Mathematics Department
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WiSe 2021/2022

Mathematics III Exam
(Module: Differential Equations I)

February 28, 2022

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I was instructed about the fact that the exam performance will only be assessed if the Central Examination Office of TUHH verifies my official admission before the exam's beginning in retrospect.

(Signature)

Exercise	**Points**	**Evaluater**
1		
2		
3		
4		
5		

$$\sum =$$

Exercise 1: (3+2 points)

a) Solve the initial value problem

$$y' - y = 2 \quad \text{with} \quad y\,(0) = 3.$$

b) Compute the general solution of Euler's differential equation

$$x^2 y'' + xy' - 4y = 0\,.$$

Hint: There exist solutions of the form $y(x) = x^\alpha$ with $\alpha \in \mathbb{R}$.

Solution:

a) (3 points)

Compute the general solution of the homogeneous linear differential equation:

$$y' - y = 0 \quad \Rightarrow \quad p(\lambda) = \lambda - 1 = 0 \quad \Rightarrow \quad \lambda = 1 \quad \Rightarrow \quad y_h(x) = ce^x.$$

The ansatz $y_p(x) = a$ for a particular solution of the inhomogeneous differential equation yields

$$0 - a = 2 \quad \Rightarrow \quad y_p(x) = -2.$$

So the general solution to the inhomogeneous equation is

$$y(x) = ce^x - 2.$$

We use the initial condition $3 = y(0) = ce^0 - 2 \;\Rightarrow\; c = 5$

and obtain the solution $y(x) = 5e^x - 2$ of the initial value problem.

b) (2 points)

We plug the ansatz $y(x) = x^\alpha$ into Euler's differential equation and obtain:

$$0 = (\alpha(\alpha-1)+\alpha-4)x^\alpha = (\alpha^2-4)x^\alpha \quad \Rightarrow \quad \alpha^2 - 4 = 0 \quad \Rightarrow \quad \alpha_1 = 2,\, \alpha_2 = -2.$$

So a fundamental system is $y_1(x) = x^2,\;\; y_2(x) = \dfrac{1}{x^2}\,.$

Then, with $c_1, c_2 \in \mathbb{R}$ we obtain the general solution

$$y(x) = c_1 x^2 + \frac{c_2}{x^2}\,.$$

Exercise 2: (5 points)

Compute the general solution of the following system of differential equations

$$\boldsymbol{y}' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \boldsymbol{y} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Hint: Note that $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$.

Solution:

(5 points)

Compute the eigenvalues:

$$\begin{aligned} p_{\boldsymbol{A}}(\lambda) &= \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 \\ &= \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda_1 = 1,\ \lambda_2 = 3 \end{aligned}$$

Computation of corresponding eigenvectors by $(\boldsymbol{A} - \lambda \boldsymbol{E})\boldsymbol{v} = \boldsymbol{0}$

Eigenvector $\boldsymbol{v}^1$ zu $\lambda_1 = 1$:

$$\left(\begin{array}{rr|r} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \boldsymbol{v}^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvector $\boldsymbol{v}^2$ zu $\lambda_2 = 3$:

$$\left(\begin{array}{rr|r} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{rr|r} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \boldsymbol{v}^2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Ansatz for particular inhomogeneous solution: $\boldsymbol{y}_p(x) = \boldsymbol{a}$

$$\Rightarrow \quad \boldsymbol{0} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \boldsymbol{y}_p - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{rr|r} 2 & -1 & 1 \\ -1 & 2 & 1 \end{array}\right) \to \left(\begin{array}{rr|r} 2 & -1 & 1 \\ 0 & 3/2 & 3/2 \end{array}\right) \Rightarrow \boldsymbol{y}_p(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general inhomogeneous solution with $c_1, c_2 \in \mathbb{R}$ is:

$$\boldsymbol{y}(x) = c_1 \boldsymbol{y}^1(x) + c_2 \boldsymbol{y}^2(x) + \boldsymbol{y}_p(x) = c_1 e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Exercise 3: (4 points)

Solve the initial value problem

$$y'' + 2y' - 3y = 7 - 6x\ , \quad y(0) = 0\ ,\ y'(0) = 3\ .$$

Hint: Note that $\lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3)$.

Solution:

(4 points)

General solution of the homogeneous differential equation $y'' + 2y' - 3y = 0$:

Characteristic polynomial:

$$p(\lambda) = \lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3) = 0 \quad \Rightarrow \quad \lambda_1 = 1\ , \lambda_2 = -3$$

General homogeneous solution: $y_h(x) = c_1 e^x + c_2 e^{-3x}$ with $c_1, c_2 \in \mathbb{R}$

Ansatz for particular inhomogeneous solution $y_p(x) = ax + b$,

We plug this into the inhomogeneous differential equation which yields

$(ax + b)'' + 2(ax + b)' - 3(ax + b) = -3ax + 2a - 3b = 7 - 6x \Rightarrow a = 2 \Rightarrow b = -1\ .$

General inhomogeneous solution:

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} + 2x - 1 \quad \Rightarrow \quad y'(x) = c_1 e^x - 3c_2 e^{-3x} + 2$$

Initial values:

$0 = y(0) = c_1 + c_2 - 1 \quad \Rightarrow \quad c_1 = 1 - c_2$

$3 = y'(0) = c_1 - 3c_2 + 2 = 1 - c_2 - 3c_2 + 2 = 3 - 4c_2 \quad \Rightarrow \quad c_2 = 0 \Rightarrow c_1 = 1$

Solution of the initial value problem: $y(x) = e^x + 2x - 1$

Exercise 4: (3 points)

Compute the real-valued general solution of the linear differential equation

$$y'' + 9y = 0.$$

Use this solution to obtain all solutions for the corresponding boundary value problem with boundary values $y'(0) = 6$ and $y\left(\frac{\pi}{6}\right) = 2$.

Solution:

(3 points)

Characteristic polynomial: $p(\lambda) = \lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$

Complex-valued fundamental system

$$e^{3ix} = \cos(3x) + i\sin(3x)\,, \quad e^{-3ix} = \cos(3x) - i\sin(3x)\,.$$

Real and imaginary part lead to the general real-valued solution

$y(x) = c_1\cos(3x) + c_2\sin(3x)$ with $c_1, c_2 \in \mathbb{R} \Rightarrow y'(x) = -3c_1\sin(3x) + 3c_2\cos(3x)$.

The boundary values yield:

$6 = y'(0) = -3c_1\sin(0) + 3c_2\cos(0) = 3c_2 \quad \Rightarrow c_2 = 2,$

$2 = y\left(\frac{\pi}{6}\right) = c_1\cos\left(\frac{\pi}{2}\right) + c_2\sin\left(\frac{\pi}{2}\right) = c_2.$

Solution of the boundary value problem: $y(x) = c_1\cos(3x) + 2\sin(3x)$ with $c_1 \in \mathbb{R}$

Exercise 5: (3 points)

Compute all equilibria for the nonlinear system of differential equations of first order

$$\begin{aligned} \dot{x} &= y - 4 \\ \dot{y} &= 4y(x-1) \end{aligned}$$

and determine their stability properties.

Solution:

(3 points)

Condition for equilibria:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y-4 \\ 4y(x-1) \end{pmatrix} = \boldsymbol{f}(x,y) \quad \Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \boldsymbol{J}\boldsymbol{f}(x,y) = \begin{pmatrix} 0 & 1 \\ 4y & 4(x-1) \end{pmatrix}$$

The eigenvalues of $\boldsymbol{J}\boldsymbol{f}(\boldsymbol{P})$ yield the stability properties

$$\boldsymbol{J}\boldsymbol{f}(1,4) = \begin{pmatrix} 0 & 1 \\ 16 & 0 \end{pmatrix} \quad \Rightarrow \quad p(\lambda) = \lambda^2 - 16 = 0 \quad \Rightarrow \quad \lambda_1 = -4 < 0 < \lambda_2 = 4$$

$\Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is a (locally) instable (saddle) point.