Analyis II
TUHH
VL 3, 28. April 2016

Partialbruchzerlegung, uneigentliche Integrale

Michael Hinze

Bsp Integration rationaler Funktionen $\qquad \int \frac{x^4+2}{x^2+2x-1}\,dx = ?$

Integrand nicht echt gebrochen rational

Polynomdivision

$$x^4 + 0x^3 + 0x^2 + 0x + 2 : (x^2+2x-1) = \underbrace{x^2-2x+5}_{S_2(x)} + \underbrace{\frac{-12x+7}{x^2+2x-1}}_{r(x)}$$

$$\begin{array}{l}
-(x^4 + 2x^3 - x^2) \\
\hline
0 \quad -2x^3 + x^2 + 0x \\
\quad -(-2x^3 - 4x^2 + 2x) \\
\hline
\quad 0 \quad 5x^2 - 2x + 2 \\
\quad\quad -(5x^2 + 10x - 5) \\
\hline
\quad\quad 0 \quad -12x + 7
\end{array}$$

Damit

$$\int \frac{x^4+2}{x^2+2x-1}\,dx = \frac{1}{3}x^3 - x^2 + 5x + \int \frac{-12x+7}{x^2+2x-1}\,dx = ?$$

$$\int \frac{-12x+7}{x^2+2x-1}\,dx = -6\underbrace{\int \frac{2x+2}{x^2+2x-1}\,dx}_{\text{„}\int \frac{f'}{f}\text{“}} + 19\int \frac{1}{x^2+2x-1}\,dx$$

$$= -6\ln|x^2+2x-1| + 19\int \frac{1}{x^2+2x-1}\,dx = ?$$

$$\int \frac{1}{x^2+2x-1}\,dx$$

NR $x^2+2x-1 = 0$

$x_{1/2} = -1 \pm \sqrt{2}$

$\Rightarrow x^2+2x-1 = (x+1-\sqrt{2})(x+1+\sqrt{2})$

PBZ $\dfrac{1}{\underbrace{x^2+2x-1}_{(x+1-\sqrt{2})(x+1+\sqrt{2})}} = \dfrac{a_{11}}{x+1-\sqrt{2}} + \dfrac{a_{12}}{x+1+\sqrt{2}}$

Damit $a_{11} = \dfrac{1}{2\sqrt{2}}$, $a_{21} = -\dfrac{1}{2\sqrt{2}}$

Damit $\displaystyle\int \frac{1}{x^2+2x-1}\,dx = \frac{1}{2\sqrt{2}} \ln|x+1-\sqrt{2}| - \frac{1}{2\sqrt{2}} \ln|x+1+\sqrt{2}| + C$

Insgesamt

$$\int \frac{x^4+2}{x^2+2x-1}\,dx = \frac{1}{3}x^3 - x^2 + 5x - 6\ln|x^2+2x-1| + \frac{19}{2\sqrt{2}}\ln|x+1-\sqrt{2}| - \frac{19}{2\sqrt{2}}\ln|x+1+\sqrt{2}| + C$$

Bsp: $\displaystyle\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = ?$

Nullstellen des Nennerpolynoms: 1 (doppelt). Damit

$$x^4+2x^3-2x^2-6x+5 = (x-1)^2(x^2+4x+5)$$

und PBZ liest sich

$$\frac{1}{\underbrace{x^4+2x^3-2x^2-6x+5}_{(x-1)^2(x^2+4x+5)}} = \frac{a_{11}}{x-1} + \frac{a_{12}}{(x-1)^2} + \frac{b_{11}x + c_{11}}{x^2+4x+5}$$

i) $\cdot (x-1)^2$ und $x=1$ setzen liefert $a_{12} = \dfrac{1}{10}$

ii) $\dfrac{a_{12}}{(x-1)^2}$ auf linke Seite, mit $(x-1)$ multiplizieren und $x=1$ setzen

liefert $a_{11} = -\dfrac{3}{50}$

$\vdots$

$c_{11} = -\dfrac{1}{5}$, $b_{11} = \dfrac{3}{50}$

Zu ii) $\frac{a_{11}}{x-1} = \frac{1}{(x-1)^2(x^2+4x+5)} - \frac{a_{12}}{(x-1)^2} - \frac{b_{11}x + c_{11}}{x^2+4x+5}$

$= \frac{1 - (x^2+4x+5)a_{12} - (x-1)^2(b_{11}x + c_{11})}{(x-1)^2(x^2+4x+5)} \quad \cdot (x-1)$

$\rightarrow \quad a_{11}(x-1)(x^2+4x+5) = \underbrace{1 - \frac{1}{10}x^2 - \frac{2}{5}x - \frac{1}{2}}_{\substack{\text{ist} = 0 \text{ für } x=1 \\ = (x-1)(--)}} - (x-1)^2(b_{11}x + c_{11}) \quad : x-1$ und $x=1$

$\rightarrow \quad a_{11} = -\frac{3}{50}$

Algorithmus wird beim nächsten mal "sauber" nachgeliefert

Zsgf:

$$\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = -\frac{3}{50}\ln|x-1| - \frac{1}{10}\frac{1}{x-1} + \int \frac{\frac{3}{50}x + \frac{1}{5}}{x^2+4x+5}\,dx$$

$$\bigcirc = \frac{3}{100}\ln(x^2+4x+5) + \frac{4}{50}\boxed{\int \frac{1}{x^2+4x+5}\,dx}$$

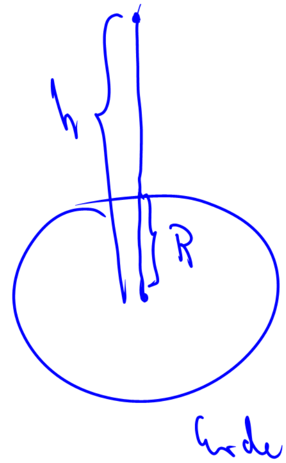
$$\square = \int \frac{1}{(x+2)^2+1}\,dx = \arctan(x+2) + c, \quad \text{also}$$

$$\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = -\frac{3}{50}\ln|x-1| - \frac{1}{10}\frac{1}{x-1} + \frac{3}{100}\ln(x^2+4x+5) + \frac{2}{25}\arctan(x+2) + c.$$

Uneigentliche Integrale

<u>Motivation</u>: Abschuss einer Rakete aus dem Schwerefeld der Erde

Frage: Welche Arbeit ist zu verrichten?



Newton'sches Gravitationsgesetz

$$\text{Arbeit} = \int_R^h G\,\frac{mM}{x^2}\,dx$$

m Raketenmasse

M Erdmasse

G Gravitationskonstante

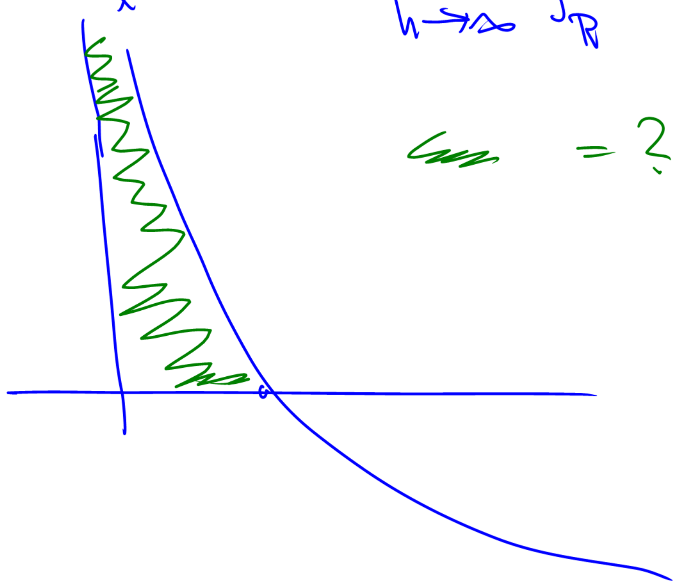
$$\underbrace{\lim_{h\to\infty}\int_R^h G\,\frac{mM}{x^2}\,dx}_{} = ?$$

$$GmM\left(-\frac{1}{x}\right)\Big|_R^h = GmM\left(\frac{1}{R}-\frac{1}{h}\right) \xrightarrow{h\to\infty} GmM\,\frac{1}{R}$$

$$\int_R^\infty G\,\frac{mM}{x^2}\,dx := \lim_{h\to\infty}\int_R^h G\,\frac{mM}{x^2}\,dx$$

(Bsp für ein uneigentliches Integral).

Bsp



$$= ? = \int_0^1 -\ln x\,dx$$

$$= -\lim_{\varepsilon\to 0}\; x\ln x - x\,\Big|_\varepsilon^1 = 1$$

graph $(-\ln x)$

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Integrand nicht echt gebrochen rational $\Rightarrow$ Polynomdivision

$$x^4+0x^3+0x^2+0x+2 : (x^2+2x-1) = \underbrace{x^2-2x+5}_{S_2(x)} + \underbrace{\frac{-12x+7}{x^2+2x-1}}_{r(x)}$$

$$\begin{array}{l} -(x^4+2x^3-x^2) \\ \hline 0 \quad -2x^3+x^2+0x \\ \quad -(-2x^3-4x^2+2x) \\ \hline \quad 0 \quad 5x^2-2x+2 \\ \qquad -(5x^2+10x-5) \\ \hline \qquad 0 \quad -12x+7 \end{array}$$

Damit

$$\int \frac{x^4+2}{x^2+2x-1}\,dx = \frac{1}{3}x^3 - x^2 + 5x + \int \frac{-12x+7}{x^2+2x-1}\,dx$$

$$\int \frac{-12x+7}{x^2+2x-1}\,dx = -6 \overbrace{\int \frac{2x+2}{x^2+2x-1}}^{\text{„}\int \frac{f'}{f}\text{"}}\,dx + 19\int \frac{1}{x^2+2x-1}\,dx$$

$$= -6\ln|x^2+2x-1| + 19\int \frac{1}{x^2+2x-1}\,dx$$

$\int \frac{1}{x^2+2x-1}\,dx = ?$ PBZ: bestimme Nullstellen von x^2+2x-1. Diese sind $x_{1/2} = -1 \pm \sqrt{2}$

Damit $\frac{1}{x^2+2x-1} \overset{PBZ}{=} \frac{a_{11}}{x+1-\sqrt{2}} + \frac{a_{21}}{x+1+\sqrt{2}} \longrightarrow a_{11} = \frac{1}{2\sqrt{2}}, \; a_{21} = -\frac{1}{2\sqrt{2}}$

Insgesamt ergibt sich

$$\int \frac{x^4+2}{x^2+2x-1}\,dx = \frac{1}{3}x^3 - x^2 + 5x - 6\ln|x^2+2x-1| + \frac{19}{2\sqrt{2}}\ln|x+1-\sqrt{2}| - \frac{19}{2\sqrt{2}}\ln|x+1+\sqrt{2}| + C$$

Weiteres Bsp mit Nennerpolynom besitzt komplexe und reelle Nullstellen

$$\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = ?$$

Nullstellen von $\underbrace{x^4+2x^3-2x^2-6x+5}_{}$ $x=1$ Nullstelle (doppelt)

$$= (x-1)^2(x^2+4x+5)$$

Damit PBZ gegeben durch

Bestimme $a_{11}, a_{12}, b_{11}, c_{11}$

$$\frac{1}{\underbrace{x^4+2x^3-2x^2-6x+5}_{(x-1)^2(x^2+4x+5)}} = \frac{a_{11}}{x-1} + \frac{a_{12}}{(x-1)^2} + \frac{b_{11}x+c_{11}}{x^2+4x+5} \quad \cdot (x-1)^2(x^2+4x+5)$$

$$\longrightarrow \quad 1 = a_{11}(x-1)(x^2+4x+5) + a_{12}(x^2+4x+5) + (b_{11}x+c_{11})(x-1)^2$$

$a_{12} = \frac{1}{10}$ (setze $x=1$ ein!) $\longrightarrow 1 - \frac{1}{10}(x^2+4x+5)$

$$= a_{11}(x-1)(x^2+4x+5) + (b_{11}x+c_{11})(x-1)^2$$

Koeffizientenvergleich $a_{11} = -\frac{3}{50}, \; b_{11} = \frac{3}{50}, \; c_{11} = \frac{1}{5}$

Damit

$$\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = -\frac{3}{50}\ln|x-1| - \frac{1}{10}\frac{1}{x-1} + \int \frac{\frac{3}{50}x + \frac{1}{5}}{x^2+4x+5}\,dx$$

$$\bigcirc = \frac{3}{100}\int \underbrace{\frac{2x+4}{x^2+4x+5}}_{\text{"}\int f'/f\text{"}}\,dx + \frac{2}{25}\int \frac{1}{x^2+4x+5}\,dx$$

$$= \frac{3}{100}\ln(x^2+4x+5) + \frac{2}{25}\arctan(x+2) + C_1 .$$

Insgesamt

$$\int \frac{1}{x^4+2x^3-2x^2-6x+5}\,dx = -\frac{3}{50}\ln|x-1| - \frac{1}{10}\frac{1}{x-1} + \frac{3}{100}\ln(x^2+4x+5) + \frac{2}{25}\arctan(x+2) + C_1$$

Uneigentliche Integrale

<u>Motivation</u>: Rakete aus dem Schwerefeld der Erde heraustransportieren $\longrightarrow$ Arbeit ?

Mit Hilfe des Newton'schen Gravitationsgesetzes ergibt sich die Arbeit für den Transport bis zur Höhe h über dem Erdmittelpunkt zu

$$\int_R^h G\,\frac{mM}{x^2}\,dx$$

mit
- m Raketenmasse
- M Erdmasse
- G Gravitationskonstante

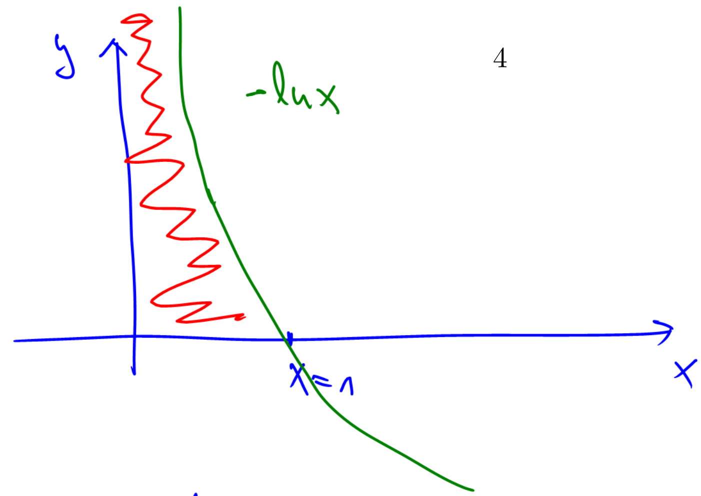
$$\lim_{h\to\infty} \underbrace{\int_R^h G\,\frac{mM}{x^2}\,dx}_{\underbrace{GmM\left(-\frac{1}{x}\right)\Big|_R^h}_{GmM\left(\frac{1}{R}-\frac{1}{h}\right)}} = ?$$

$$GmM\left(\frac{1}{R}-\frac{1}{h}\right) \xrightarrow{h\to\infty} GmM\,\frac{1}{R}$$

Definition

$$\int_R^\infty GmM\,\frac{1}{x^2}\,dx := \lim_{h\to\infty}\int_R^h GmM\,\frac{1}{x^2}\,dx$$

Bsp eines uneigentlichen Integrals



$$? = \int_0^1 (-\ln x)\,dx$$

$$\text{"} = -(x\ln x - x)\Big|_0^1 \text{"}$$

Betrachte statt dessen

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1(-\ln x)\,dx = -\lim_{\varepsilon\to 0}(x\ln x - x)\Big|_\varepsilon^1 = 1 + \underbrace{\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon}_{=0} - \lim_{\varepsilon\to 0}\varepsilon$$

$$= 1$$

Bsp

$$\int_1^\infty \frac{1}{x^s}\,dx = ? \qquad (s>1)$$

$$\shortparallel$$

$$\lim_{R\to\infty}\int_1^R \frac{1}{x^s}\,dx = \lim_{R\to\infty}\frac{1}{1-s}x^{1-s}\Big|_1^R = \lim_{R\to\infty}\left(-\frac{1}{1-s}+\frac{1}{1-s}R^{1-s}\right)$$

$$= \begin{cases} -\frac{1}{1-s}, & s>1 \\ \text{divergent}, & s<1 \end{cases}$$