210408 ①

PBZ von

$$r(x) = \frac{p_n(x)}{q_m(x)} \quad \text{mit} \quad \deg p_n = n < \deg q_m = m$$

$$= \frac{a_{11}}{x-x_1} + \frac{a_{12}}{(x-x_1)^2} + \cdots + \frac{a_{1m_1}}{(x-x_1)^{m_1}}$$

$$+ \frac{a_{21}}{x-x_2} + \frac{a_{22}}{(x-x_2)^2} + \cdots + \frac{a_{2m_2}}{(x-x_2)^{m_2}}$$

$$+ \vdots$$

$$+ \frac{a_{r1}}{x-x_r} + \frac{a_{r2}}{(x-x_r)^2} + \cdots + \frac{a_{rm_r}}{(x-x_r)^{m_r}}$$

$$+ \frac{b_{11}x + c_{11}}{x^2+p_1x+q_1} + \cdots + \frac{b_{1n_1}x + c_{1n_1}}{(x^2+p_1x+q_1)^{n_1}}$$

$$+ \vdots$$

$$+ \frac{b_{s1}x + c_{s1}}{x^2+p_sx+q_s} + \cdots + \frac{b_{sn_s}x + c_{sn_s}}{(x^2+p_sx+q_s)^{n_s}}$$

Dabei

$$q_m(x) = (x-x_1)^{m_1} \cdots (x-x_r)^{m_r} (x^2+p_1x+q_1)^{n_1} \cdots (x^2+p_sx+q_s)^{n_s}$$

$$m = \sum_{i=1}^{r} m_i + 2 \sum_{i=1}^{s} n_s$$

<u>Bsp</u>: $\int \frac{x^4+2}{x^2+2x-1}\,dx = ?$

Integrand <u>nicht</u> echt gebrochen rational

$\rightarrow$ Polynomdivision

$$x^4+2 : x^2+2x-1 = x^2-2x+5+\frac{-12x+7}{x^2+2x-1}$$

$$\int \frac{x^4+2}{x^2+2x-1}\,dx = \frac{1}{3}x^3 - x^2 + 5x + \int \frac{-12x+7}{x^2+2x-1}\,dx$$

$$\int \frac{-12x+7}{x^2+2x-1}\,dx = -6\int \frac{2x-\frac{7}{6}}{x^2+2x-1}\,dx$$

$$= -6\int \frac{2x+2}{x^2+2x-1}\,dx + 19\int \frac{1}{x^2+2x-1}\,dx$$

$\int \frac{g'}{g} = \ln|g|$

$$= -6\ln|x^2+2x-1| + 19\int \frac{1}{x^2+2x-1}\,dx$$

$\int \frac{1}{x^2+2x-1}\,dx = ?$ $\qquad x^2+2x-1=0$

$x_{1,2} = -1 \pm \sqrt{2}$

$$\Rightarrow \frac{1}{x^2+2x-1} = \frac{a_{11}}{x+1-\sqrt{2}} + \frac{a_{21}}{x+1+\sqrt{2}}$$

mit $a_{11} = \frac{1}{2\sqrt{2}}$, $a_{21} = -\frac{1}{2\sqrt{2}}$

Damit

$$\int \frac{1}{x^2+2x-1}\,dx = \frac{1}{2\sqrt{2}} \ln|x+1-\sqrt{2}|$$
$$- \frac{1}{2\sqrt{2}} \ln|x+1+\sqrt{2}|$$

Insgesamt:

$$\int \frac{x^4+2}{x^2+2x-1}\,dx = \frac{1}{3}x^3 - x^2 + 5x$$
$$- 6 \ln|x^2+2x-1|$$
$$+ \frac{19}{2\sqrt{2}} \ln|x+1-\sqrt{2}|$$
$$- \frac{19}{2\sqrt{2}} \ln|x+1+\sqrt{2}| + C$$

Bsp: $\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = ?$

$$x^4+2x^3-2x^2-6x+5 = (x-1)^2(x^2+4x+5)$$

PBZ

$$\frac{1}{x^4+2x^3-2x^2-6x+5} = \frac{a_{11}}{x-1} + \frac{a_{12}}{(x-1)^2}$$
$$+ \frac{b_{11}x + c_{11}}{x^2+4x+5} \qquad (*)$$

$\cdot (x-1)^2$ und $x=1$ liefert $a_{12} = \frac{1}{10}$

$a_{11} = -\frac{3}{50}$, $b_{11} = \frac{3}{50}$, $c_{11} = \frac{1}{5}$

$a_{12} = \frac{1}{10}$ liefert in $\circledast$

$$1 - \frac{1}{10}x^2 - \frac{2}{5}x - \frac{1}{2} = a_{11}(x-1)(x^2+4x+5) + (b_{11}x + c_{11})(x-1)^2$$

$:(x-1)$ und $x=1$: $a_{11} = -\frac{3}{50}$

Damit

$$\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = \circledast$$

$$= -\frac{3}{50}\ln|x-1| - \frac{1}{10}\frac{1}{x-1} + \int \frac{\frac{3}{50}x + \frac{1}{5}}{x^2+4x+5}$$

$$\bigcirc = \frac{3}{100}\int \frac{2x+4+\frac{8}{3}}{x^2+4x+5}\,dx$$

$$= \frac{3}{100}\ln(x^2+4x+5) + \frac{3}{100}\cdot\frac{8}{3}\boxed{\int \frac{1}{x^2+4x+5}\,dx}$$

$$\square = \int \frac{1}{(x+2)^2+1}\,dx = \arctan(x+2) + C$$

Insges.: $\frac{2}{25}\arctan(x+2)$

$$\circledast = -\frac{3}{50}\ln|x-1| - \frac{1}{10(x-1)} + \frac{3}{100}\ln(x^2+4x+5) + \frac{2}{25}\arctan(x+2) + C$$

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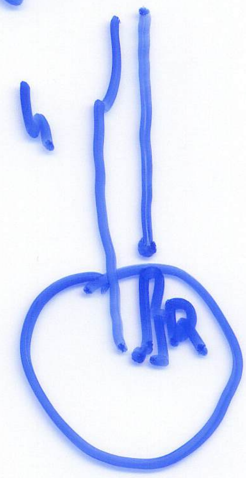
Uneigentliche Integrale

<u>Motivation</u>: Rakete aus dem Schwerefeld der Erde transportieren

$\rightsquigarrow$ Arbeit ?

Newton'sches Gravitationsgesetz



Arbeit

$$= \int_R^h G \frac{mM}{x^2}\, dx$$

mit m Raketenmasse, M Erdmasse
G Gravitationskonstante, h Höhe

$$\lim_{h \to \infty} \int_R^h G \frac{mM}{x^2}\, dx = ?$$

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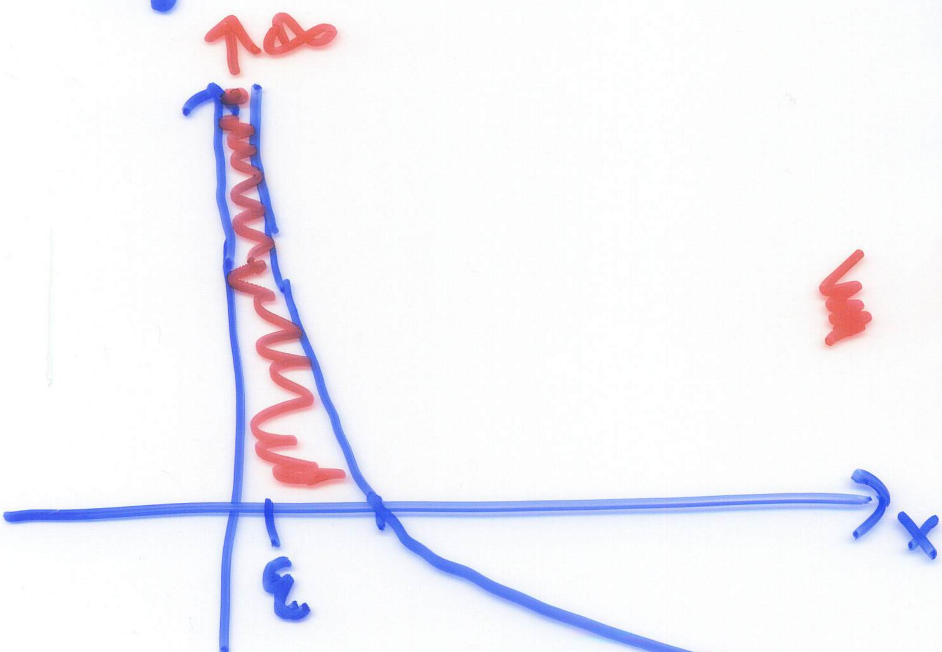
$$\int_R^h G\frac{mM}{x^2}\,dx = GmM\left(-\frac{1}{x}\right)\Big|_R^h$$

$$= GmM\left(\frac{1}{R}-\frac{1}{h}\right)$$

$$\Rightarrow \underbrace{\lim_{h\to\infty}\int_R^h G\frac{mM}{x^2}\,dx}_{} = GmM\frac{1}{R}$$

$$=: \int_R^\infty G\frac{mM}{x^2}\,dx$$

ii) $\int_0^1 -\ln x\,dx = ?$ Achtung: $\ln(0)$ nicht erklärt



besitzt endliche Fläche ?

Untersuche

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 -\ln x\,dx = -\lim_{\varepsilon\to 0}\left.\left(\cancel{(x+1)\ln x}\ x\ln x - x\right)\right|_\varepsilon^1$$

$$\cancel{= \lim_{\varepsilon\to 0}(1+\varepsilon)\ln\varepsilon}$$

$$\cancel{= \lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon}$$

$$= -\lim_{\varepsilon\to 0}\left[-1 - \varepsilon\ln\varepsilon - \varepsilon\right]$$

$$= +1 + \lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon + \lim_{\varepsilon\to 0}\varepsilon$$

$$= 1$$

$$\int_0^1 -\ln x\,dx := \lim_{\varepsilon\to 0}\int_\varepsilon^1 -\ln x\,dx$$

$$= 1$$

<u>Bsp</u> $\int_1^\infty \frac{1}{x^s}\,dx = ?$ $\quad$ <u>$s>1$</u>

$$\int_1^R \frac{1}{x^s}\,dx = \frac{1}{1-s}\, x^{1-s}\Big|_1^R$$

$$= -\frac{1}{1-s} + \frac{1}{1-s} R^{1-s}$$

Damit

$$\lim_{R\to\infty} \int_1^R \frac{1}{x^s}\,dx = \begin{cases} -\frac{1}{1-s} & , s>1 \\ \text{divergent} & , s\le 1 \end{cases}$$