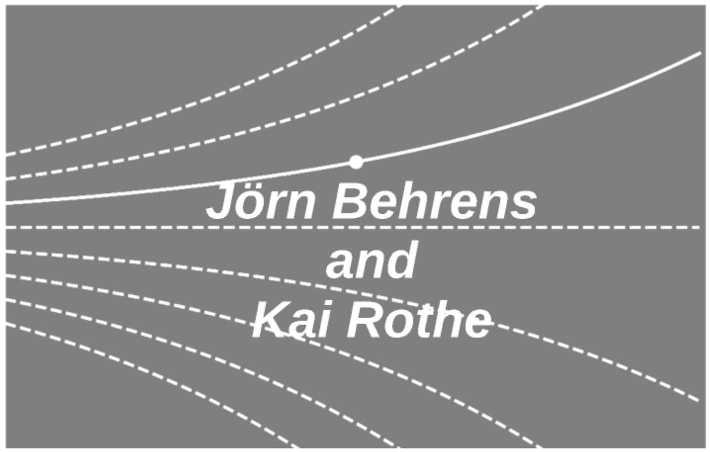
Differential Equations I



Introduction

Chapter 6.1-6.2

Your Prof

Coordinates:

Prof. Dr. Jörn Behrens
Uni Hamburg Mathematik/ CEN
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Bundesstraße 55, Room 120 (1st floor)
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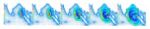
Background

Short CV

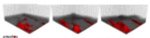
since 2009 Prof. @ Uni Hamburg, KlimaCampus/Dept. Mathematics
2006-2009 Head Tsunami Group @ AWI, Adjunct @ Uni Bremen
2005 Habilitation (Mathematics) @ TUM
2003-2004 Visiting Scientist @ NCAR, Boulder, CO, USA
1998-2006 Assistent Prof. (Akad. Rat) @ TUM, Sci. Computing
1996-1998 Post-Doc @ AWI
1991-1996 Dr. rer. nat. (Mathematics) @ AWI/Uni Bremen
1991 Diploma Mathematics @ Uni Bonn

Research Acitivities

Adaptive Tsunami Modelling



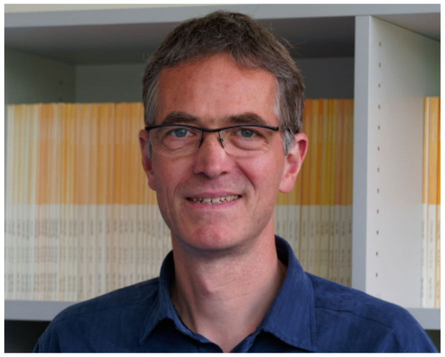
Adaptive Atmosphere Modelling



Mesh Generation

Multi-Scale Simulation

Coordinates:



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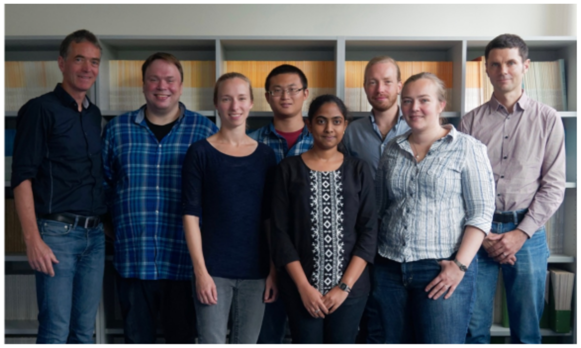

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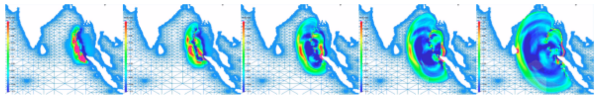

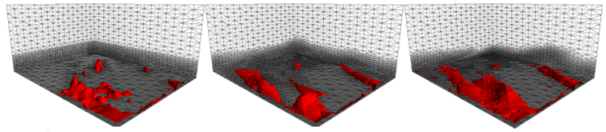

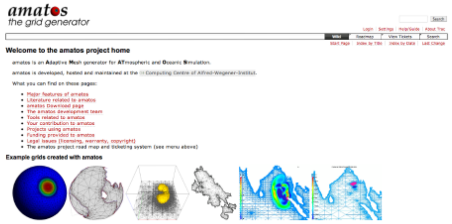

Course Info

Literature

Examples!

G. Bärwolff: *Höhere Mathematik für Naturwissenschaftler und Ingenieure* (2. Aufl.), Springer, Berlin/Heidelberg, 2009.

Formelsammlung

K.Vetters: *Formeln und Fakten im Grundkurs Mathematik*, Vieweg+Teubner Verlag, Wiesbaden, 2004.

Übungsleitung und Übungen

Dr. Kai Rothe/Dr. Claudine von Hallern

https://www.math.uni-hamburg.de/home/rothe/
https://www.math.uni-hamburg.de/en/numgeo/team/vonhallern-claudine-dr.html

Material:

https://www.math.uni-hamburg.de/teaching/export/tuhh/
Stud.IP

Please come prepared!

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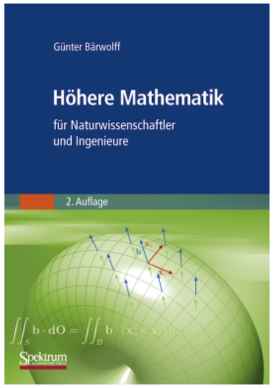

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A Bit of History

History:

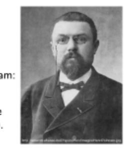
- Differential equations appear first in works of Leibniz and Newton.
- The equations were motivated by time dependent physical problems.

Gottfried Wilhelm Leibniz (1646-1716)

Isaac Newton (1642-1726)

Philosophy:

- Solution of a differential equation means predicting a state in the future!
- This paradigm influenced the philosophical main-stream: *Determinism*.
- Only in the early 20th century Poincaré destroyed the deterministic viewpoint (basis for dynamical systems).



Henri Poincaré (1854-1912)

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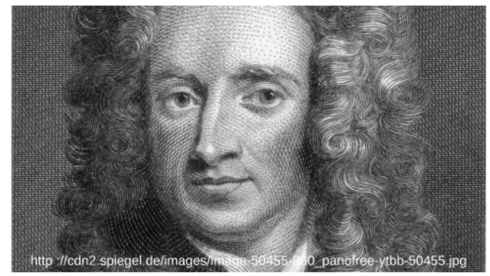

Isaac Newton (1642-1726)

Idea:

- Observe a system at two stages/times t_1 and t_2, obtain states $s(t_1), s(t_2)$
- Determine difference of states and obtain principles: $\frac{s(t_2)-s(t_1)}{t_2-t_1} \approx g(t)$.
- Conduct a limit process and obtain differential equation: $\lim_{t_1 \to t_2} \frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{ds}{dt} = g(t)$.

Philosophy:

- Solution of a differential equation means predicting a state in the future!
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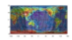
Remark:

- In the 18th century the concept of diff. equations was extended to several dimensions: *Partial Differential Equations*.
- By this differential equations (in one variable) became *ordinary*.

Applications



Newton's Mechanics

The principle first law of mechanics is given by

$$\text{force} = \text{mass} \times \text{acceleration}.$$

We can write this law as an ODE:

$$\boxed{m\ x''(t) = F(t,x,x'),}$$

with m the mass, x'' the acceleration, F a force function depending on $t \in I \subset \mathbb{R}$, the time in an interval I, $x \in \mathbb{R}^3$, the location, x', the velocity of a mass particle. This is an ODE of second order, since x'' is involved.

Fields of application

- satellite trajectories, planetary or astro-mechanics
- ballistic problems
- multi-body systems
- robotics

Reaction Kinetics

Assume 3 chemical species, S_1, S_2, and S_3, with concentration densities or mass densities x_1, x_2, and x_3 respectively. We assume reactions

$$\begin{aligned} S_1 &\xrightarrow{k_1} S_2, \\ S_2 + S_3 &\xrightarrow{k_2} S_1 + S_3, \\ S_2 + S_2 &\xrightarrow{k_3} S_3 + S_2, \end{aligned}$$

with reaction constants k_1, k_2, and k_3 respectively. These reaction equations represent a *catalytic converter*.

This transforms into a system of ODEs, when we consider the mass effects:

$$\begin{aligned} x_1' &= -k_1 x_1 + k_2 x_2 \cdot x_3, \\ x_2' &= k_1 x_1 - k_2 x_2 \cdot x_3 - k_3 x_2^2, \\ x_3' &= + k_3 x_2^2. \end{aligned}$$

We need further conditions for a unique solution. For conservation of mass, we assume that

$$x_1' + x_2' + x_3' = \sum_{i=1,3} x_i' = 0,$$

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$$x_1(0) = x_{1,0}; \quad x_2(0) = x_{2,0},$$

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As a Principle: Application of differential equations to technical problems

1. Mathematical modeling of problem by setting up differential equation
2. Formulation of suitable initial and boundary conditions
3. Solution of differential equation
4. Transfer of mathematical solution to original problem

Biological Systems

Consider the simple model of exponential growth of a population (think of cells that duplicate in each time interval τ). Since the exponential function is its own derivative (up to a constant) this can be written as

$$x' = ax; \quad x(t_0) = x_0,$$

analytical solution:

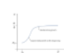
$$x(t) = x_0 \cdot e^{a(t-t_0)}.$$

We can improve this model by assuming a maximum number of individuals (a saturation) by exchanging

$$a \longrightarrow a - bx.$$

This yields a non-linear equation

$$x' = (a - bx)x; \quad x(t_0) = x_0.$$



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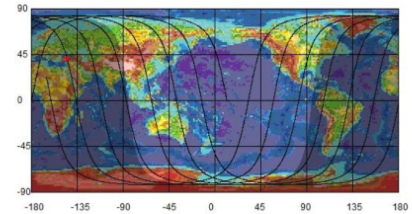
This represents a *system of ODEs*, called the *Lottka-Voltera equations.*

1



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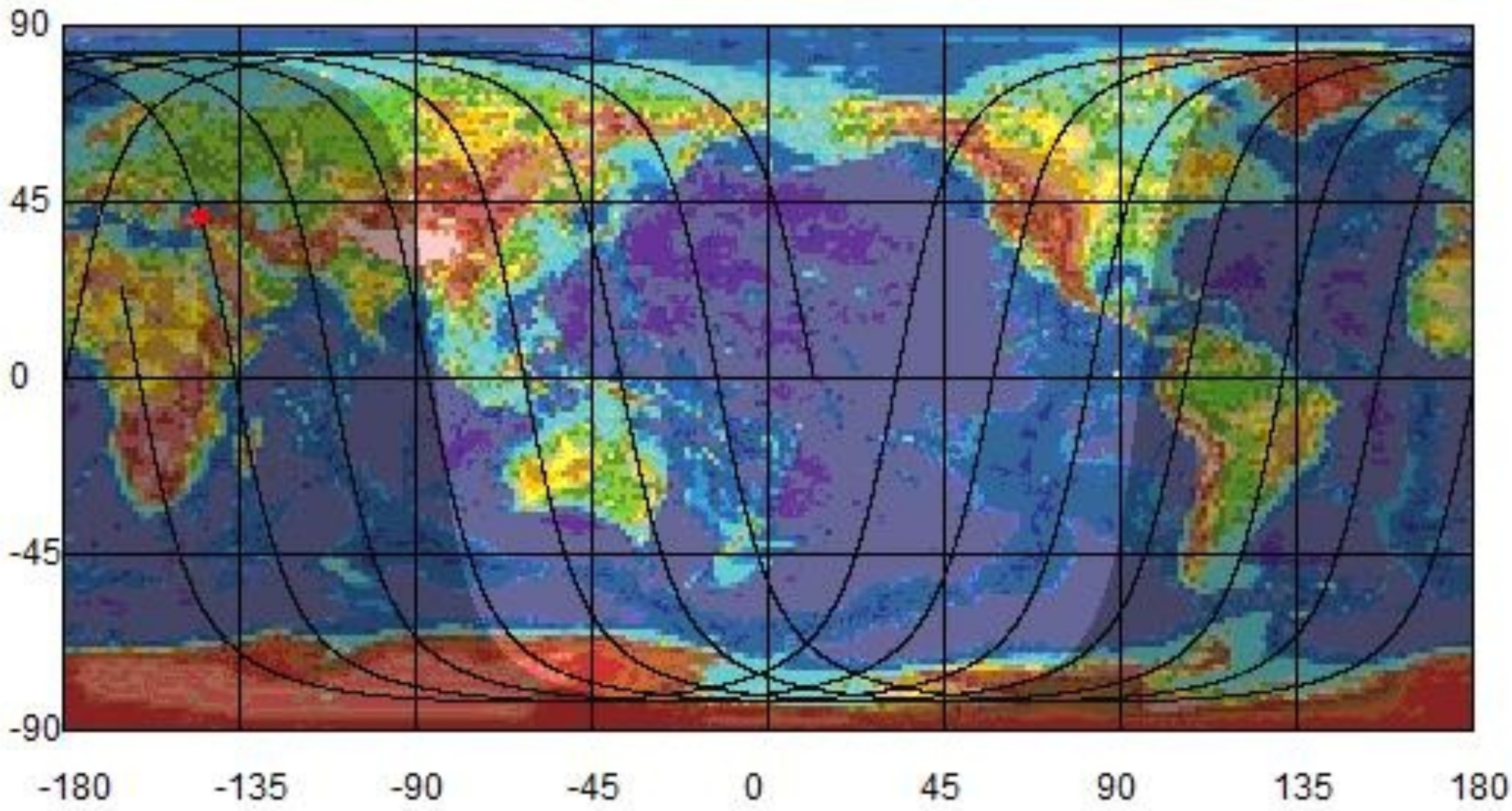
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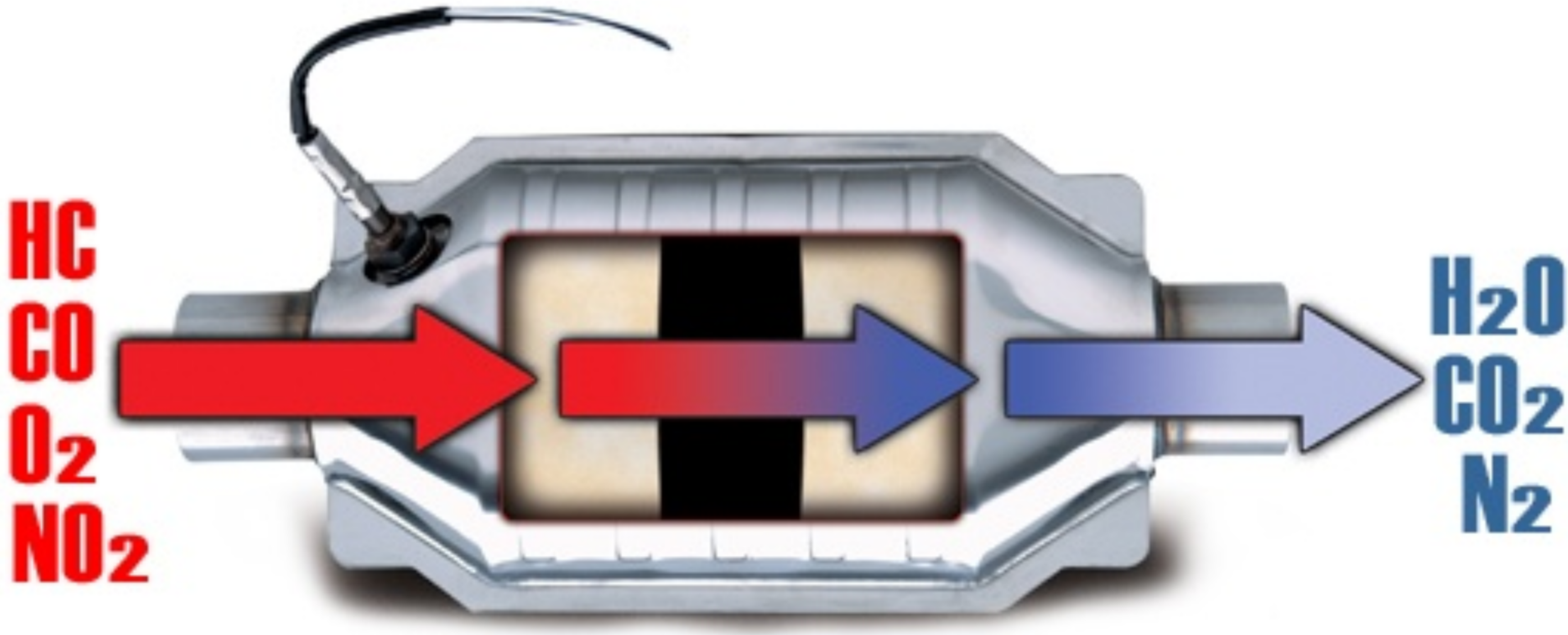

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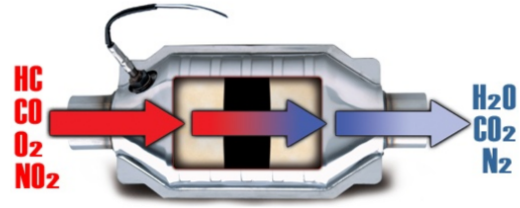

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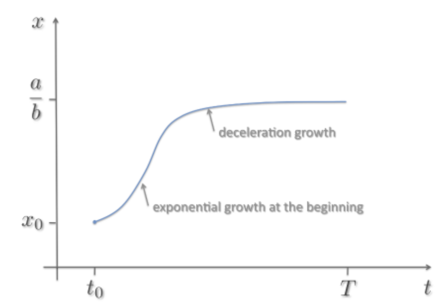

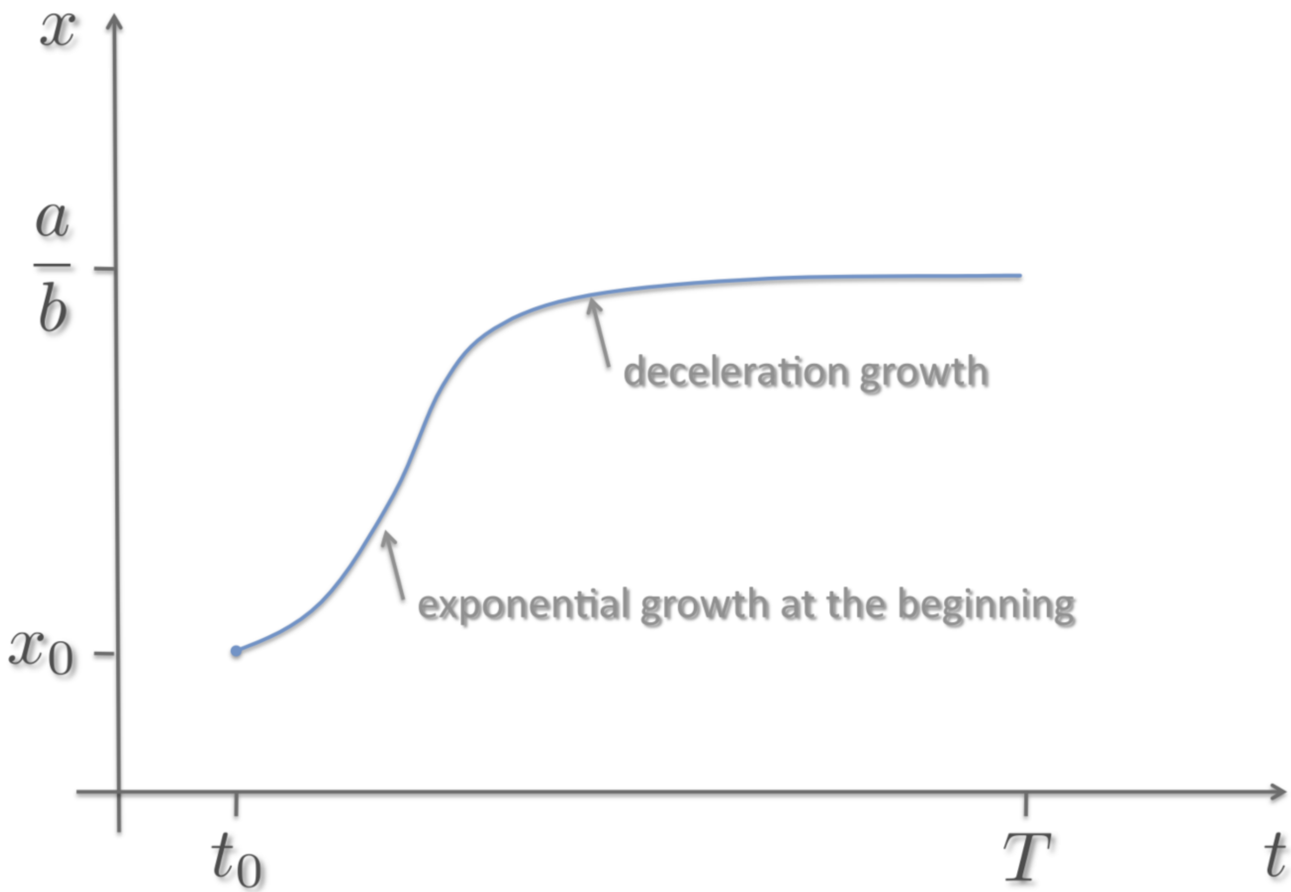
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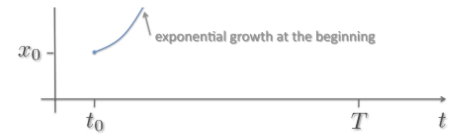

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As a Principle: Application of differential equations to technical problems

1. Mathematical modeling of problem by setting up differential equation
2. Formulation of suitable initial and boundary conditions
3. Solution of differential equation
4. Transfer of mathematical solution to original problem

Basic Terms

Definition (Ordinary Differential Equation):
An ordinary differential equation of oder n (n-th order ODE) for a function $y = y(x)$ is an equation of x, y and the derivatives of y up to (including) n-th order:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \quad \text{(implicit form)}$$

If the equation can be solved for the highest derivative in y, then we obtain the form:

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}) \quad \text{(explicot form)}.$$

We call the function y, solving the ODE solution or integral of the ODE.

2

Definition (Initial and Boundary Values):
Conditions on the solution of the ODE that apply to exactly one value of the independent variable x are called initial conditions, otherwise boundary conditions.

If the solution of an ODE is required to fulfill initial conditions, we call this problem an initial value problem (IVP).
Correspondingly, a boundary value problem (BVP) is given, when the solution is required to fulfill boundary conditions.

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Note Instead of $y = y(x)$, we may also use $y = y(t)$,
since many ODEs describe behavior of a system in time.

ODEs of order 1

Preliminary Remark:

- Let the ODE be given in explicit form: $y' = f(x, y)$.
- By this, the ODE prescribes in each $x, y \in D_f$, the domain of f, a "slope" y' of the solution graph.
- We will call a short section with slope y' in (x, y) as line element.

Solvability of Differential Equations:

- Let $f(x, y)$ be defined in a domain $D_f \subset \mathbb{R}^2$ and let $(x_0, y_0) \in D_f$.
- An initial value problem (IVP) is given by

$$y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0.$$

Further Terms:

- The function $y = \phi(x, C)$ is called general solution of $y' = f(x, y)$.
- For a certain C_0 one obtains a particular solution $y = \phi(x, C_0)$.
- If a solution $y = \Phi(x)$ has the property that at least one other solution runs through each of its points, then it is called singular solution.

Remarks:

- The solution $y : I \to \mathbb{R}$ of the IVP is maximal in the following sense: $y(x)$ runs to the boundary of D_f and cannot be further extended in D_f as a continuously differentiable curve. Under condition 2 of the proposition there exists exactly one such maximal solution to the IVP and I is the maximal domain/interval.
- Continuity is sufficient for existence only, uniqueness follows from condition 2.
- If f is continuously partially differentiable in D_f, then the solutions to $y' = f(x, y)$ form a family $y = \phi(x, C)$, where each initial condition corresponds to one value of C.

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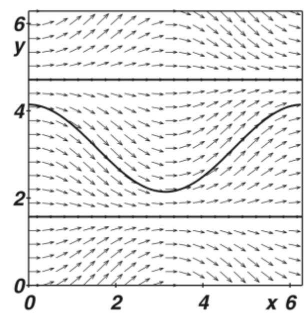
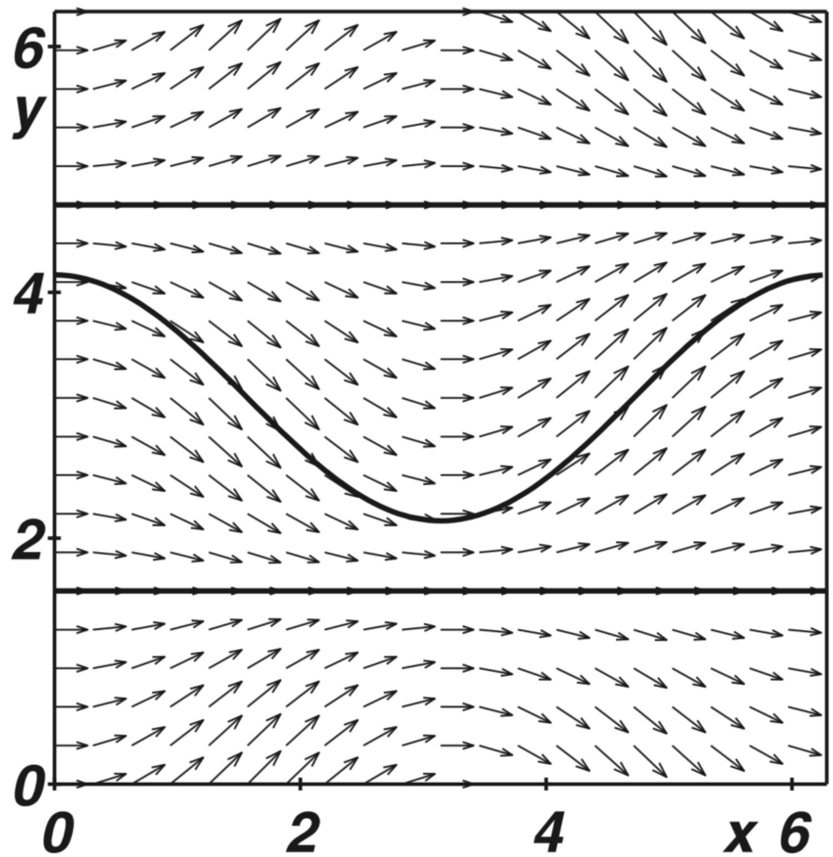
Definition (Slope Field):
The slope field of an ODE is given by the entire set of line elements.

Remark: the solutions of $y' = f(x, y)$ represent curves thet "fit" into the slope field.



Slope field of the ODE $y' = \sin x \cos y$



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Proposition (Existence and Uniquenes):

1. Let $f(x, y)$ in D_f be continuous. Then in some interval $I = \{x : x_0 - a < x < x_0 + b\}$ around x_0 $(a, b > 0$ suitable) there exists at least one solution $y(x)$ of the IVP.
2. Let the function $f(x, y)$ and its partial derivative $\frac{\partial f}{\partial y}(x, y)$ in D_f be continuous. Then there is exactly one solution $y(x)$ to the IVP through each point $(x_0, y_0) \in D_f$, which exists in an interval I around x_0.
3. Each solution curve $y(x)$ of the IVP can be extended to both directions (i.e.. for $x < x_0$ and $x > x_0$) until it reaches or gets arbitrarily close to the boundary of the domain D_f, resp.

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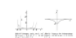

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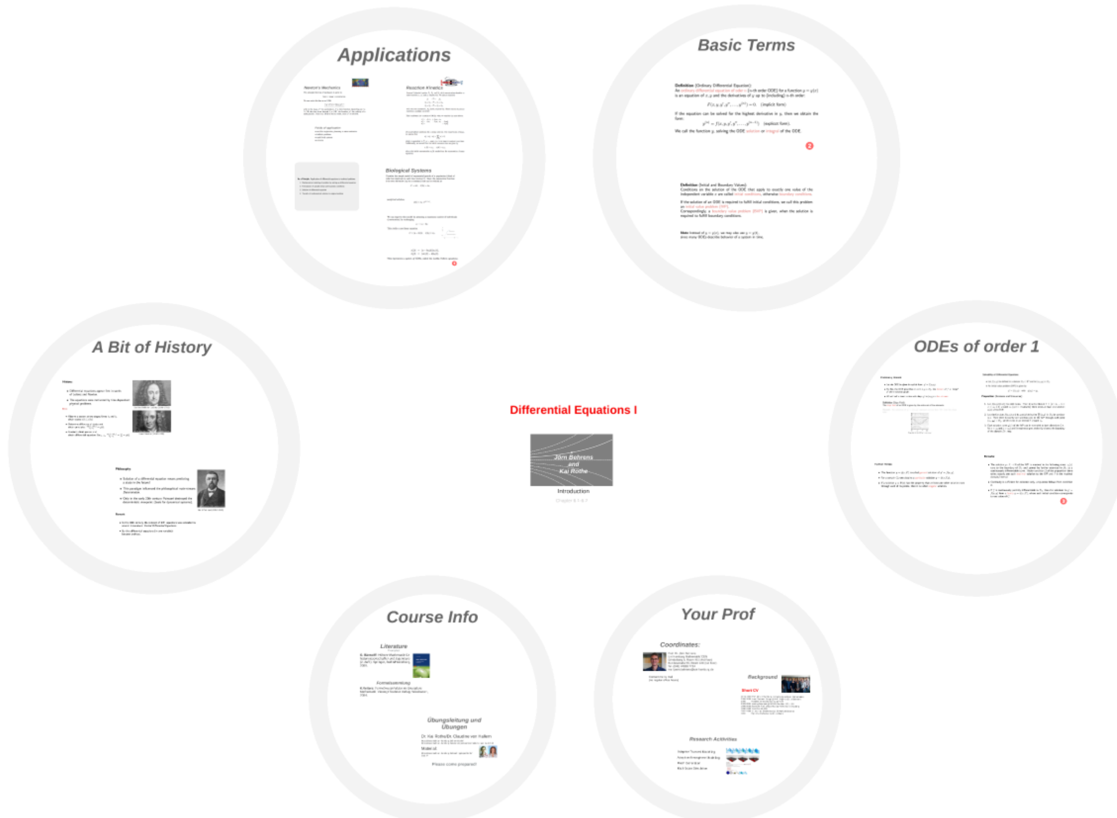
Applications
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