Remark.

For the vector product of two vectors $a, b \in \mathbb{R}^3$ we have

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - \langle a, b\rangle^2$$

Thus we have

$$\left\| \frac{\partial p}{\partial x_1} \times \frac{\partial p}{\partial x_2} \right\|^2 = \left\| \frac{\partial p}{\partial x_1} \right\|^2 \left\| \frac{\partial p}{\partial x_2} \right\|^2 - \left\langle \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2} \right\rangle^2$$

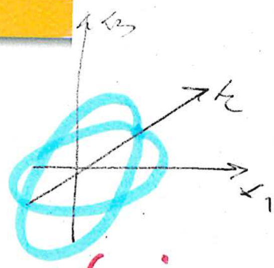
If we define

$$E := \left\| \frac{\partial p}{\partial x_1} \right\|^2, \quad F := \langle \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2} \rangle^2, \quad G := \left\| \frac{\partial p}{\partial x_2} \right\|^2,$$

we obtain the relation

$$do = \sqrt{EG - F^2}\, d(u_1, u_2)$$

Example.



R fixed

For the surface element of the sphere

$$S_r^2 = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = r^2\}$$

we obtain using the parameterisation via spherical coordinates

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = r \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix} \qquad \text{für } (\varphi, \theta) \in [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

the relations

$$\frac{\partial p}{\partial \varphi} = r \begin{pmatrix} -\sin\varphi\cos\theta \\ \cos\varphi\cos\theta \\ 0 \end{pmatrix} \quad \overset{\perp}{\text{und}} \quad \frac{\partial p}{\partial \theta} = r \begin{pmatrix} -\cos\varphi\sin\theta \\ -\sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

Thus we have

$$E = r^2\cos^2\theta, \quad F = 0, \quad G = r^2$$

Continuation of the examples.

With

$$E = r^2 \cos^2 \theta, \quad F = 0, \quad G = r^2$$

we obtain the relation

$$do = \sqrt{EG - F^2}\, d(u_1, u_2)$$

and therefore

$$do = r^2 \cos\theta \, d(\varphi, \theta) \qquad \text{für } (\varphi, \theta) \in [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We can calculate the surface of the sphere as follows (In spherical coordinates it is a rectangle)

$$\begin{aligned} O &= \int_{S_r^2} do = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2 \cos\theta \, d\varphi \, d\theta \\ &= 2\pi r^2 \sin\theta \Big|_{-\pi/2}^{\pi/2} = 4\pi r^2 \end{aligned}$$

Surface integrals of scalar and vector fields.

Definition: Let $x = p(u)$ be a C^1–parametrisation of a surface $F = p(K)$, where $K \subset D$ is compact, measurable and connected.

- For a continuous function $f : F \to \mathbb{R}$ the surface integral of a scalar field is defined as

$$\int_F f(x)\, do := \int_K f(p(u)) \underbrace{\left\| \frac{\partial p}{\partial u_1} \times \frac{\partial p}{\partial u_2} \right\| du}_{do}$$

- For a continuous vector field $f : F \to \mathbb{R}^3$ the surface integral of a vector field is defined as

$$\int_F f(x)\, do := \int_K \left\langle f(p(u)), \overbrace{\frac{\partial p}{\partial u_1} \times \frac{\partial p}{\partial u_2}}^{\text{normal vector on the surface}} \right\rangle du$$

Alternative representation of surface integrals.

Othere representations of surface integrals of vector fields

The unit normal vector n(x) on a surface F is given by

$$\mathrm{n}(\mathrm{x}) = \mathrm{n}(\mathrm{p}(\mathrm{u})) = \frac{\dfrac{\partial \mathrm{p}}{\partial u_1} \times \dfrac{\partial \mathrm{p}}{\partial u_2}}{\left\| \dfrac{\partial \mathrm{p}}{\partial u_1} \times \dfrac{\partial \mathrm{p}}{\partial u_2} \right\|}.$$

Therefore we can write

$$\begin{aligned}
\int_F \mathrm{f}(\mathrm{x})\, do &= \int_K \left\langle \mathrm{f}(\mathrm{p}(\mathrm{u})), \frac{\partial \mathrm{p}}{\partial u_1} \times \frac{\partial \mathrm{p}}{\partial u_2} \right\rangle du \\
&= \int_K \langle \mathrm{f}(\mathrm{p}(\mathrm{u})), \mathrm{n}(\mathrm{p}(\mathrm{u})) \rangle \underbrace{\left\| \frac{\partial \mathrm{p}}{\partial u_1} \times \frac{\partial \mathrm{p}}{\partial u_2} \right\| du}_{do} \\
&= \int_F \langle \mathrm{f}(\mathrm{x}), \mathrm{n}(\mathrm{x}) \rangle\, do
\end{aligned}$$

Interpretation of surface integrals.

Remark: ·) f(x) = 1 size of surface

- If $f(\mathrm{x})$ is the mass density of a surface with a mass distribution, the the surface integral of the scalar field (mass density) gives the total mass of the surface.
- If f(x) is the velocity field of a stationary flow, then the surface integral of the vector field (velocity field) gives the amount of flow which passes the surface F per time unit, i.e. the flow of f(x) through the surface F.
- If F is a closed surface, i.e. surface (boundary) of a compact and simply connected region (body) in $\mathbb{R}^3$, we write

$$\oint_F f(\mathrm{x})\, do \qquad \text{bzw.} \qquad \oint_F \mathrm{f}(\mathrm{x})\, do$$

The parameterisation is chosen such that the unit normal vector n(x) is pointing outwards.

The divergence theorem (Gauß theorem).

sphere

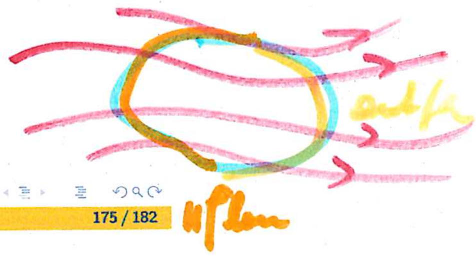
cube

Theorem: (divergence theorem/Gauß theorem) Let $G \subset \mathbb{R}^3$ a compact and measurable standard domain, i.e. G is projectable with respect to all coordinates. The boundary ∂G consists of finite many smooth surfaces with outer normal vector $\mathrm{n}(\mathrm{x})$.
If $\mathrm{f} : D \to \mathbb{R}^3$ is a C^1–vector field with $G \subset D$, then

$$\boxed{\int_G \operatorname{div} \mathrm{f}(\mathrm{x})\, dx = \oint_{\partial G} \mathrm{f}(\mathrm{x})\, do}$$

Interpretation of the Gauß theorem: The left side is an integral of the scalar function $g(\mathrm{x}) := \operatorname{div} \mathrm{f}(\mathrm{x})$ over G. The right hand side is a surface integral of the vector field $\mathrm{f}(\mathrm{x})$. If $\mathrm{f}(\mathrm{x})$ is the vectorfield of an incompressible flow, then $\operatorname{div} \mathrm{f}(\mathrm{x}) = 0$ and therefore

$$\oint_{\partial G} \mathrm{f}(\mathrm{x})\, do = 0$$

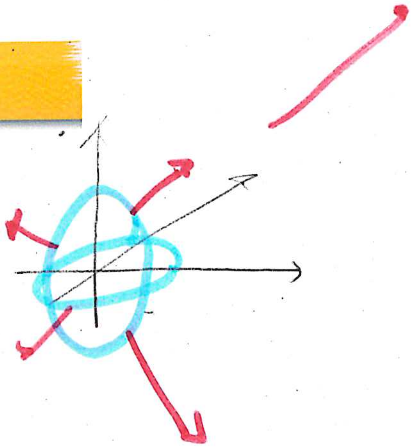


Example.

Consider the vector field

$$\mathrm{f}(\mathrm{x}) = \mathrm{x} = (x_1, x_2, x_3)^T$$

and the sphere K:



$$K := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

We have

Gauss

$$\operatorname{div} \mathrm{f}(\mathrm{x}) = 3 = \sum_{i=1}^{3} \partial_{x_i} f_i$$

and thus

$$4\pi = \oint_{\partial K} \underbrace{\langle f(x), n \rangle}_{1} do = \oint_{\partial K} f(x)\, do = \int_K \operatorname{div} \mathrm{f}(\mathrm{x})\, dx = 3 \cdot \operatorname{vol}(K) = 4\pi \quad \Rightarrow \operatorname{vol}(K) = \frac{4\pi}{3}$$

The related surface integral can be calculated easily using spherical coordinates.

The Green formulas.

Theorem: (Green formulas) Let the set $G \subset \mathbb{R}^3$ satisfy the prerequisites of the Gauß theorem. For C^2–functions $f, g : D \to \mathbb{R}$, $G \subset D$ we have the relations:

$f \Leftrightarrow g$
Differenz

i) $$\int_G (f\Delta g + \langle \nabla f, \nabla g\rangle)\, dx = \oint_{\partial G} f \frac{\partial g}{\partial n}\, do$$ $= \int g \frac{\partial f}{\partial n}\, do$

$\int (g \Delta f) + \langle \nabla g, \nabla f\rangle\, dx$

ii) $$\int_G (f\Delta g - g\Delta f)\, dx = \oint_{\partial G} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) do$$

We denote by
$$\frac{\partial f}{\partial n}(x) = D_n f(x) = n \cdot \nabla f(x) \quad \text{for } x \in \partial G$$
the directional derivative of $f(x)$ in the direction of the outer unit normal vector $n(x)$.

1d. $\int f g_{xx} = f g_x \Big| - \int f_x g_x$

$$\int_a^b (f g_{xx} + f_x g_x)\, dx = f g_x \Big|_a^b$$

Proof of the Green formulas.

We set
$$F(x) = f(x) \cdot \nabla g(x)$$
Then we have
$$\begin{aligned} \operatorname{div} F(x) &= \frac{\partial}{\partial x_1}\left(f \cdot \frac{\partial g}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(f \cdot \frac{\partial g}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(f \cdot \frac{\partial g}{\partial x_3}\right) \\ &= f \cdot \Delta g + \langle \nabla f, \nabla g\rangle \end{aligned}$$
Now we apply the Gauß theorem:

Gauss

$$\begin{aligned} \int_G (f\Delta g + \langle \nabla f, \nabla g\rangle)\, dx &= \int_G \operatorname{div} F(x)\, dx = \oint_{\partial G} \langle F, n\rangle\, do \\ &= \oint_{\partial G} f \langle \nabla g, n\rangle\, do = \oint_{\partial G} f\frac{\partial g}{\partial n}\, do \end{aligned}$$

$\langle f \nabla g, n \rangle$

The second formula follows directely by exchanging f and g.

The Stokes theorem.

Theorem: (Stokes theorem)

Let $\mathrm{f} : D \to \mathbb{R}^3$ be a C^1–vector field on a domain $D \subset \mathbb{R}^3$.

Let $F = \mathrm{p}(K)$ be a surface in D, $F \subset D$, with parameterisation $\mathrm{x} = \mathrm{p}(\mathrm{u})$, $\mathrm{u} \in \mathbb{R}^2$. Let $K \subset \mathbb{R}^2$ be a Green area.

The boundary ∂K is parameterised by a piecewise smooth C^1–curve c and the image $\tilde{c}(t) := p(c(t))$ parameterises the boundary ∂F of the surface F.

The orientation of the boundary curve $\tilde{c}(t)$ is chosen such that $\mathrm{n}(\tilde{c}(t)) \times \dot{\tilde{c}}(t)$ points in the direction of the surface.

Then we have

$$\boxed{\int_F \operatorname{curl} \mathrm{f}(\mathrm{x})\, do = \oint_{\partial F} \mathrm{f}(\mathrm{x})\, d\mathrm{x}}$$

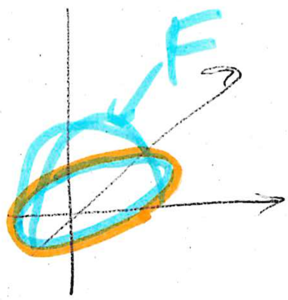
Example.

Given the vector field

$$\mathrm{f}(x, y, z) = (-y, x, -z)^T$$

and let the closed curve $\mathrm{c} : [0, 2\pi] \to \mathbb{R}^3$ be parameterised by

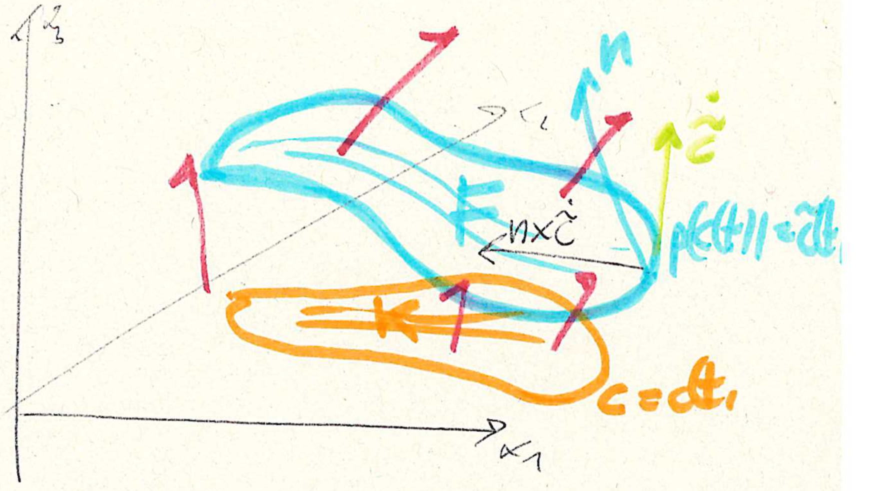
$$\mathrm{c}(t) = (\cos t, \sin t, 0)^T \qquad \text{für } 0 \le t \le 2\pi$$



Then:

right hand side of Stokes

$$\begin{aligned} \oint_c \mathrm{f}(\mathrm{x})\, d\mathrm{x} &= \int_0^{2\pi} \langle \mathrm{f}(\mathrm{c}(t)), \dot{\mathrm{c}}(t) \rangle\, dt \\ &= \int_0^{2\pi} \left\langle \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \right\rangle dt \\ &= \int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{1}\, dt = 2\pi \end{aligned}$$



x_3
x_2
x_1
n
$\dot{\tilde{c}}$
F
$n \times \dot{\tilde{c}}$
K
$c = ct_1$

Continuation of the example.

We define a surface $F \subset \mathbb{R}^3$, bounded by the curve $c(t)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix} =: p(\varphi,\psi)$$

R=1 const.

with $(\varphi,\psi) \in K = [0, 2\pi] \times [0, \pi/2]$, i.e. the surface F is the upper half sphere.

In spherical coordinates the upper half sphere is a rectangle

Stokes theorem tells us:

$$\int_F \operatorname{curl} f(x)\, do = \oint_{c=\partial F} f(x)\, dx = 2\pi$$

We have already calculated the right side, a **surface integral of a vector field**:

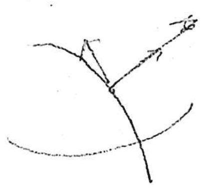
$$\oint_{c=\partial F} f(x)\, dx = 2\pi$$

Completion of the example.

It remains a **surface integral of a vector field**:

$$\int_F \overset{\text{curl}}{\cancel{\operatorname{rot}}} f(x)\, do = \int_K \left\langle \overset{\text{curl}}{\cancel{\operatorname{rot}}} f(p(\varphi,\psi)), \frac{\partial p}{\partial\varphi} \times \frac{\partial p}{\partial\psi} \right\rangle d\varphi d\psi$$

Attention: the right hand side is an **intergal over a domain**.



We have $\operatorname{curl} f(x) = (0, 0, 2)^T$ and

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \sin\psi\cos\psi \end{pmatrix} \right\rangle =$$

$$\frac{\partial p}{\partial\varphi} \times \frac{\partial p}{\partial\psi} = \begin{pmatrix} \cos\varphi\cos^2\psi \\ \sin\varphi\cos^2\psi \\ \sin\psi\cos\psi \end{pmatrix}$$

Thus:

$$\int_F \operatorname{curl} f(x)\, do = \int_0^{\pi/2}\int_0^{2\pi} \underbrace{2\sin\psi\cos\psi}_{\sin 2\psi}\, d\varphi d\psi = 2\pi \int_0^{\pi/2} \sin(2\psi)\, d\psi = \boxed{2\pi}$$