ANALYSIS III

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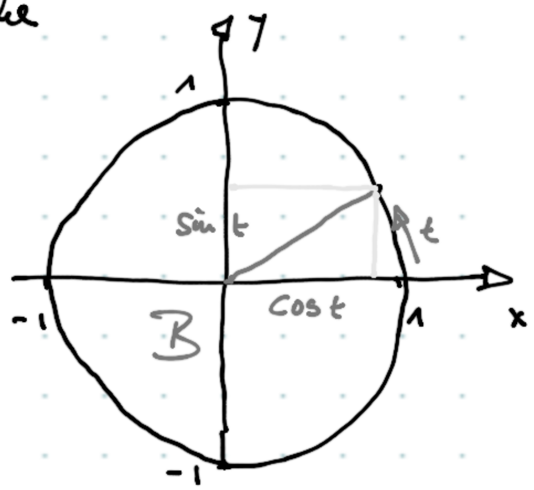
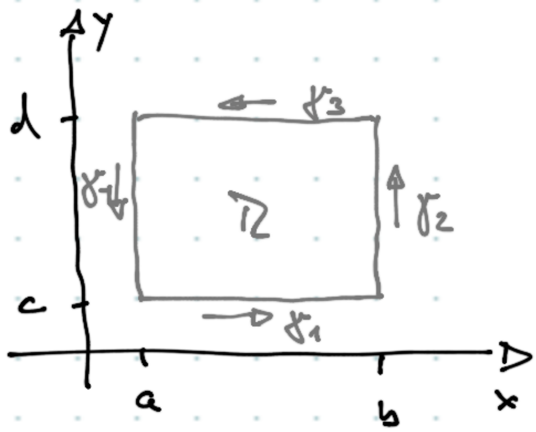
① Beispiele für orientierte Randbereiche

a) Einheitskreis

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$$



b) Rechteck $R = [a,b] \times [c,d]$



$$\gamma = [\gamma_1, \gamma_2, \gamma_3, \gamma_4]$$

$$\gamma_1(t) = \begin{pmatrix} t \\ c \end{pmatrix} \quad t \in [a,b]$$

$$\gamma_2(t) = \begin{pmatrix} b \\ t \end{pmatrix} \quad t \in [c,d]$$

$$\gamma_3(t) = \begin{pmatrix} b+a-t \\ d \end{pmatrix} \quad t \in [a,b]$$

$$\gamma_4(t) = \begin{pmatrix} a \\ c+d-t \end{pmatrix} \quad t \in [c,d]$$

② Herleitung des Satzes von Green

- Betrachte: stetig diff'baren Vektorfeld $\vec{v} : \underset{\text{Rechteck}}{R} \to \mathbb{R}^2$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ Komponentenfkt.}$$

- Ableitungen der Randkurven: Erinnerung $\dot{\gamma} = \frac{d\gamma}{dt}$

$$\dot{\gamma}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \dot{\gamma}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \dot{\gamma}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \; \dot{\gamma}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

- Kurvenintegral:

$$\underline{\int_\gamma \vec{v} \cdot \vec{dx}} = \sum_{j=1}^{4} \int_{\gamma_j} \vec{v} \cdot \vec{dx}$$

$$= \int_a^b v_1(x,c)\,dx + \int_c^d v_2(b,y)\,dy - \int_a^b v_1(x,d)\,dx - \int_c^d v_2(a,y)\,dy$$

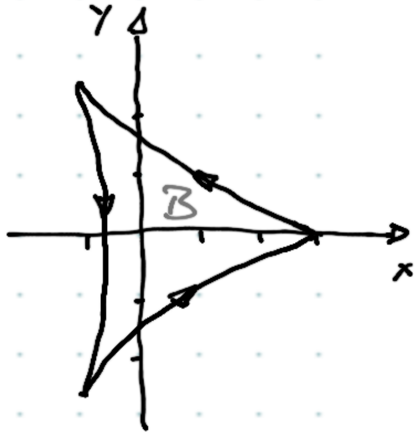
$$= \int_a^b [v_1(x,c) - v_1(x,d)]\,dx + \int_c^d [v_2(b,y) - v_2(a,y)]\,dy$$

$$= -\int_a^b \left[\int_c^d \frac{\partial v_1(x,y)}{\partial y}\,dy\right]dx + \int_c^d \left[\int_a^b \frac{\partial v_2(x,y)}{\partial x}\,dx\right]dy$$

$$= \underline{\int_R \left[\frac{\partial v_2(x,y)}{\partial x} - \frac{\partial v_1(x,y)}{\partial y}\right] dF}$$

③ <u>Beispiel</u> für Anwendung Satz von Green

- Berechne Flächeninhalt des Bereiches B, der durch eine Hypozykloide berandet ist



$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2\cos t + \cos(2t) \\ 2\sin t - \sin(2t) \end{pmatrix}$$

$t \in [0, 2\pi]$ positiv orientiert

- Führe $\vec{v}(x,y) = \frac{1}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$ ein

- Es gilt: $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 1$

$$\dot{\gamma}(t) = \begin{pmatrix} -2(\sin t + \sin(2t)) \\ 2(\cos t - \cos(2t)) \end{pmatrix}$$

- Anwendung Greenscher Satz:

$$F(B) = \int_B dF = \int_B \left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right] dF$$

$$= \int_{\partial B} \vec{v} \cdot d\vec{x} = \frac{1}{2}\int_0^{2\pi} [-y(t)\,\dot{x}(t) + x(t)\cdot\dot{y}(t)]\, dt$$

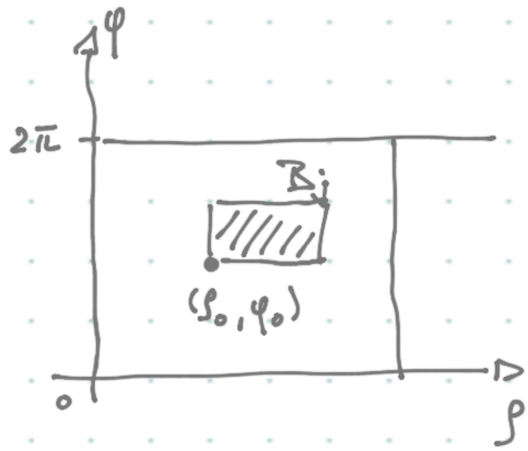
$$= \frac{1}{2}\int_0^{2\pi} \Big[-(2\sin t - \sin(2t))(-2(\sin t + \sin(2t))) + (2\cos t + \cos(2t))(2(\cos t - \cos(2t)))\Big]\, dt$$

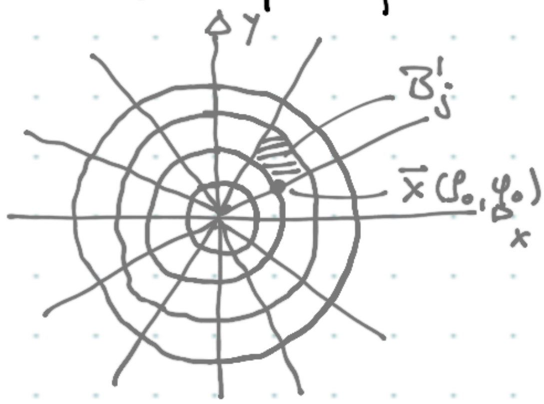
$$= \frac{1}{2}\int_0^{2\pi} [-8\cos^3 t + 6\cos t + 2]\, dt = 2\pi$$

④ **Beispiel** Kartesische → Polar-Koordinaten

- Transformation: $\vec{x}(\rho,\varphi) = \begin{pmatrix} x(\rho,\varphi) \\ y(\rho,\varphi) \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix}$

$\rho > 0, \quad 0 \le \varphi \le 2\pi$





- Determinante: $\dfrac{\partial(x,y)}{\partial(\rho,\varphi)} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & -\rho\cos\varphi \end{vmatrix} = \rho$

- Transformationsregel:

$$\int_{B'} f \, dF = \int_{B'} f(x,y)\,dx\,dy = \int_{B} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho\,d\varphi$$