Requirements for potentials.

"curl" ≙ "rot"

$\text{curl} f = \nabla \times f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = (*)$

$= \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix}$

Remark: If f(x), $x \in D \subset \mathbb{R}^3$ is a C^1–vector field with potential $\varphi(x)$, then

$$\text{curl f(x)} = \text{curl}\,(\nabla\varphi(x)) = 0 \qquad \text{für alle } x \in D$$

Thus curl f(x) = 0 is a necessary condition for the existence of a potential.

If we define for a vector field $f : D \to \mathbb{R}^2$, $D \subset \mathbb{R}^2$, the **scalar** curl

$$\text{curl f}(x,y) := \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y) = \partial_x \overbrace{\partial_y \varphi}^{f_2} - \partial_y \overbrace{\partial_x \varphi}^{f_1} = 0$$

then curl f(x, y) = 0 is a necessary condition even in 2 dimensions.

The condition

$$\text{curl f(x)} = 0$$

is a sufficient condition, if the domain D is **simply connected**, i.e. if D has no "holes".

$(*) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ \partial_1\varphi & \partial_2\varphi & \partial_3\varphi \end{vmatrix} = \begin{pmatrix} \partial_2\partial_3 f - \partial_3\partial_2 f \\ \partial_3\partial_1 f - \partial_1\partial_3 f \\ \partial_1\partial_2 f - \partial_2\partial_1 f \end{pmatrix} = 0$

$f = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \Rightarrow \tilde{f}(x,y,z) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ 0 \end{pmatrix}$

$\text{curl}\,\tilde{f} = \begin{pmatrix} 0 \\ 0 \\ \boxed{\partial_x f_2 - \partial_y f_1} \end{pmatrix}$

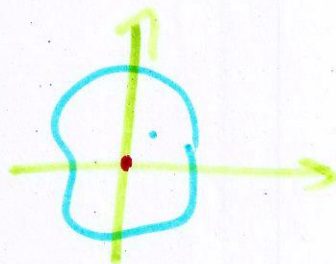
$\nabla\varphi = f \quad C^2$

$\varphi \quad C^1$

Example.

We consider the vector field

$$f(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{with } (x,y)^T \in D = \mathbb{R}^2 \setminus \{0\}$$



Calculating the curl gives

$$\begin{aligned} \text{curl}\left[\frac{1}{r^2}\begin{pmatrix} -y \\ x \end{pmatrix}\right] &= \frac{\partial}{\partial x}\underbrace{\left(\frac{x}{x^2+y^2}\right)}_{f_2} + \frac{\partial}{\partial x}\underbrace{\left(\frac{y}{x^2+y^2}\right)}_{-f_1} \\ &= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \\ &= 0 \end{aligned}$$

The curl of f(x, y) vanishes.
But f(x, y) has on the set $D = \mathbb{R}^2 \setminus \{0\}$ no potential.
The domain is not simply connected.

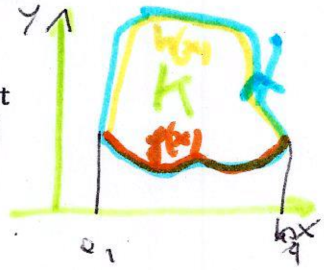
The integral theorem of Green for vector fields in $\mathbb{R}^2$.

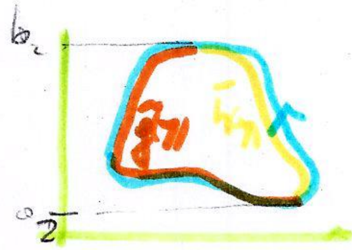
Theorem: (Integral theorem of Green)

Let f(x) be a C^1–vector field on a domain $D \subset \mathbb{R}^2$. Let $K \subset D$ be compact and projectable with respect to both coordinates, such that K is bounded by a closed and piecewise C^1–curve c(t).

The parameterisation of c(t) is chosen such that K is always on the left when going along the curve with increasing parameter (positive circulation). Then:

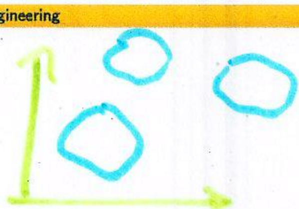
$$\boxed{\oint_c \mathrm{f}(x)\,dx = \int_K \mathrm{curl}\,\mathrm{f}(x)\,dx}$$



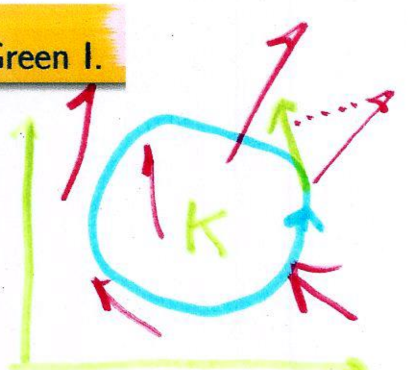


Remark:

The integral theorem is also valid for domains which can be splittet in *finite* many domains which all are projectable with respect to both coordinate directions, so called Green domains.



Alternative formulation of the integral theorem of Green I.



We have seen that the relation

$$\oint_c \mathrm{f}(x)\,dx = \oint_c \langle \mathrm{f}, \mathrm{T}\rangle\, ds$$

holds, where $\mathrm{T}(t) = \frac{\dot{c}(t)}{\|\dot{c}(t)\|}$ denotes the tangent unit vector.

With the intergral thoerem of Green we obtain

$$\int_K \mathrm{curl}\,\mathrm{f}(x)\,dx = \oint_{\partial K} \langle \mathrm{f}, \mathrm{T}\rangle\, ds$$

Is f(x) a velocity field, then the fluid motion described by f is curl free if curl f(x) = 0, since

$$\oint_c \mathrm{f}(x)dx$$

is the circulation of f(x).

$\operatorname{div} f = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i$

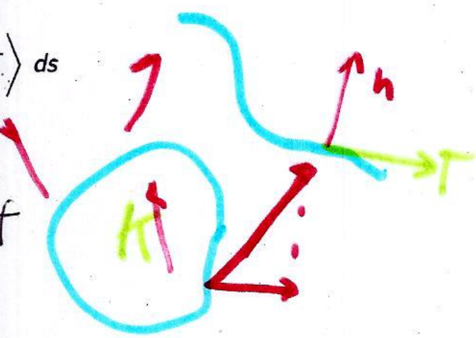
Alternative formulation of the integral theorem of Green II.

If we substitute in the above equations the vector T by the outer normal vector $n = (T_2, -T_1)^T$, we obtain $n^T \cdot T = \langle n, T \rangle = 0$

$T = (T_1, T_2)$

$$\oint_{\partial K} \langle f, n \rangle\, ds = \oint_{\partial K} (f_1 T_2 - f_2 T_1) ds = \oint_{\partial K} \left\langle \underbrace{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}}_{g}, T \right\rangle ds$$

$$\overset{\text{Green}}{=} \int_K \text{curl}\, \cancel{\text{rot}} \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} dx = \int_K \operatorname{div} f\, dx$$



and thus the relation $\quad \text{curl}\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} = \partial_x f_1 - \partial_y(-f_2) = \operatorname{div} f$

$$\int_K \operatorname{div} f(x)\, dx = \oint_{\partial K} \langle f, n \rangle\, ds$$

If f(x) is the velocity field of a fluid motion, then the right side describes describes the total flow of the fluid through the boundary of K. Therefore if $\operatorname{div} f(x) = 0$, then the fluid motion is is source and sink free (or divergence free).

$\operatorname{div} f < 0$ in K Sink
$\operatorname{div} f = 0$ in K no total outflow
$\operatorname{div} f > 0$ in K Source

Back again to the existence of potentials.

Conclusion: If $\operatorname{curl} f(x) = 0$ for all $x \in D$, $D \subset \mathbb{R}^2$ a domain, then we have

$$\oint_c f(x)\, dx = 0$$

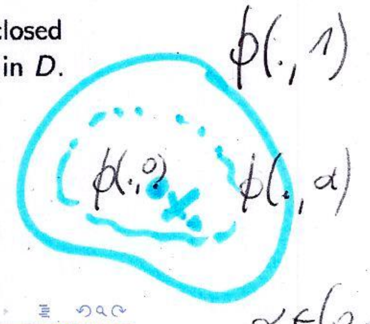
Simply connected

for every closed piecewise C^1–curve, which surounds a Green domain $B \subset D$ completely.

not S.C.

Definition: A domain $D \subset \mathbb{R}^n$ is called simply connected, if any closed curve $c : [a, b] \to D$ can be shrinked continuously in D to a point in D. More precise: There is a continuous map for $x^0 \in D$

$$\Phi : [a, b] \times [0, 1] \to D$$



with $\Phi(t, 0) = c(t)$, for all $t \in [a, b]$ and $\Phi(t, 1) = x^0 \in D$, for all $t \in [a, b]$. The map $\Phi(t, s)$ is called a homotopy.

$\alpha \in (0,1)$

Criteria for integrability for potentials.

Theorem: Let $D \subset \mathbb{R}^n$ be a simply connected domain. A C^1–vector field $f : D \to \mathbb{R}^n$ has a potential on D if and only if the integrability criteria

$$J\,f(x) = (J\,f(x))^T \qquad \text{for all } x \in D$$

are satisfied, i.e. if

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k} \qquad \forall j, k$$

$$n=3 \quad \text{curl} f = \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 \partial_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix} = 0$$

Remark: For $n = 2, 3$ the integrability criteria coincide with

$$\operatorname{rot} f(x) = 0$$

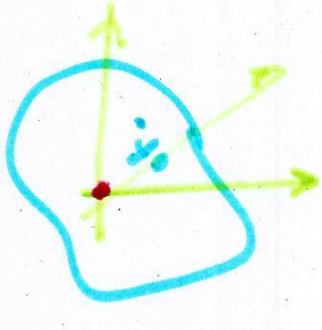
$$n=2 \quad \text{curl} f = \partial_1 f_2 - \partial_2 f_1 = 0$$

$$Jf = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{pmatrix}$$

Example.

For $x \in \mathbb{R}^3 \setminus \{0\}$ let the vector field be

$$f(x) = \begin{pmatrix} \dfrac{2xy}{r^2} + \sin z \\ \ln r^2 + \dfrac{2y^2}{r^2} + z e^y \\ \dfrac{2yz}{r^2} + e^y + x \cos z \end{pmatrix} \qquad \text{with } r^2 = x^2 + y^2 + z^2.$$



We would like to study the existence of a potential for $f(x)$.

The set $D = \mathbb{R}^3 \setminus \{0\}$ is apparentely simply connected. In addition we have

$$\text{curl}\, f(x) = 0$$

Thus $f(x)$ has a potential.

Calculation of the potential.

We need to have: $\boxed{f(x) = \nabla\varphi(x)}$. Thus:

$r^2 = x^2 + y^2 + z^2$

First step
$$\frac{\partial\varphi}{\partial x} = f_1(x,y,z) = \frac{2xy}{r^2} + \sin z$$

$(y \ln r^2)_x = y\,\frac{2x}{r^2}$

By integration with respect to the variable x we obtain

$$\varphi(x) = y\ln r^2 + x\sin z + c(y,z)$$

!

with an unknown function $c(y,z)$.

Pluging into the equation

$$\frac{\partial\varphi}{\partial y} = f_2(x,y,z) = \ln r^2 + \frac{2y^2}{r^2} + ze^y$$

gives

Second step
$$\cancel{\ln r^2} + \cancel{\frac{2y^2}{r^2}} + \frac{\partial c}{\partial y} = \cancel{\ln r^2} + \cancel{\frac{2y^2}{r^2}} + ze^y$$

Calculation of the potential (continuation).

From this we get the condition

$$\frac{\partial c}{\partial y} = ze^y$$

and therefore

$$c(y,z) = ze^y + d(z)$$

!

for an unknown function $d(z)$. So far we know:

$$\varphi(x) = y\ln r^2 + x\sin z + ze^y + d(z)$$

$\varphi_z = \frac{2yz}{r^2} + e^y + x\cos z + d'(z)$

The last condition is

3 step.
$$\frac{\partial\varphi}{\partial z} = f_3(x,y,z) = \frac{2yz}{r^2} + e^y + x\cos z$$

Therefore $d'(z) = 0$ and the potential is given by

$$\boxed{\varphi(x) = y\ln r^2 + x\sin z + ze^y + c} \qquad \text{for } c \in \mathbb{R}$$

$t \in [a,b)$

3.3 Surface integrals

$$p(u) = \begin{pmatrix} p_1(u_1,u_2) \\ p_2(u_1,u_2) \\ p_3(u_1,u_2) \end{pmatrix}$$

Definition: Let $D \subset \mathbb{R}^2$ be a domain and $\mathrm{p} : D \to \mathbb{R}^3$ a C^1–map

$$\mathrm{x} = \mathrm{p}(\mathrm{u}) \quad \text{with } \mathrm{x} \in \mathbb{R}^3 \text{ and } \mathrm{u} = (u_1, u_2)^T \in D \subset \mathbb{R}^2$$

If for all $\mathrm{u} \in D$ the two vectors

$$\begin{pmatrix} p_{1u_1} \\ p_{2u_2} \\ p_{3u_2} \end{pmatrix} = \frac{\partial \mathrm{p}}{\partial u_1} \quad \text{and} \quad \frac{\partial \mathrm{p}}{\partial u_2} = \begin{pmatrix} p_{1u_2} \\ p_{2u_2} \\ p_{3u_2} \end{pmatrix}$$

are linear independent, we call

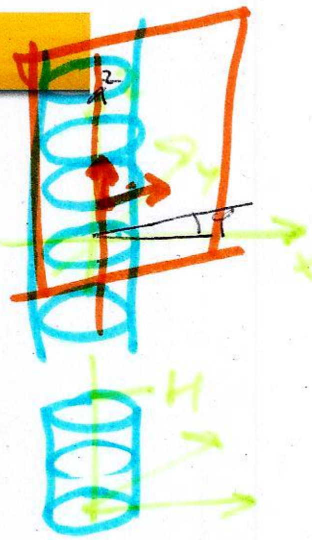
$$F := \{\mathrm{p}(\mathrm{u}) \mid \mathrm{u} \in D\}$$

a surface or a piece o surface. The map $\mathrm{x} = \mathrm{p}(\mathrm{u})$ is called a parameterisation or parameter representation of the surface F.

Example I.

We consider for a given $r > 0$ the map **r fixed**

$$\mathrm{p}(\varphi, z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \text{for } (\varphi, z) \in \mathbb{R}^2.$$



The corresponding parameterized surface is an unbounded cylinder in $\mathbb{R}^3$.
If we restrict the area of definition, e.g.

$$(\varphi, z) \in K := [0, 2\pi] \times [0, H] \subset \mathbb{R}^2$$

we obtain a bounded cylinder of height H.
The partial derivatives

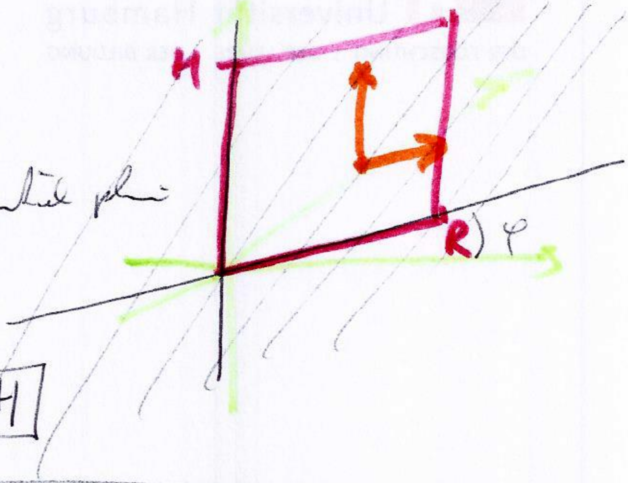
$$\frac{\partial \mathrm{p}}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}, \qquad \frac{\partial \mathrm{p}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

of $\mathrm{p}(\varphi, z)$ are linearly independent on $\mathbb{R}^2$.

φ fixed

$$\tilde{p}(r,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \tilde{p}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \qquad \tilde{p}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

surface = plane = tangential plane



$(r,z) \in [0,R] \times [0,H]$

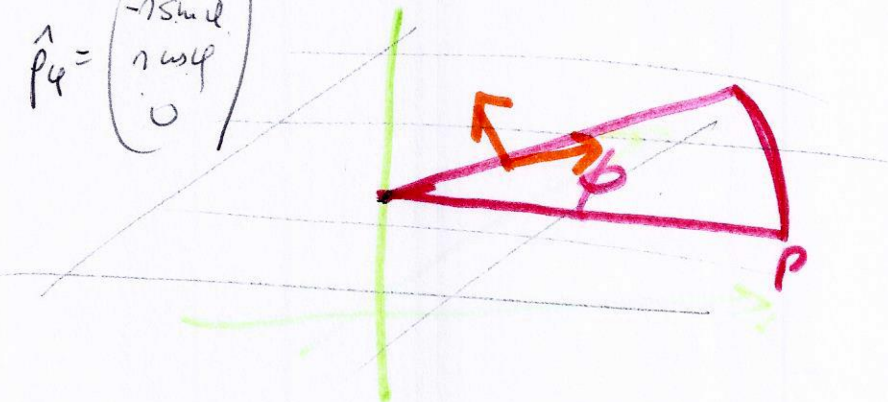
$$\|\tilde{p}_r \times \tilde{p}_z\| = \left\| \begin{pmatrix} \sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \right\| = 1 \qquad A = \int_0^R \int_0^H 1 \cdot dr\,dz = \boxed{R \cdot H}$$

---

z fixed

$$\hat{p}(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \hat{p}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \qquad \cancel{\hat{p}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\hat{p}_\varphi = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$$



$(r,\varphi) \in [0,R] \times [0,\phi]$

$$\|\hat{p}_r \times \hat{p}_\varphi\| = \left\| \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right\| = r$$

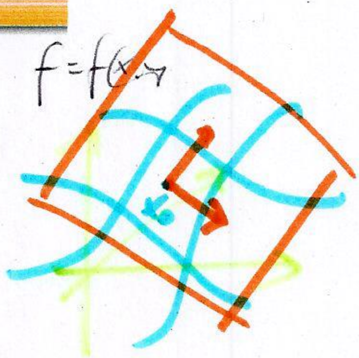
$$A = \int_H do = \int_0^R \int_0^\phi r\,dr\,d\varphi = \frac{R^2}{2} \cdot \phi$$

Example II.

The graph of a scalar C^1–function $\varphi : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, is a surface.

A parametrisation is given by

$$\mathrm{p}(u_1, u_2) := \begin{pmatrix} u_1 \\ u_2 \\ \varphi(u_1, u_2) \end{pmatrix} \qquad \text{for } \mathrm{u} \in D$$



The partial derivatives

$$\frac{\partial \mathrm{p}}{\partial u_1} = \begin{pmatrix} 1 \\ 0 \\ \varphi_{u_1} \end{pmatrix}, \qquad \frac{\partial \mathrm{p}}{\partial u_2} = \begin{pmatrix} 0 \\ 1 \\ \varphi_{u_2} \end{pmatrix}$$

are linear independent.

The tangential plane on a surface.

The two linear independent vectors

$$\frac{\partial \mathrm{p}}{\partial u_1}(\mathrm{u}^0) \qquad \text{und} \qquad \frac{\partial \mathrm{p}}{\partial u_2}(\mathrm{u}^0)$$

are tangential on the surface F.

The two vectore span the tangential plane $T_{\mathrm{x}^0}F$ of the surface F at the point $\mathrm{x}^0 = \mathrm{p}(\mathrm{u})$.

The tangential plane has a parameter representation

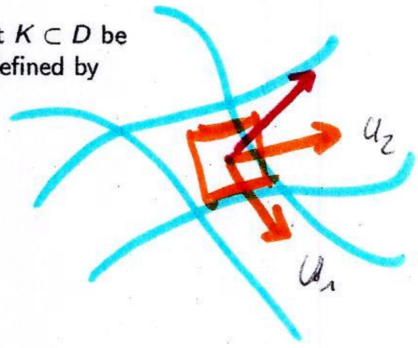
$$T_{\mathrm{x}^0}F \;:\; \mathrm{x} = \mathrm{x}^0 + \lambda \frac{\partial \mathrm{p}}{\partial u_1}(\mathrm{u}^0) + \mu \frac{\partial \mathrm{p}}{\partial u_2}(\mathrm{u}^0) \qquad \text{for } \lambda, \mu \in \mathbb{R}.$$

Question: How can wie calculate the size of a given surface F?

The surface integral of a piece of surface.

Definition: Let $\mathrm{p} : D \to \mathbb{R}^3$ be a parameterisation of a surface, and let $K \subset D$ be compact, measurable and connected. Then the "content" of $\mathrm{p}(K)$ is defined by the surface integral

$$\int_{\mathrm{p}(K)} do := \int_K \left\| \frac{\partial \mathrm{p}}{\partial u_1}(\mathrm{u}) \times \frac{\partial \mathrm{p}}{\partial u_2}(\mathrm{u}) \right\| du$$



We call

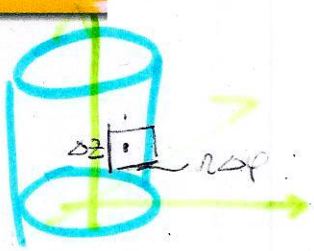
$$do := \left\| \frac{\partial \mathrm{p}}{\partial u_1}(\mathrm{u}) \times \frac{\partial \mathrm{p}}{\partial u_2}(\mathrm{u}) \right\| du$$

the surface element of the surface $\mathrm{x} = \mathrm{p}(\mathrm{u})$.

Remark: The surface integral is **independent** of the particular parameterisation of the surface. This follows from the theorem of transformation.

Example.

For the lateral surface of a cylinder $Z = \mathrm{p}(K)$ with r fixed



$$K := [0, 2\pi] \times [0, H] \subset \mathbb{R}^2$$

and

$$\mathrm{x} = \mathrm{p}(\varphi, z) := \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \text{for } (\varphi, z) \in \mathbb{R}^2$$

we obtain

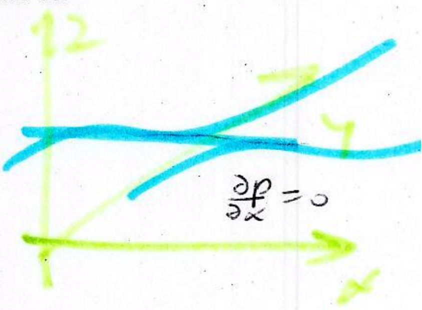
$$\left\| \frac{\partial \mathrm{p}}{\partial \varphi} \times \frac{\partial \mathrm{p}}{\partial z} \right\| = r \quad = \left\| \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} r\cos\varphi \\ -r\sin\varphi \\ 0 \end{pmatrix} \right\| = r$$

the value

$$O(Z) = \int_Z do = \int_K r d(\varphi, z) = \int_0^{2\pi} \int_0^H \underbrace{r dz d\varphi}_{r\Delta z\Delta\varphi} = \boxed{2\pi r H}$$

Example.

If the surface is the graph of a scalar function, i.e. $x_3 = \varphi(x_1, x_2)$, the for the related tangential vectors we have



$$\frac{\partial p}{\partial x_1} \times \frac{\partial p}{\partial x_2} = \begin{pmatrix} 1 \\ 0 \\ \varphi_{x_1} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \varphi_{x_2} \end{pmatrix} = \begin{pmatrix} -\varphi_{x_1} \\ -\varphi_{x_2} \\ 1 \end{pmatrix}$$

$\frac{\partial \varphi}{\partial x} = 0$

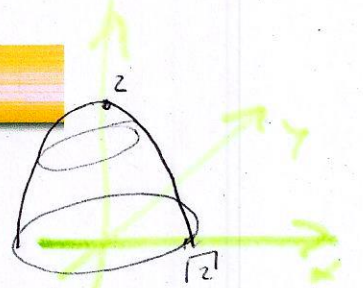
Thus we obtain

$$\left\| \frac{\partial p}{\partial x_1} \times \frac{\partial p}{\partial x_2} \right\| = \sqrt{1 + \varphi_{x_1}^2 + \varphi_{x_2}^2}$$

and

$$\begin{aligned} O(p(K)) &= \int_{p(K)} do \\ &= \int_K \overbrace{\sqrt{1 + \varphi_{x_1}^2 + \varphi_{x_2}^2}\, \underbrace{d(x_1, x_2)}_{dx_1 dx_2}}^{do} \end{aligned}$$

Example.



For the surface of the paraboloid P, given by

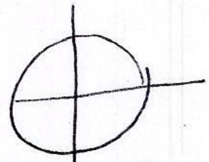
$$P := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_3 = 2 - x_1^2 - x_2^2,\ x_1^2 + x_2^2 \le 2\},$$

$x_3 = \varphi(x_1, x_2)$ $\varphi_{x_1} = -2x_1,\ \varphi_{x_2} = -2x_2$

we have

$$O(P) = \int_{x_1^2 + x_2^2 \le 2} \sqrt{1 + 4x_1^2 + 4x_2^2}\, d(x_1, x_2)$$

Polar Koord.



Polar coordinates

$$\begin{aligned} &= \int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{1 + 4r^2}\, r\, d\varphi\, dr = \pi \int_0^2 \sqrt{1 + 4s}\, ds \\ &= \pi \left[\frac{1}{6}(1 + 4s)^{3/2} \right]_0^2 = \pi \left(\frac{1}{6}(27 - 1) \right) = \boxed{\frac{13}{3}\pi} \end{aligned}$$

$O \rightarrow \square$

$\cancel{A+4a^2}$ $r^2 = s$
$2r\,dr = ds$