Chapter 3. Integration over general areas

3.2 Line integrals

We already had a defintion of a line integral of a scalar field for a piecewise C^1–curve $c : [a, b] \to D$, $D \subset \mathbb{R}^n$, and a continuous scalar function $f : D \to \mathbb{R}$

$$\int_c f(x)\, ds := \int_a^b f(c(t)) \|\dot{c}(t)\|\, dt$$

where $\|\cdot\|$ denotes the Euklidian norm.

Generalisation: Line integrals of vector valued functions, i.e.

$$\int_c f(x) dx := ?$$

Application: A point mass is moving along $c(t)$ in a force field $f(x)$.

Question: How much physical work has to be done along the curve?

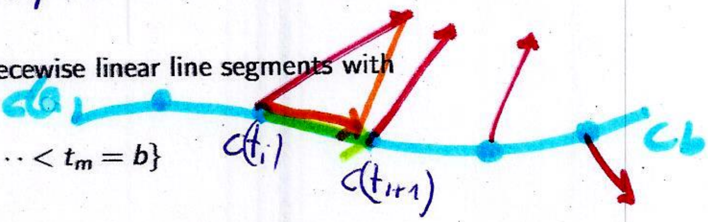
Work = force x path

Line integral on vector fields.

Definition: For a continuous vector field $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ open, and a piecewise C^1–curve $c : [a, b] \to D$ we define the line integral on vector fields by

$$\int_c f(x) dx := \int_a^b \langle f(c(t)), \dot{c}(t)\rangle\, dt$$

Derivation: Approximate the curve by piecewise linear line segments with corners $c(t_i)$, where



$$Z = \{a = t_0 < t_1 < \cdots < t_m = b\}$$

is a partition of the interval $[a, b]$.

Then the workload along the curve $c(t)$ in the force field $f(x)$ is approximately given by :

$$A \approx \sum_{i=0}^{m-1} \langle f(c(t_i)), c(t_{i+1}) - c(t_i)\rangle$$

Continuation of the derivation.

Thus:

components   partition

$$A \approx \sum_{j=1}^{n}\sum_{i=0}^{m-1} f_j(c(t_i))\underbrace{(c_j(t_{i+1}) - c_j(t_i))}_{t_{i+1}-t_i}\ (t_{i+1}-t_i)$$

$\tau_{ij} \in [t_i, t_{i+1}]$

$$= \sum_{j=1}^{n}\sum_{i=0}^{m-1} f_j(c(t_i))\dot{c}_j(\tau_{ij})(t_{i+1} - t_i)$$

$\int_a^b \langle f(c(t)), \dot{c}(t)\rangle \, dt$

For a sequence of partitions Z with $\|Z\| \to 0$ the left side converges to the above defined line integral on vector fields.

Remarks: For a closed curve $c(t)$, i.e. $c(a) = c(b)$, we use the notation

$$\oint_c f(x)\,dx$$

Properties of the line integral on vector fields.

- Linearity:

$$\int_c (\alpha f(x) + \beta g(x))\,dx = \alpha \int_c f(x)\,dx + \beta \int_c g(x)\,dx$$

$t: a \to b$

$c(t): c(a) \to c(b)$

$(-c)(t): c(b) \to c(a)$

$\dot{c}(t) = -(-c)^{\cdot}(t)$

- It is:

$$\int_{-c} f(x)\,dx = -\int_c f(x)\,dx,$$

where $(-c)(t) := c(b + a - t)$, $a \le t \le b$, denotes the inverted path.

- It is

$$\int_{c_1+c_2} f(x)\,dx = \int_{c_1} f(x)\,dx + \int_{c_2} f(x)\,dx$$

where $c_1 + c_2$ denotes the path composed by c_1 and c_2 such that the end point of c_1 coincides with the starting point of c_2.

Further properties of the line integral on vector fields.

- The line integral on vector fields is invariant under paramterisation.
- It is

$$\int_c \mathrm{f}(\mathrm{x})\,d\mathrm{x} = \int_a^b \langle \mathrm{f}(\mathrm{c}(t)), \mathrm{T}(t)\rangle\, \|\dot{\mathrm{c}}(t)\|\, dt = \int_c \langle \mathrm{f}, \mathrm{T}\rangle\, ds$$

with the tangent unit vector $\mathrm{T}(t) := \dfrac{\dot{\mathrm{c}}(t)}{\|\dot{\mathrm{c}}(t)\|}$.

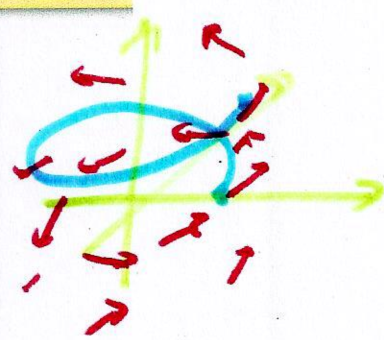
- Formal notation:

$$\int_c \mathrm{f}(\mathrm{x})\,d\mathrm{x} = \int_c \sum_{i=1}^n f_i(\mathrm{x})\,d\mathrm{x}_i = \sum_{i=1}^n \int_c f_i(\mathrm{x})\,d\mathrm{x}_i$$

with

$$\int_c f_i(\mathrm{x})\,d\mathrm{x}_i := \int_a^b f_i(\mathrm{c}(t))\dot{c}_i(t)\,dt$$

Example.



Let $\mathrm{x} \in \mathbb{R}^3$ and

$$\begin{aligned} \mathrm{f}(\mathrm{x}) &:= (-y, x, z^2)^T \\ \mathrm{c}(t) &:= (\cos t, \sin t, at)^T \qquad \text{with } 0 \le t \le 2\pi \end{aligned}$$

We calculate

$$\begin{aligned} \int_c \mathrm{f}(\mathrm{x})\,d\mathrm{x} &= \int_c (-ydx + xdy + z^2dz) \\ &= \int_0^{2\pi} (\underbrace{(-\sin t)(-\sin t) + \cos t\cos t}_{1} + a^2t^2a)\,dt \\ &= \int_0^{2\pi} (1 + a^3t^2)\,dt \\ &= 2\pi + \frac{a^3}{3}(2\pi)^3 \end{aligned}$$

$$\int f(x)\,dx = \int_a^b \langle f(c(t)), \dot{c}(t)\rangle\, dt = \left[\begin{array}{l} t = \varphi(s) \\ [a,b] \to [\varphi^{-1}(a), \varphi^{-1}(b)] \\ \frac{dt}{ds} = \varphi'(s) \\ c(t) = c(\varphi(s)) = \tilde{c}(s) \end{array}\right.$$

$$= \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} \langle f(c(\varphi(s))), \frac{d}{dt}\underbrace{c(\varphi(s))}_{t}\rangle\, \varphi'(s)\, ds$$

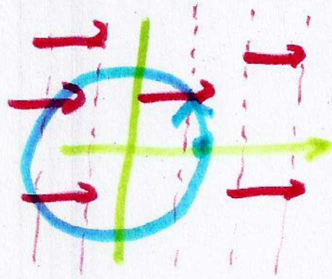
$$= \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} \langle f(\tilde{c}(s)), \frac{d}{ds}\underbrace{c(\varphi(s))}_{\tilde{c}(s)}\rangle\, ds = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} \langle f(\tilde{c}(s)), \tilde{c}'(s)\rangle\, ds$$

Ex1

$$f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \nabla x \qquad c(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$$

$\varphi(x) = x$

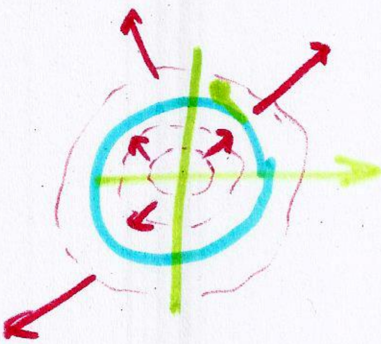
$$\dot{c}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$



$$\int_C f(x)\,dx = \int_0^{2\pi} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle dt = \int_0^{2\pi} -\sin t \, dt = 0$$

Ex2

$$f = \begin{pmatrix} x \\ y \end{pmatrix} = \nabla\left(\frac{x^2+y^2}{2}\right) \qquad \varphi(x,y) = \frac{x^2+y^2}{2}$$

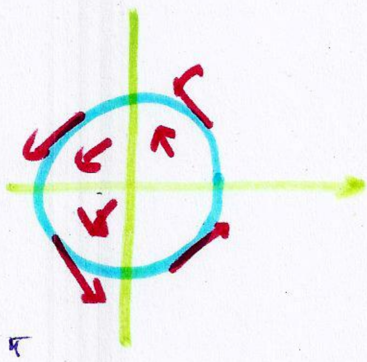


$$\int_C f(x)\,dx = \int_0^{2\pi} \underbrace{\left\langle \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle}_{0} dt = 0$$

Ex3

$$f = \begin{pmatrix} -y \\ x \end{pmatrix} = ?$$

no potential



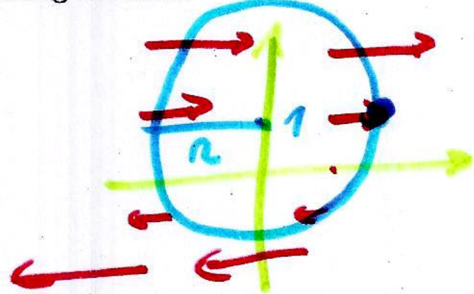
$$\int_C f(x)\,dx = \int_0^{2\pi} \underbrace{\left\langle \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle}_{1} dt = 2\pi$$

not curl free

The circulation of a field along a curve.

Definition: Let u(x) be the velocity field of a moving fluid. We call the line integral $\oint_c$ u(x)dx along a closed curve the circulation of the field u(x). relevant to c.

Example: For the field $u(x,y) = (y,0)^T \in \mathbb{R}^2$ we obtain along the curve $c(t) = (r\cos t, 1 + r\sin t)^T$, $0 \le t \le 2\pi$ the circulation



$$\int_0^{2\pi} \left\langle \begin{pmatrix} 1+r\sin t \\ 0 \end{pmatrix}, \begin{pmatrix} -r\sin t \\ \cdot \end{pmatrix} \right\rangle = \oint_c u(x)\,dx = \int_0^{2\pi} (1 + r\sin t)(-r\sin t)dt$$

$$= \int_0^{2\pi} (-r\sin t - r^2 \sin^2 t)dt$$

$$= \left[r\cos t - \frac{r^2}{2}(t - \sin t\cos t) \right]_0^{2\pi} = -\pi r^2$$

Curl free vector fields.

Definition: A continuous vector field f(x), $x \in D \subset \mathbb{R}^n$, is called curl free if the line integral along all closed and piecewise C^1–curves $c(t)$ in D vanishes, i.e.

$$\oint_c f(x)\,dx = 0 \qquad \text{for all closed c.}$$

depends only on f!
not on c!

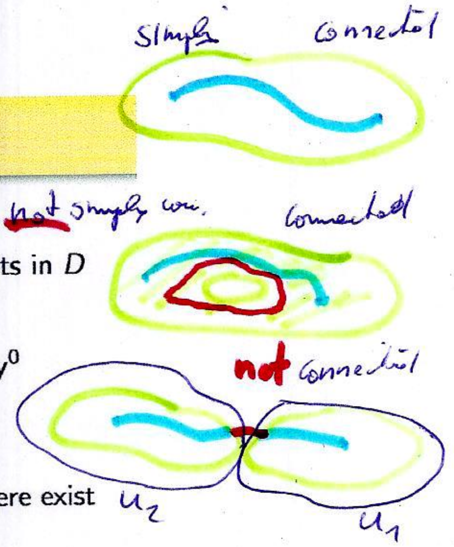
Remark: A vector field is curl free if an only if the value of the line integral $\int_c$ f(x)dx depends only from the starting and the end point of the path, but not on the specific path c. In this case we call the line integral path independent.

not path independent

$$0 \neq \int_{c_1} f(x)dx - \int_{c_2} f(x)dx = \int_{c_1} + \int_{-c_2} = \oint_{c_1 - c_2} f(x)dx$$

Question: Which criteria on the vector field f(x) guarantee the path independency of the line integral?

Connected sets.



Definition: A subset $D \subset \mathbb{R}^n$ is called connected, if any two points in D can be connected by a piecewise C^1–curve: in D

$$\forall x^0, y^0 \in D \; : \; \exists c : [a,b] \to D \quad : \quad c(a) = x^0 \wedge c(b) = y^0$$

An open and connected set $D \subset \mathbb{R}^n$ is called domain in $\mathbb{R}^n$.

Remark: An **open** set $D \subset \mathbb{R}^n$ is **not** connected if and only if there exist **disjoint** and open sets $U_1, U_2 \subset \mathbb{R}^n$ with

$$U_1 \cap D \neq \emptyset, \quad U_2 \cap D \neq \emptyset, \quad D \subset U_1 \cup U_2$$

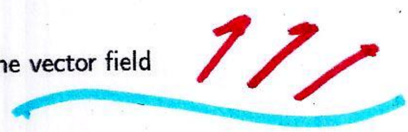
Not connected sets are – in contrary to connected sets – a separable in at least two disjoint open sets.

simply connected: you can shrink any closed curve in D to a point

Gradient fields, antiderivatives, potentials.

Definition: Let $f : D \to \mathbb{R}^n$ be a vector field on a domain $D \subset \mathbb{R}^n$. The vector field is called gradient field, if there is a scalar C^1–function $\varphi : D \to \mathbb{R}$ with

$$f(x) = \nabla\varphi(x)$$



The function $\varphi(x)$ is called antiderivative or potential of $f(x)$, and the vector field $f(x)$ is called conservativ.

Remark: Suppose a mass point is moving in a conservative force field $K(x)$, i.e. K has a potential $\varphi(x)$ such that $K(x) = \nabla\varphi(x)$. The the function $U(x) = -\varphi(x)$ gives the potential energy:

$m\ddot{x} + \nabla U = 0 \;|\, \dot{x}$

$\underbrace{m\langle \ddot{x}, \dot{x}\rangle}_{\frac{d}{dt}\frac{\langle \dot{x},\dot{x}\rangle}{2}} + \underbrace{\langle \nabla U, \dot{x}\rangle}_{\frac{dU}{dt}} = 0$

force, mass × accell.

$$\Leftarrow \quad K(x) \underset{\text{Newton}}{=} m\ddot{x} = -\nabla U(x) \; = \nabla\varphi(x)$$

Multiplying this relation with $\dot{x}$ we obtain

kinetic energy, potential

$$\Longrightarrow \quad m\langle \ddot{x}, \dot{x}\rangle + \langle \nabla U(x), \dot{x}\rangle = \frac{d}{dt}\underbrace{\left(\frac{1}{2}m\|\dot{x}\|^2 + U(x)\right)}_{\text{total energy}} = 0$$

Fundamental theorem on line integrals.

Theorem: (Fundamental theorem on line integrals)
Let $D \subset \mathbb{R}^n$ be a domain and f(x) a continuous vector field on D.

1) If f(x) has a potential $\varphi(x)$, then for all piecewise C^1–curves $c : [a, b] \to D$ we have:

$$\boxed{\int_c f(x)\, dx = \varphi(c(b)) - \varphi(c(a))}$$

In particular the line integral is path independent and f(x) is curl free.

2) In the opposite direction we have: If f(x) is curl free, then f(x) has a potential $\varphi(x)$.
Let $x^0 \in D$ be a fixed point and c_x (for $x \in D$) denotes an arbitrary piecewise C^1–curve in D connecting the points x^0 and x, then $\varphi(x)$ is given by:

$$\varphi(x) = \int_{c_x} f(x)\, dx + \text{const.}$$

Example I.

Gravitation: $K(x) = -c \frac{1}{r^2} u$

Coulomb: $K(x) = \pm c \frac{1}{r^2} u$

The central force field

repulsive

$$K(x) := \frac{x}{\|x\|^3} = \frac{1}{\|x\|^2} \cdot \underbrace{\frac{x}{\|x\|}} = \frac{1}{r^2} \cdot u$$

has the potential

$$U(x) \overset{= \frac{1}{r}}{=} -\frac{1}{\|x\|} = -(x_1^2 + x_2^2 + x_3^2)^{-1/2}$$

since

$$\nabla U(x) = (x_1^2 + x_2^2 + x_3^2)^{-3/2}(x, y, z)^T = \frac{x}{\|x\|^3}$$

The workload along a piecewise C^1–curve $c : [a, b] \to \mathbb{R}^3 \setminus \{0\}$ is given by

$$A = \int_c K(x)\, dx = \left(\frac{1}{\|c(a)\|} - \frac{1}{\|c(b)\|} \right)$$

Example II.

The vector field

$$f(x) := \begin{pmatrix} 2xy + z^3 \\ x^2 + 3z \\ 3xz^2 + 3y \end{pmatrix} = \nabla\varphi(x)$$

has the potential

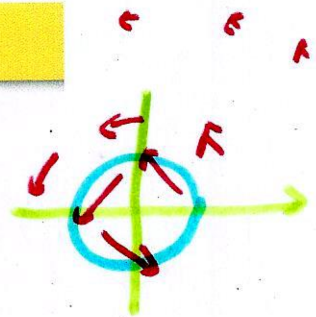
$$\varphi(x) = x^2y + xz^3 + 3yz$$

For an arbitrary C^1–curve $c(t)$ from $P = (1, 1, 2)$ to $Q = (3, 5, -2)$ we have

$$\int_c f(x)\,dx = \varphi(Q) - \varphi(P) = -9 - 15 = -24$$

If we interpret $f(x)$ as electrical field, then the line integral on vector fields represents the electrical voltage between the two points P and Q.

Example III.



Consider the vector field

$$f(x, y) = \frac{1}{x^2 + y^2}\begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{mit } (x, y)^T \in D = \mathbb{R}^2 \setminus \{0\}$$

For the unit sphere $c(t) := (\cos t, \sin t)^T$, $0 \le t \le 2\pi$, we obtain

$$\begin{aligned} \int_c f(x)\,dx &= \int_0^{2\pi} \langle f(c(t)), \dot{c}(t)\rangle\,dt \\ &= \int_0^{2\pi} \underbrace{\left\langle \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle}_{1} dt \\ &= \int_0^{2\pi} 1\,dt = 2\pi \end{aligned}$$

$f(x, y)$ is therefore not curl free and has no potential on D.