinements as described in Theorem 5.17?*

6 Using Local Refinements towards a Tree of Tangles

In this section we present some applications of the results from Section 5. We first consider a universe of separations equipped with a submodular order function, and reobtain a non-canonical version of Theorem 1.2. In order to get canonicity, we need to ensure that our construction commutes with isomorphisms of separation systems [6]. But this is a global phenomenon and it cannot be guaranteed by our local refinement approach without further global assumptions. In the end of this section, we discuss a possible choice of such global assumptions that ensure canonicity. Before doing so, we investigate conditions that allow us to build a tree of tangles with an iterative process using Theorem 5.17. In particular, we obtain a sequential tree-of-tangles theorem, and compare it to a similar approach in [10].

Let $\vec{U}$ be a submodular universe of separations as assumed in Theorem 1.2. Now we use Theorem 5.17 to reobtain a non-canonical version of Theorem 1.2. Note that the set of profiles $\mathcal{P}$ in Theorem 1.2 is a set of profiles *in* $\vec{U}$, so each profile $P \in \mathcal{P}$ is an ℓ-profile in $\vec{U}$ for some $\ell \in \mathbb{N}$. Moreover, the set $\mathcal{P}$ of profiles is required to be *robust*. The definition of robustness of a set of profiles is quite technical, so we do not include it here. But we shall note that a robust set $\mathcal{P}$ of profiles is distinguishable. Furthermore, the robustness of $\mathcal{P}$ guarantees the applicability of the following Lemma 6.1 in the situations that are relevant to us (see [6]).

Lemma 6.1 ([6, Lemma 3.5])**.** *Let U be a submodular universe of separations, and let n be a positive integer. Let $t \in U$ be a separation that efficiently distinguishes two n-robust profiles P, P' in $\vec{U}$, and let $s \in U$ be a separation that efficiently distinguishes two profiles $\hat{P}, \hat{P}'$ in $\vec{U}$. If $|t| < |s| < n$, then t has an orientation $\vec{t}$ such that either $\vec{t} \wedge \vec{s}$ or $\vec{t} \wedge \overleftarrow{s}$ efficiently distinguishes $\hat{P}$ from $\hat{P}'$.*

For the proof and any details about robustness, we refer the reader to [6]. Note that since $(\overleftarrow{t} \vee \overleftarrow{s})^* = \vec{t} \wedge \vec{s}$ and $(\overleftarrow{t} \vee \vec{s})^* = \vec{t} \wedge \overleftarrow{s}$, we may assume, by replacing $\vec{t}$ with $\overleftarrow{t}$ if necessary, that Lemma 6.1 gives us an orientation $\vec{t}$ of t such that either $\vec{t} \vee \vec{s}$ or $\vec{t} \vee \overleftarrow{s}$ efficiently distinguishes $\hat{P}$ from $\hat{P}'$.

In order to simplify the wording in the remainder of this section, let us make the following definition: given a nested set T in a universe U and a set $\mathcal{P}$ of profiles within U, we call a location $\vec{L} \in \mathcal{L}(\mathcal{P}, T)$ *trivial* if $\mathcal{P}_{\vec{L}}$ contains only one profile; otherwise we say that $\vec{L}$ is *non-trivial*.

Theorem 6.2 (Non-Canonical Version of Theorem 1.2)**.** *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge, |\ |)$ be a submodular universe of separations. Then for every robust set $\mathcal{P}$ of profiles in $\vec{U}$, there is a nested set $T = T(\mathcal{P}) \subseteq U$ of separations such that:*

(a) *every two profiles in $\mathcal{P}$ are efficiently distinguished by some separation in T;*

(b) *every separation in T efficiently distinguishes a pair of profiles in $\mathcal{P}$;*

(c) *if all the profiles in $\mathcal{P}$ are regular, then T is a regular tree set.*

Proof. We are going to build $T = T(\mathcal{P})$ iteratively by constructing a sequence of nested sets T_k where T_{k+1} arises from T_k by a local refinement as described in Theorem 5.17. To ensure the efficiency of T in (a) and (b), we have to keep track of certain properties of the T_k. More precisely, we will inductively construct nested sets T_k and integers $\ell_k \geq 0$ with the following properties:

- $T_k \supsetneq T_j$ and $\ell_k > \ell_j$ for $0 \leq j < k$;
- T_k strongly partitions $\mathcal{P}$;
- T_k consists of separations of order $< \ell_k$;
- T_k efficiently distinguishes every pair of profiles which is distinguished by a separation of order $< \ell_k$;
- every separation in T_k efficiently distinguishes a pair of profiles in $\mathcal{P}$.

Since $\mathcal{P}$ is finite and ℓ_k strictly increases with growing k, this iterative process terminates after finitely many steps with a nested set $T = T_n$ satisfying (a) and (b). Part (c) of the statement follows directly from the fact that small separations cannot distinguish regular profiles. So if $\mathcal{P}$ consists of regular profiles, then $\vec{T}$ does not contain any small separations by (b), and hence T is a regular tree set.

We start our construction with $T_0 = \emptyset$ and $\ell_0 = 0$ which trivially have all the above properties. Now suppose that $T_0 \subsetneq \cdots \subsetneq T_k$ and corresponding $\ell_0 < \cdots < \ell_k$ were already constructed. If T_k does not yet distinguish $\mathcal{P}$ completely, then we are going to apply Theorem 5.17 to suitably chosen separation systems $S_{\vec{L}}$ based on a suitably chosen $\ell_{k+1} > \ell_k$, and thereby construct the nested set $T_{k+1} \supsetneq T_k$.

Let $\mathcal{L}_k$ be the set of locations of $\mathcal{P}$ with respect to T_k. In order to apply Theorem 5.17, we have to choose for each location $\vec{L} \in \mathcal{L}_k$ a set $S_{\vec{L}} \subseteq U$ of separations which satisfies the conditions (i)-(iii) of Theorem 5.17. Let $\ell_{k+1} > \ell_k$ be the smallest integer such that some two profiles in $\mathcal{P}$ are efficiently distinguished by a separation of size $\ell_{k+1} - 1$; such an integer exists since $\mathcal{P}$ is not yet distinguished by T_k. By the minimal choice of ℓ_{k+1}, we are guaranteed that if a separation $s \in S_{\ell_{k+1}}$ distinguishes some two profiles P and P' that are not yet distinguished by S_{ℓ_k}, then s distinguishes P and P' efficiently.

If $\vec{L} \in \mathcal{L}_k$ is trivial, then $\mathcal{P}_{\vec{L}}$ contains only one profile P, and T_k efficiently distinguishes P from all other profiles in $\mathcal{P}$ since T_k strongly partitions $\mathcal{P}$ by the induction hypothesis. Therefore, we set $S_{\vec{L}} = \emptyset$ for every such trivial location $\vec{L} \in \mathcal{L}_k$. For every non-trivial location $\vec{L} \in \mathcal{L}_k$, let $S_{\vec{L}}$ be the set $S_{\ell_{k+1}}$ of all the separations of order less than ℓ_{k+1}. (It is not possible to set $S_{\vec{L}} = S_{\ell_{k+1}}$ for a trivial location $\vec{L}$, too: the profile $P \in \mathcal{P}_{\vec{L}}$ may have order $< \ell_{k+1}$, and hence P does not need to orient $S_{\ell_{k+1}}$ which would contradict condition (ii) of Theorem 5.17.)

Let us check that these sets $S_{\vec{L}}$ satisfy the conditions (i)-(iii) of Theorem 5.17. For trivial locations, the conditions (i)-(iii) are clearly satisfied. Let $\vec{L}$ be a non-trivial location. By the submodularity of the order function of U, the separation system $\vec{S}_{\vec{L}} = \vec{S}_{\ell_{k+1}}$ is structurally submodular, so condition (i) holds. For condition (ii), note that every two profiles in $\mathcal{P}_{\vec{L}}$ are not yet distinguished by T_k. So by the induction hypothesis about T_k, they cannot be distinguished by any separation of order $< \ell_k$. In particular, since $\mathcal{P}_{\vec{L}} \subseteq \mathcal{P}$ is distinguishable, all the profiles in $\mathcal{P}_{\vec{L}}$ must have order $\geq \ell_{k+1}$ by the minimal choice of ℓ_{k+1}, so they orient $S_{\vec{L}} = S_{\ell_{k+1}}$.

Finally, condition (iii) holds by Lemma 6.1: Let $\vec{t} \in \vec{L}$, and let $s \in S_{\vec{L}}$ be $\mathcal{P}_{\vec{L}}$-relevant. Then t efficiently distinguishes two profiles $P, P' \in \mathcal{P}$ by the induction hypothesis, and since s is $\mathcal{P}_{\vec{L}}$-relevant, it distinguishes two profiles $\hat{P}, \hat{P}' \in \mathcal{P}_{\vec{L}}$. In particular, s distinguishes $\hat{P}$ and $\hat{P}'$ efficiently by the minimal choice of ℓ_{k+1}.

By the definition of the robustness for sets of profiles, the profiles P and P' in the robust set $\mathcal{P}$ of profiles are n-robust for some $n > |s|$ (see [6]). Moreover, we have $|t| < |s|$ by construction. So we can apply Lemma 6.1. Since $\vec{t} \in \hat{P}, \hat{P}'$ by $\vec{t} \in \vec{L}$, the profiles $\hat{P}$ and $\hat{P}'$ cannot be distinguished by the separations $\overleftarrow{t} \vee \vec{s}$ and $\overleftarrow{t} \vee \overleftarrow{s}$ by consistency. Therefore, Lemma 6.1 guarantees the existence of $\vec{t} \vee \vec{s}$ or $\vec{t} \vee \overleftarrow{s}$ in $S_{\vec{L}}$: if one of them efficiently distinguishes $\hat{P}$ and $\hat{P}'$, then it has order $\leq |s| < \ell_{k+1}$, and thus it must be in $S_{\vec{L}} = S_{\ell_{k+1}}$.

Now since the $S_{\vec{L}}$ satisfy the conditions (i)-(iii) of Theorem 5.17, we can apply Theorem 5.17 to the set $\mathcal{P}$ of profiles, the nested set T_k, and the collection of $S_{\vec{L}}$ for $\vec{L} \in \mathcal{L}_k$, and we obtain a refined nested set $T_{k+1} \supseteq T_k$. Then, by construction, T_{k+1} and ℓ_{k+1} clearly have the first four properties listed above. In particular, T_{k+1} is a proper superset of T_k as the choice of ℓ_{k+1} guarantees that the set $S_{\vec{L}} = S_{\ell_{k+1}}$ of separations and - hence T_{k+1} - distinguishes some two profiles which live in the same non-trivial location $\vec{L} \in \mathcal{L}_k$. For the fifth and final property, we can assume that every separation $t \in T_{k+1} \setminus T_k$ distinguishes some two profiles $P, P' \in \mathcal{P}$ that were not distinguished by $T_k \subseteq S_{\ell_k}$ (otherwise remove such separations from T_{k+1}). So by the minimal choice of ℓ_{k+1}, the separation t distinguishes P and P' efficiently. This completes the proof. $\square$

While our proof of Theorem 6.2 proceeds similar to the original proof of the canonical Theorem 1.2 in [6], let us discuss one key difference here: In the above proof of Theorem 6.2, the profiles which live in trivial locations with respect to T_k do not need to orient $S_{\ell_{k+1}}$ since they may have order $< \ell_{k+1}$. Therefore, we could not set $S_{\vec{L}} = S_{\ell_{k+1}}$ for a trivial location $\vec{L}$, but instead we set $S_{\vec{L}} = \emptyset$. This distinction between trivial and non-trivial locations is circumvented in the proof of Theorem 1.2 given in [6] as follows: since T_k efficiently distinguishes every pair of profiles which is distinguished by a separation of order $< \ell_k$, it especially distinguishes every profile of order $\leq \ell_k$ from all the other profiles in $\mathcal{P}$.

Let $\mathcal{P}_{k+1}$ be the set of all ℓ_{k+1}-profiles which are induced by profiles in $\mathcal{P}$ (and which are 'n-robust' for some fixed n depending only on $\mathcal{P}$). Then our conditions on the nested set $T_{k+1} \supseteq T_k$ can be reformulated in that T_{k+1} should efficiently distinguish $\mathcal{P}_{k+1}$, and every separation in $T_{k+1} \setminus T_k$ should efficiently distinguish some two profiles in $\mathcal{P}_{k+1}$. Now the advantage of considering $\mathcal{P}_{k+1}$ instead of $\mathcal{P}$ in the construction of T_{k+1} is that every profile in $\mathcal{P}_{k+1}$ orients $S_{\ell_{k+1}}$, and hence we can choose $S_{\vec{L}} = S_{\ell_{k+1}}$ for every location $\vec{L}$ of $\mathcal{P}_{k+1}$ with respect to T_k.

The application of Theorem 5.17 to the set $\mathcal{P}_{k+1}$ of profiles, the nested set T_k, and the collection of $S_{\vec{L}} = S_{\ell_{k+1}}$ for $\vec{L} \in \mathcal{L}(\mathcal{P}_{k+1}, T_k)$ then yields a nested set $T_{k+1} \supseteq T_k$ with the same properties as the one constructed in our proof - except for the fact, that this T_{k+1} only strongly partitions $\mathcal{P}_{k+1}$. But this is no real issue since T_{k+1} differs from T_k only by local nested sets $T_{\vec{L}}$ inside every location $\vec{L} \in \mathcal{L}(\mathcal{P}_{k+1}, T_k)$. So T_k already distinguishes a profile $P \in \mathcal{P}$ living in a location in $\mathcal{L}(\mathcal{P}, T_k) \setminus \mathcal{L}(\mathcal{P}_{k+1}, T_k)$ from every profile in a location in $\mathcal{L}(\mathcal{P}_{k+1}, T_k)$ by the induction hypothesis. Hence, this T_{k+1} strongly partitions $\mathcal{P}$ as well.

Our proof of Theorem 6.2 illustrates the iterative application of Theorem 5.17 to the same set $\mathcal{P}$ of distinguishable profiles and growing nested sets T_k in order to build a tree of tangles for $\mathcal{P}$. In particular, the above discussed difference to the proof in [6] shows how the variability in the choice of the $S_{\vec{L}}$ can helpful.

In the proof of Theorem 6.2 given above, we needed robustness to ensure condition (iii) of Theorem 5.17 in every iteration step for all non-trivial locations. If we return to a structural setup where we do not assume the existence of a global submodular order function, then we can also formulate a structural requirement on the set $\mathcal{P}$ of profiles within $\vec{U}$ which always ensures condition (iii). More precisely, we have the following:

Theorem 6.3. *Let $\vec{U} = (\vec{U}, \leq, *, \vee, \wedge)$ be a universe of separations, and let $\mathcal{P}$ be a distinguishable set of profiles within $\vec{U}$.*
Then we can build a sequence $\emptyset = T_0 \subsetneq \cdots \subsetneq T_n = T$ of nested sets in U by local refinements as in Theorem 5.17 such that T is a tree of tangles for $\mathcal{P}$ if the following holds:

(a) *For every set $\mathcal{P}' \subseteq \mathcal{P}$ of at least two profiles within $\vec{U}$, the set $S = \bigcap_{P \in \mathcal{P}'} S(P)$ is a structurally submodular separation system which contains a separation $s \in S$ that distinguishes some two profiles in $\mathcal{P}'$.*

(b) *For every $P, Q, Q' \in \mathcal{P}$ with $\vec{t} \in Q \cap Q'$ and $\overleftarrow{t} \in P$ and for every $s \in U$ distinguishing Q and Q', there is an orientation $\vec{s}$ of s such that $\vec{t} \vee \vec{s} \in \vec{S}(R)$ for every profile $R \in \mathcal{P}$ with $s, t \in S(R)$.*

Proof sketch. As in the proof of Theorem 6.2, we build a sequence $\emptyset = T_0 \subsetneq \cdots \subsetneq T_n = T$ of nested sets in U such that every T_k strongly partitions $\mathcal{P}$, and such that every T_k contains only $\mathcal{P}$-relevant separations. Suppose that T_k was already constructed. If T_k distinguishes $\mathcal{P}$, then we are done setting $n = k$. Otherwise, there exists a non-trivial location in $\mathcal{L}_k = \mathcal{L}(\mathcal{P}, T_k)$.

For each trivial location $\vec{L} \in \mathcal{L}_k$, we set $S_{\vec{L}} = \emptyset$ which satisfies the conditions (i)-(iii) of Theorem 5.17. For each non-trivial location $\vec{L} \in \mathcal{L}_k$, we set $S_{\vec{L}} = \bigcap_{P \in \mathcal{P}_{\vec{L}}} S(P)$ which is clearly oriented by every profile in $\mathcal{P}_{\vec{L}}$. By assumption (a), this $S_{\vec{L}}$ is structurally submodular, and it distinguishes some two profiles in $\mathcal{P}_{\vec{L}}$. Now the assumption (b) is exactly constructed to guarantee the existence of a suitable corner as in condition (iii) of Theorem 5.17: Suppose that $\vec{t} \in \vec{L}$ and a $\mathcal{P}_{\vec{L}}$-relevant separation $s \in S_{\vec{L}}$ cross. Since every separation in T_k is $\mathcal{P}$-relevant, there exists a profile $P \in \mathcal{P}$ with $\overleftarrow{t} \in P$. Moreover, s distinguishes some two profiles $Q, Q' \in \mathcal{P}_{\vec{L}}$ and both these profiles orient t as $\vec{t}$. So by assumption (b), there exists an orientation $\vec{s}$ of s with $\vec{t} \vee \vec{s} \in S_{\vec{L}}$ since all the profiles in $\mathcal{P}_{\vec{L}}$ orient both s and t. This ensures condition (iii) of Theorem 5.17.

Applying Theorem 5.17 to T_k, $\mathcal{P}$, and the collection of the $S_{\vec{L}}$ for $\vec{L} \in \mathcal{L}_k$, we obtain a refined nested set $T_{k+1} \supsetneq T_k$. Note that T_{k+1} is a proper superset of T_k: since there exists some non-trivial location $\vec{L} \in \mathcal{L}_k$, the refined nested set T_{k+1} distinguishes some previously not distinguished profiles in $\mathcal{P}_{\vec{L}}$ by assumption (a). Since $\mathcal{P}$ is finite, the iterative process terminates with a nested set T_n distinguishing $\mathcal{P}$. This completes the proof. $\square$

Condition (a) in Theorem 6.3 is in particular satisfied if the separation systems $S(P)$ form a sequence $S_1 \subseteq \cdots \subseteq S_n$ of structurally submodular separation systems in that for every $P \in \mathcal{P}$, we have $S(P) = S_i$ for some $i \in [n]$. For example, this is the case if we consider profiles *in* some submodular universe $\vec{U}$ since the S_ℓ with $\ell \in \mathbb{N}$ form such a sequence. However, even for a robust set $\mathcal{P}$ of profiles, the application of Lemma 6.1 as in the proof of Theorem 6.2 does not necessarily guarantee the existence of a suitable corner as in condition (b) of Theorem 6.3: in order to apply Lemma 6.1, we additionally needed that $|t| < |s|$, that t efficiently distinguishes some two profiles in $\mathcal{P}$, and that s efficiently distinguishes Q and Q'. From this point of view, Theorem 6.3 is not a direct generalization of Theorem 6.2 since condition (b) does not need to hold in its full generality.

But in a sequential setup like in the order function-case, we can relax condition (b) to these 'efficient cases' in that we find a good notion of efficiency. This approach was made precise by Elbracht, Kneip, and Teegen in [10], when they showed a sequential tree-of-tangles theorem, Theorem 6.4 below. Let us have a closer look on the relation of Theorem 6.3 to Theorem 6.4.

In order to state Theorem 6.4, we need the following definitions: we call $\mathcal{S} = (S_1, \dots, S_n)$ a *sequence of structurally submodular separation systems in* U if $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq U$ and each S_i is structurally submodular. Such a sequence $\mathcal{S}$ is *compatible* if for every $s_i \in S_i$ and $s_j \in S_j$ with $i \leq j$, there are either at least two corner separations of s_i and s_j in S_i, or at least three of them in S_j. A profile in $\mathcal{S} = (S_1, \dots, S_n)$ is a profile of some S_i. Two profiles in $\mathcal{S}$ are distinguished *efficiently* by $s \in U$ if $s \in S_i$ for all S_i distinguishing the two profiles.

Moreover, Elbracht, Kneip, and Teegen introduced the following structural notion of robustness in the context of sequences of separation systems. A set $\mathcal{P}$ of profiles in $\mathcal{S}$ is called *robust* if for all $P, Q, Q' \in \mathcal{P}$ the following holds: for every $\vec{t} \in Q \cap Q'$ with $\overleftarrow{t} \in P$ and every s which distinguishes Q and Q' efficiently, if $s \in S_j$, then there is an orientation $\vec{s}$ of s such that either $\overleftarrow{t} \vee \vec{s} \in P$ or $\vec{t} \vee \vec{s} \in \vec{S}_j$.

Now Elbracht, Kneip, and Teegen proved the following:

Theorem 6.4 ([10, Theorem 1.4])**.** *If $\mathcal{S} = (S_1, \dots, S_n)$ is a compatible sequence of structurally submodular separation systems inside a universe U, and $\mathcal{P}$ is a robust set of profiles in $\mathcal{S}$, then there is a nested set T of separations in U which efficiently distinguishes all the distinguishable profiles in $\mathcal{P}$.*

The notion of efficiency used in Theorem 6.4 can be ensured as in the proof of Theorem 6.2. So the difference between Theorem 6.4 and Theorem 6.3 for sequences boils down to the question how the combination of the *compatibility* of S and the *robustness* of $\mathcal{P}$ in Theorem 6.4 relates to condition (b) in Theorem 6.3. As mentioned above, condition (b) is much stronger than the combination of compatibility and robustness used in Theorem 6.4. This is mainly due to the fact that condition (b) needs to allow us to build the nested set T by local refinements.

Let us take a closer look on how the interplay of compatibility and robustness is used in the proof of [10, Theorem 1.4] if condition (b) does not hold. So suppose that we are in the setup of Theorem 6.4, and that condition (b) fails in that none of the corners $\vec{t} \vee \vec{s}$ and $\vec{t} \vee \overleftarrow{s}$ exists in $\vec{S}_{\vec{L}}$. Since we are in a sequential setup, and ensure efficiency throughout our construction of the T_k, we may assume that there are minimal $i, j \in [n]$ with $i \leq j$ such that $t \in S_i$ and $s \in S_j = S_{\vec{L}}$. Then in the proof of Theorem 6.4 given in [10], the compatibility of $\mathcal{S}$ ensures that there are two corners of s and t in S_i, namely the corners $\overleftarrow{t} \vee \vec{s}$ and $\overleftarrow{t} \vee \overleftarrow{s}$. This is the place where we can invoke the robustness of $\mathcal{P}$ to see that one of these corners still distinguishes P from Q and Q' (for the details, see the proof of [10, Theorem 1.4]).

But this does not help us in our iterative approach: We seek to extend the nested set T_k by local refinements. So in particular, while refining a location $\vec{L}$, we only add separations from $S_{\vec{L}}$ to T_k which are inside $\vec{L}$ (and not outside $\vec{L}$). However, both corners $\overleftarrow{t} \vee \vec{s}$ or $\overleftarrow{t} \vee \overleftarrow{s}$ are clearly outside $\vec{L}$. So condition (b) in Theorem 6.3 is exactly designed to avoid this case in that it ensures the existence of $\vec{t} \vee \vec{s}$ or $\vec{t} \vee \overleftarrow{s}$ in $\vec{S}_{\vec{L}}$. In this sense, condition (b) is a strengthening of the robustness of $\mathcal{P}$ that avoids the use of compatibility, and allows us to build the nested set T by local refinements.

Let us conclude this section by addressing the question which global assumptions on the choice of the $S_{\vec{L}}$ would guarantee the existence of a canonical refinement. The key building block in our approach is the following canonical tree-of-tangles theorem for structurally submodular separation systems which was recently proven by Elbracht and Kneip:

Theorem 6.5 ([9]). *Let $\vec{U}$ be a finite universe of separations, $\vec{S} \subseteq \vec{U}$ structurally submodular, and $\mathcal{P}$ a set of profiles of S. Then there is a nested set $T = T(\mathcal{P}) \subseteq S$ which distinguishes $\mathcal{P}$. Moreover, every separation in $T(\mathcal{P})$ is $\mathcal{P}$-relevant. This $T(\mathcal{P})$ can be chosen canonically: if $\alpha\colon \vec{U} \to \vec{U'}$ is an isomorphism of universes, then we have $\alpha(T(\mathcal{P})) = T(\alpha(\mathcal{P}))$.*

In the spirit of Theorem 5.5, we obtain the following canonical local refinement theorem:

Theorem 6.6. *Let $\vec{U}$ be a universe of separations and $\mathcal{P}$ a set of profiles within U. Let $T = T(\mathcal{P}) \subseteq U$ be a nested set of separations, $\mathcal{L}$ the set of locations of $\mathcal{P}$ in T, and let $S_{\vec{L}}$ be a set of separations for each location $\vec{L} \in \mathcal{L}$. Assume that $\mathcal{L}$ satisfies (L) and all the $S_{\vec{L}}$ satisfy (i)-(iii) as in Theorem 5.5.*
Then there exists a nested set $T' = T \cup \bigcup_{\vec{L} \in \mathcal{L}} T_{\vec{L}}$ in U where for each location $\vec{L} \in \mathcal{L}$, the nested set $T_{\vec{L}} \subseteq S_{\vec{L}} \setminus T$ is a set of $\mathcal{P}_{\vec{L}}$-relevant separations inside $\vec{L}$ such that every pair of profiles in $\mathcal{P}_{\vec{L}}$ which is distinguished by $S_{\vec{L}}$ is also distinguished by $T_{\vec{L}}$. The set $\mathcal{L}' = \mathcal{L}(\mathcal{P}, T')$ satisfies (L) again.
Moreover, this $T'(\mathcal{P})$ can be chosen canonical: if, for some isomorphism $\alpha\colon \vec{U} \to \vec{U'}$ of universes, we have $T(\alpha(\mathcal{P})) = \alpha(T(\mathcal{P}))$ and $S_{\alpha(\vec{L})} = \alpha(S_{\vec{L}})$ for all $\vec{L} \in \mathcal{L}$, then we also have $\alpha(T'(\mathcal{P})) = T'(\alpha(\mathcal{P}))$.

Proof. The proof proceeds as the one of Theorem 5.5 with the only difference that we use Theorem 6.5 instead of Theorem 3.8 to find the local tree sets $T_{\vec{L}} = T(\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}})$. So all the conclusions hold except the last claim that if, for some isomorphism $\alpha\colon \vec{U} \to \vec{U'}$ of universes, we have $T(\alpha(\mathcal{P})) = \alpha(T(\mathcal{P}))$ and $S_{\alpha(\vec{L})} = \alpha(S_{\vec{L}})$ for all $\vec{L} \in \mathcal{L}$, then we also have $\alpha(T'(\mathcal{P})) = T'(\alpha(\mathcal{P}))$.

By the definition of a location, the following holds: if $\vec{L}$ is the location of $P \in \mathcal{P}$ with respect to T, then $\alpha(\vec{L})$ is the location of $\alpha(P)$ with respect to $\alpha(T)$. So by definition of the $\mathcal{P}_{\vec{L}}$, we have

$$(\alpha(\mathcal{P})_{\alpha(\vec{L})} \restriction S_{\alpha(\vec{L})}) = \alpha(\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}}) \text{ for all } \vec{L} \in \mathcal{L}.$$

Theorem 6.5 yields that for all $\vec{L} \in \mathcal{L}$,

$$T_{\alpha(\vec{L})} = T(\alpha(\mathcal{P})_{\alpha(\vec{L})} \restriction S_{\alpha(\vec{L})}) = T(\alpha(\mathcal{P})_{\alpha(\vec{L})} \restriction \alpha(S_{\vec{L}})) = \alpha(T(\mathcal{P}_{\vec{L}} \restriction S_{\vec{L}})) = \alpha(T_{\vec{L}}).$$

Finally, the above equalities combine to

$$T'(\alpha(\mathcal{P})) = T(\alpha(\mathcal{P})) \cup \bigcup_{\alpha(\vec{L}) \in \alpha(\mathcal{L})} T_{\alpha(\vec{L})} = \alpha(T(\mathcal{P})) \cup \bigcup_{\vec{L} \in \mathcal{L}} \alpha(T_{\vec{L}}) = \alpha(T'(\mathcal{P})).$$

□

The part of this proof dealing with the canonicity of T' is essentially the same as the corresponding part of the proof of Theorem 1.2 in [6].

If we use Theorem 6.6 together with the arguments leading to Theorem 5.17 in the proof of Theorem 6.2, then we can indeed reobtain Theorem 1.2 completely: The choice of the sets $S_{\vec{L}}$ in the above proof of Theorem 6.2 clearly commutes with an isomorphism of universes as required in Theorem 6.6. By Proposition 5.6 and Lemma 6.1, we can transition to the set $S'_{\vec{L}} \subseteq S_{\vec{L}}$ of separations nested with T_k without loosing any power in distinguishing $\mathcal{P}_{\vec{L}}$, and this transition is again canonical since an isomorphism of universes respects the corresponding partial orders. To ensure the efficiency of the nested sets T_k in each iteration step, we then use the fact that the local tree sets $T_{\vec{L}}$ constructed using Theorem 6.5 only contain $\mathcal{P}_{\vec{L}}$-relevant separations.

7 Local Refinements and Tangle-Tree Duality

In Theorem 3.11, we showed that it is possible to use tangle-tree duality to witness the absence of local refinements in the algorithmic setup of Section 3. As we did in Section 5 for tangle distinguishing refinements, we now investigate tangle-tree duality refinements in more general setups in this section. In particular, we derive local refinement results in general universes of separations where we relax the assumption that the locally chosen set $S_{\vec{L}}$ of separations is inside $\vec{L}$. For an in-depth discussion of the definitions around tangle-tree duality, we refer the reader to [7] and [11].

Let us recall the tangle-tree duality theorem:

Theorem 3.10. ([7, Theorem 4.3]) *Let $\vec{U}$ be a universe of separations containing a finite separation system $\vec{S}$. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for $\vec{S}$. If $\vec{S}$ is $\mathcal{F}$-separable, exactly one of the following holds:*

(i) *There exists an $\mathcal{F}$-tangle of S.*

(ii) *There exists an S-tree over $\mathcal{F}$.*

In contrast to Section 3, we do not restrict ourselves to regular tree sets in universes of set bipartitions here. So let U be an arbitrary universe of separations. Now what can we do if $T \subseteq U$ is a nested set and not a regular tree set any more? Or what if the locally chosen set $S_{\vec{L}} \subseteq U$ of separations is not inside $\vec{L}$?

In [11], Erde investigated these questions under the assumption that U is equipped with a submodular order function. His key lemma [11, Lemma 8] can be adapted to universes without an order function by requiring the $S_{\vec{L}}$ to satisfy certain structural assumptions that were otherwise implied by the submodular order function. Our adaption of Erde's lemma, Proposition 7.1 below, is a direct generalization of Theorem 3.11 (since a universe of bipartitions of a finite set V only contains one small separation, namely the trivial $(\emptyset, V)$). Note that our additional assumptions on the $S_{\vec{L}}$ are stronger than what the submodular order function provides in Erde's setup; so our adapted version is not a direct generalization of Erde's Lemma (see the discussion below Proposition 7.1). Nevertheless, the following proof of the modified statement is very similar to Erde's proof of [11, Lemma 8].

For notational simplicity, we are going to assume that we have a star $\vec{L}$ of separations contained in a separation system $\vec{S} \subseteq \vec{U}$; this is equivalent to considering $S = S_{\vec{L}} \cup L$ in our previous notation. Now our structural version of [11, Lemma 8] reads as follows:

Proposition 7.1. *Let $\vec{U} = (\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations, and let $\vec{S} \subseteq \vec{U}$ be a separable separation system. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for $\vec{S}$, which contains $\{\vec{r}\}$ for all co-small $\vec{r}$, and which is closed under shifting in $\vec{S}$. Suppose that $\vec{L} = \{\vec{t_i} \mid i \in [n]\} \subseteq \vec{S}$ is a star of separations such that one of the following holds for each $\vec{t_i}$, and (ii) holds for either none or at least two of the $\vec{t_i}$:*

(i) *$\vec{t_i}$ points towards S;*

(ii) *there exists an $\mathcal{F}$-tangle O_i of $\vec{S}$ with $\overleftarrow{t_i} \in O_i$.*

Moreover, assume that each $\vec{t_i}$ emulates every $\vec{x} \leq \vec{t_i}$:

$$\forall \vec{t_i} \in \vec{L},\ \vec{x}, \vec{r} \in \vec{S} : (\vec{x} \leq \vec{r},\ \vec{x} \leq \vec{t_i} \text{ and } \vec{x} \neq \overleftarrow{r}) \implies (\vec{t_i} \vee \vec{r} \in \vec{S}).$$

Let $\mathcal{F}' = \mathcal{F} \cup \{\{\overleftarrow{t_i}\} | \vec{t_i} \in \vec{L}\}$. Then either there is an $\mathcal{F}'$-tangle of S, or there is an S-tree (G, α) over $\mathcal{F}'$ such that the nested set of unoriented separations underlying $\alpha(\vec{E}(G))$ is inside $\vec{L}$.

Proof. Let

$$\bar{\mathcal{F}} = \mathcal{F} \cup \{\{\overleftarrow{x}\} \subseteq \vec{S} \mid \overleftarrow{t_i} \leq \overleftarrow{x} \text{ for some } i \in [n]\}.$$

We first show that we can apply Theorem 3.10 to $\vec{S}$ and $\bar{\mathcal{F}}$, i.e. that $\vec{S}$ is $\bar{\mathcal{F}}$-separable, and that $\bar{\mathcal{F}}$ is standard. The latter is clear: since $\mathcal{F}$ is standard for $\vec{S}$, so is $\bar{\mathcal{F}}$. To see that $\vec{S}$ is $\bar{\mathcal{F}}$-separable, it is enough to show that $\bar{\mathcal{F}}$ is closed under shifting since $\vec{S}$ is separable. By assumption, $\mathcal{F}$ is closed under shifting. Further, every element of $\bar{\mathcal{F}} \setminus \mathcal{F}$ is a singleton star $\{\overleftarrow{x}\}$. So its image under some relevant shift is a singleton star $\{\overleftarrow{y}\}$ for some separation $\overleftarrow{x} \leq \overleftarrow{y}$, and hence, $\{\overleftarrow{y}\} \in \bar{\mathcal{F}}$ by definition. Thus, $\bar{\mathcal{F}}$ is closed under shifting.

By Theorem 3.10, there is either an $\bar{\mathcal{F}}$-tangle of $\vec{S}$ or an S-tree over $\bar{\mathcal{F}}$. In the first case, we are done since $\mathcal{F}' \subseteq \bar{\mathcal{F}}$ implies that every $\bar{\mathcal{F}}$-tangle is an $\mathcal{F}'$-tangle, too. So let us assume that there exists an S-tree (G, α) over $\bar{\mathcal{F}}$. By [7, Lemma 2.4], we may assume that (G, α) is irredundant and tight.

By assumption, there exists some $k \neq 1$ with $0 \leq k \leq n$ such that, after potentially renumbering the $\vec{t_i}$, each separation $\vec{t_i}$ with $i \in [k]$ is oriented as $\overleftarrow{t_i}$ by some $\mathcal{F}$-tangle O_i. The remaining separations $\vec{t}_{k+1}, \ldots, \vec{t}_n$ point towards S.

Let $i \geq k+1$, and consider $\vec{x} \in \vec{S}$ with $\vec{x} < \vec{t_i}$. Since $\vec{t_i}$ points towards x, we must have $\vec{t_i} < \overleftarrow{x}$, so $\vec{x}$ is trivial in $\vec{S}$ witnessed by t_i. Now by assumption, $\mathcal{F}$ is standard, and thus, $\{\overleftarrow{x}\} \in \mathcal{F}$. This implies that (G, α) is already an S-tree over

$$\bar{\mathcal{F}}_{k+1} = \mathcal{F} \cup \{\{\overleftarrow{t_j}\} \mid j \geq k+1\} \cup \{\{\overleftarrow{x}\} \subseteq \vec{S} \mid \overleftarrow{t_j} \leq \overleftarrow{x} \text{ for some } j \in [k]\}.$$

We now aim to modify (G, α) into an S-tree (G', α') over $\mathcal{F}'$ such that the set S' of unoriented separations underlying $\alpha(\vec{E}(G'))$ is inside $\vec{L}$ and nested. Given an S-tree (G', α') over $\mathcal{F}'$, the $\vec{t_i}$ with $i \geq k+1$ point towards $S' \subseteq S$ by assumption; we have to make sure that the $\vec{t_i}$ with $i \leq k$ point towards S', too. To achieve this, we are going to move the set of unoriented separations underlying $\alpha(\vec{E}(G))$ into $\vec{L}$ step-by-step in that we construct a sequence $(G, \alpha) = (G_{k+1}, \alpha_{k+1}), (G_k, \alpha_k), \ldots, (G_1, \alpha_1)$ of tight and irredundant S-trees over $\bar{\mathcal{F}}$ such that (G_i, α_i) is also an S-tree over

$$\bar{\mathcal{F}}_i = \mathcal{F} \cup \{\{\overleftarrow{t_j}\} \mid j \geq i\} \cup \{\{\overleftarrow{x}\} \subseteq \vec{S} \mid \overleftarrow{t_j} \leq \overleftarrow{x} \text{ for some } j \in [i-1]\},$$

and such that $\vec{t}_n, \ldots, \vec{t_i}$ point towards the nested set S_i of unoriented separations which underlies $\alpha(\vec{E}(G_i))$. Then, since $\bar{\mathcal{F}}_1 = \mathcal{F}'$, the S-tree (G_1, α_1) is as desired.

Suppose that (G_{i+1}, α_{i+1}) for some $1 \leq i \leq k$ is already constructed. By assumption, there exists an $\mathcal{F}$-tangle O_i of S that orients t_i as $\overleftarrow{t_i}$. Then this $\vec{t_i}$ is the unique $\vec{t_j} \in \vec{L}$ which is oriented as $\overleftarrow{t_j}$ by O_i: Since $\mathcal{F}$ contains $\{\vec{r}\}$ for every co-small separation $\vec{r}$, the $\mathcal{F}$-tangle O_i is regular. Hence, as $\vec{L}$ is a star and O_i is consistent, the $\vec{t_i}$ is unique as claimed. Next, we will show that there exists a leaf separation $\vec{x_i}$ of (G_{i+1}, α_{i+1}) with $\overleftarrow{t_i} \leq \overleftarrow{x_i}$ and $\overleftarrow{x_i} \in O_i$.

Since O_i is a consistent orientation of $\vec{S}$, it is contained in some vertex of (G_{i+1}, α_{i+1}). The star of separations at that vertex cannot lie in $\mathcal{F}$ as O_i is an $\mathcal{F}$-tangle. So it must lie in $\bar{\mathcal{F}} \setminus \mathcal{F}$. Each of these stars is a singleton, so the vertex containing O_i must be a leaf. In particular, there exists a leaf separation $\vec{x_i}$ of (G_{i+1}, α_{i+1}) with $\overleftarrow{x_i} \in O_i$. Since $\{\overleftarrow{x_i}\} \in \bar{\mathcal{F}} \setminus \mathcal{F}$, we must have $\overleftarrow{t_j} \leq \overleftarrow{x_i}$ for some $j \in [n]$. So $\overleftarrow{t_j} \in O_i$ by consistency, and, since $\vec{t_i}$ is unique in $\vec{L}$ with this property, we have $i = j$ as desired.

By assumption, $\vec{t}_i$ emulates $\vec{x}_i$ in $\vec{S}$, and, since $\bar{\mathcal{F}}$ is closed under shifting, $\vec{t}_i$ even emulates $\vec{x}_i$ in $\vec{S}$ for $\bar{\mathcal{F}}$. Moreover, $\vec{t}_i$ and $\vec{x}_i$ are both non-trivial and non-degenerate because they distinguish the $\mathcal{F}$-tangle O_i from all the other $\mathcal{F}$-tangles O_j with $i \neq j \in [k]$ (this is where we need $k \neq 1$).

Now if $\overleftarrow{t}_i = \overleftarrow{x}_i$, then $\vec{t}_i$ points towards the set S_{i+1} of unoriented separations underlying $\alpha_{i+1}(\vec{E}(G_{i+1}))$ since (G_i, α_i) is irredundant and over stars, and hence order-respecting by [3, Lemma 6.3 (i)]. In this case, we are done by setting $(G_i, \alpha_i) = (G_{i+1}, \alpha_{i+1})$.

So suppose that $\overleftarrow{t}_i < \overleftarrow{x}_i$. Then we use [7, Lemma 4.2], and shift (G_{i+1}, α_{i+1}) onto $\vec{t}_i$ to get an order-respecting S-tree $(\bar{G}_i, \bar{\alpha}_i)$ over $\bar{\mathcal{F}}$ which contains $\vec{t}_i$ as a unique leaf separation. Moreover, every leaf separation $\vec{r}$ of $(\bar{G}_i, \bar{\alpha}_i)$ with $\overleftarrow{t}_i < \overleftarrow{r}$ is trivial: since $(\bar{G}_i, \bar{\alpha}_i)$ is order-respecting, the leaf separations of $(\bar{G}_i, \bar{\alpha}_i)$ point towards each other. So as $\overleftarrow{t}_i$ is the image of a unique leaf, we also have $\vec{t}_i < \overleftarrow{r}$. Hence, $\vec{r}$ must be trivial and $\{\overleftarrow{r}\} \in \mathcal{F}$ since $\mathcal{F}$ is standard. In particular, $(\bar{G}_i, \bar{\alpha}_i)$ is also an S-tree over

$$\bar{\mathcal{F}}_i = \mathcal{F} \cup \{\{\overleftarrow{t}_j\} \mid j \geq i\} \cup \{\{\overleftarrow{x}\} \subseteq \vec{S} \mid \overleftarrow{t}_j \leq \overleftarrow{x} \text{ for some } j \in [i-1]\}.$$

Next, we show that $\vec{t}_n, \ldots, \vec{t}_{i+1}$ also point towards the set $\bar{S}_i$ of unoriented separations underlying $\bar{\alpha}_i(\vec{E}(\bar{G}_i))$. This is clear for $j > k$, so consider $\vec{t}_j$ for some $j \leq k$ with $j > i$. By the induction hypothesis, $\vec{t}_j$ points towards the set S_{i+1} of unoriented separations underlying $\alpha_{i+1}(\vec{E}(G_{i+1}))$, and $\vec{t}_j$ also points towards t_i since $\vec{L}$ is a star. Thus, $\vec{t}_j$ points towards all the corners of t_i and each separation in S_{i+1}. But by the definition of the shift of (G_{i+1}, α_{i+1}) onto $\vec{t}_i$, the set S_i of separations contains only separations from S_{i+1} and some of their corners with t_i. So $\vec{t}_j$ points towards $\bar{S}_i$ as well.

By [7, Lemma 2.4], there exists an S-tree (G_i, α_i) over $\bar{\mathcal{F}}$ with G_i a minor of $\bar{G}_i$ and $\alpha_i = \bar{\alpha}_i \restriction \vec{E}(\bar{G}_i)$, such that (G_i, α_i) is tight and irredundant, and contains $\vec{t}_i$ as a leaf separation. Now $\vec{t}_i$ points towards the set $S_i \subseteq \bar{S}_i$ of unoriented separations underlying $\alpha_i(\vec{E}(G_i))$ since (G_i, α_i) is order-respecting by [3, Lemma 6.3 (i)]. So $\vec{t}_n, \ldots, \vec{t}_i$ point towards S_i. Moreover, S_i is nested since (G_i, α_i) is order-respecting. This completes the induction step and hence the proof. □

In Proposition 7.1, we have the somewhat technical-looking condition on the interplay of S and $\vec{L}$ that each $\vec{t}_i \in \vec{L}$ emulates every $\vec{x} \leq \vec{t}_i$. It says that if a separation $\vec{r} \in \vec{S}$ crosses some $\vec{t}_i \in \vec{L}$, then the corner $\vec{r} \vee \vec{t}_i$ has to be in S. In comparison to the existence of some corner of r and t_i inside $\vec{L}$ as in condition (ii) of Proposition 5.6, we consider an orientation $\vec{r}$ of $r \in S$ here, and require that $\vec{S}$ contains the specific corner $\vec{r} \vee \vec{t}_i$. Since this condition is used to guarantee that our preliminary tree with a leaf separation $\vec{x}_i \leq \vec{t}_i$ can be shifted onto $\vec{t}_i$ to make $\vec{t}_i$ a leaf separation of the shifted tree, it is enough to require the condition for such $\vec{r} \in \vec{S}$ that satisfy $\vec{r} \geq \vec{x}$ for some $\vec{x} \in \vec{S}$ which is itself $\leq \vec{t}_i$.

In the setup of Erde's [11, Lemma 8], the assumptions only guarantee that if there exists any leaf separation $\leq \vec{t}_i$, then there exists *some* leaf separation $\vec{x}_i \leq \vec{t}_i$ which is emulated by $\vec{t}_i$. The respective key assumptions are the existence of a submodular order function together with the requirement that each $\vec{t}_i$ in case (ii) efficiently distinguishes some two $\mathcal{F}$-tangles of $\vec{S}$. Due to the lack of similar tools in the absence of an order function, we imposed the stronger structural assumption that *each* separation $\leq \vec{t}_i$ is emulated by $\vec{t}_i$, and this is why Proposition 7.1 does not directly generalize [11, Lemma 8].

Let us view Proposition 7.1 from the perspective of local refinements as in Section 3: Let $T \subseteq U$ be a nested set and $\vec{L} \subseteq \vec{T}$ a star which corresponds to the maximal set of separations of a consistent orientation of T. Suppose we chose a set $S_{\vec{L}} \subseteq \vec{U}$ of separations and a set $\mathcal{F}_{\vec{L}}$ of stars in $\vec{U}$ such that $S_{\vec{L}} \cup L$ and $\mathcal{F}_{\vec{L}}$ satisfy the assumptions of Proposition 7.1.Then we apply Proposition 7.1 to the star $\vec{L} \subseteq \vec{S}_{\vec{L}} \cup \vec{L}$ and $\mathcal{F}_{\vec{L}}$, and obtain either an $\mathcal{F}_{\vec{L}}$-tangle of $S_{\vec{L}} \cup L$ extending $\vec{L}$, or an $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}}$ which witnesses the non-existence of such tangles.

In the latter case, this $(S_{\vec{L}} \cup L)$-tree over $\mathcal{F}'_{\vec{L}}$ is constructed such that the nested set $T_{\vec{L}}$ of unoriented separations underlying $\alpha(\vec{E}(G))$ is inside $\vec{L}$. Since every separation of T is outside $\vec{L}$, this is enough to guarantee that $T_{\vec{L}}$ is nested with T. So $T' = T \cup T_{\vec{L}}$ is a refined nested set which witnesses the non-existence of $\mathcal{F}_{\vec{L}}$-tangles of $S_{\vec{L}} \cup L$ extending $\vec{L}$.

We conclude this section with the following version of Proposition 7.1 in which we relax condition (i) to nestedness with t_i under the assumptions of Section 3. More precisely, we require that $\mathcal{F}$ contains its essential core and that all the separations in $\vec{L}$ are non-trivial:

Proposition 7.2. *Let $\vec{U} = (\vec{U}, \leq, *, \wedge, \vee)$ be a universe of separations, and let $\vec{S} \subseteq \vec{U}$ be a separable separation system without degenerate elements. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for $\vec{S}$, which contains $\{\overleftarrow{r}\}$ for all co-small $\vec{r}$, and which is closed under shifting in $\vec{S}$. Suppose $\vec{L} = \{\vec{t}_i \mid i \in [n]\} \subseteq \vec{S}$ is a star of non-trivial separations such that one of the following holds for each $\vec{t}_i$, and (ii) holds for either none or at least two of the $\vec{t}_i$:*

(i) *S is nested with t_i;*

(ii) *there exists an $\mathcal{F}$-tangle O_i of $\vec{S}$ with $\overleftarrow{t}_i \in O$.*

Moreover, assume that each $\vec{t}_i$ emulates every $\vec{x} \leq \vec{t}_i$:

$$\forall \vec{t}_i \in \vec{L},\ \vec{x}, \vec{r} \in \vec{S} : (\vec{x} \leq \vec{r},\ \vec{x} \leq \vec{t}_i \text{ and } \vec{x} \neq \overleftarrow{r}) \implies (\vec{t}_i \vee \vec{r} \in \vec{S}).$$

Let $\mathcal{F}' = \mathcal{F} \cup \{\{\overleftarrow{t}_i\} | \vec{t}_i \in \vec{L}\}$. Then either there is an $\mathcal{F}'$-tangle of S, or there is an essential S-tree over the essential core of $\mathcal{F}'$ such that the nested set of unoriented separations underlying $\alpha(\vec{E}(G))$ is a tree set inside $\vec{L}$.

Proof sketch. We follow the same proof strategy as in Proposition 7.1, and first obtain an S-tree (G, α) over $\bar{\mathcal{F}}$ from Theorem 3.10. By [3, Corollary 6.7], we may assume that (G, α) is essential and over the essential core of $\bar{\mathcal{F}}$.

Let $k \leq n$ be chosen as above, and let $l \in \mathbb{N}$ with $k \leq l \leq n$ such that, after potentially renumbering the $\vec{t}_i$, the separation $\vec{t}_i$ does not point towards S for each t_i with $k < i \leq l$. Then we construct a sequence $(G, \alpha) = (G_{l+1}, \alpha_{l+1}), (G_l, \alpha_l), \ldots, (G_1, \alpha_1)$ of essential S-trees over $\bar{\mathcal{F}}$ similar to the proof of Proposition 7.1, such that (G_i, α_i) is also an S-tree over the essential core of

$$\bar{\mathcal{F}}_i = \mathcal{F} \cup \{\{\overleftarrow{t}_j\} \mid j \geq i\} \cup \{\{\overleftarrow{x}\} \subseteq \vec{S} \mid \overleftarrow{t}_j \leq \overleftarrow{x} \text{ for some } j \in [i-1]\},$$

and such that $\vec{t}_n, \ldots, \vec{t}_i$ point towards the tree set of unoriented separations underlying $\alpha(\vec{E}(G_i))$. We are again done for $i = 1$.

For $i \leq k$, the construction of (G_i, α_i) from (G_{i+1}, α_{i+1}) works analogously to the proof of Proposition 7.1 with one difference: we combine [7, Lemma 2.4] with [3, Lemma 6.6] to obtain an essential S-tree (G_i, α_i) over the essential core of $\bar{\mathcal{F}}$ from $(\bar{G}_i, \bar{\alpha}_i)$ where G_i is a minor of $\bar{G}_i$ and $\alpha_i = \bar{\alpha}_i \restriction \vec{E}(\bar{G}_i)$, such that (G_i, α_i) contains $\vec{t}_i$ as a leaf separation (this works since t_i is non-trivial by assumption).

It remains to consider the case $k < i \leq l$. If the separation $\vec{t}_i$ points towards the set S_{i+1} of unoriented separations underlying $\alpha_{i+1}(\vec{E}(G_{i+1}))$, then we are done by setting $(G_i, \alpha_i) = (G_{i+1}, \alpha_{i+1})$.

So suppose $\vec{t}_i$ does not point towards S_{i+1}. Similar as for $i \leq k$, we want to shift our tree (G_{i+1}, α_{i+1}) onto $\vec{t}_i$ in this case. Let us show that we can do so in that we again find a non-trivial and non-degenerate leaf separation $\overrightarrow{x}_i$ of (G_{i+1}, α_{i+1}) with $\overleftarrow{t}_i \leq \overleftarrow{x}_i$: Since $\vec{t}_i$ does not point towards S_{i+1}, and t_i is nested with S, there exists $\overrightarrow{x} \in \vec{S}_{i+1}$ such that $\overleftarrow{t}_i \leq \overleftarrow{x}$. Now $\vec{S}_{i+1}$ is finite, so there exists a minimal separation $\overrightarrow{x}_i \in \vec{S}_{i+1}$ with $\overrightarrow{x}_i \leq \overrightarrow{x}$. By [3, Lemma 6.3 (i)], this $\overrightarrow{x}_i$ must be a leaf separation of (G_{i+1}, α_{i+1}). Since (G_{i+1}, α_{i+1}) is essential, $\overrightarrow{x}_i$ is non-trivial, and by assumption on S, it is also non-degenerate. From this point on, we can continue as for $i \leq k$.

$\square$

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Eidesstattliche Erklärung

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Hamburg, den 25. August 2020

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