Differential Equations I

Separation of Variables
Variation of Constants

Chapters 6.4-6.5

Recap

Definition (Ordinary Differential Equation):
An ordinary differential equation of oder n (n-th order ODE) for a function $y = y(x)$ is an equation of x, y and the derivatives of y up to (including) n-th order:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \quad \text{(implicit form)}$$

If the equation can be solved for the highest derivative in y, then we obtain the form:

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}) \quad \text{(explicit form).}$$

We call the function y, solving the ODE solution or integral of the ODE.

Definition (Initial and Boundary Values):
Conditions on the solution of the ODE that apply to exactly one value of the independent variable x are called initial conditions, otherwise boundary conditions.

If the solution of an ODE is required to fulfill initial conditions, we call this problem an initial value problem (IVP).
Correspondingly, a boundary value problem (BVP) is given, when the solution is required to fulfill boundary conditions.

ODE of Order 1:

- Let the ODE of order 1 be given in explicit form: $y' = f(x, y)$.
- Pairs $x, y \in D_f$ are in the domain of f.

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ODE of order 1
with separable variables

Idea:
Let the ODE be given in the orm

$$y' = \frac{g(x)}{h(y)}.$$

We call this differential equation with separable variables.
Let $g(x)$ and $h(y)$ for $(x, y) \in D_f$ continuous and $h(y) \neq 0$.

- According to existence theorem there exists at least one solution.
- Let
$$G(x) = \int_a^x g(t)\,dt, \quad \text{and} \quad H(y) = \int_b^y h(t)\,dt$$
primitive functions (antiderivatives) for g and h, and H^{-1} inverse of H (i.e. $H^{-1}(H(y)) = y$).
- Write the ODE as $h(y)y' = g(x)$ then integration yield the solution:
$$H(y(x)) = G(x) + C$$
- Application of the inverse results in
$$y(x) = H^{-1}[H(y(x))] = H^{-1}[G(x) + C].$$

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Solution scheme:
Let an ODE of the form

$$y' = \frac{g(x)}{h(y)},$$

be given and let $g(x)$ and $h(y)$ for $(x, y) \in D_f$ continuous, $h(y) \neq 0$, $G(x)$, $H(y)$ as before.

1. Write the ODE in form $h(y)y' = g(x)$ resp. $h(y)dy = g(x)dx$.
2. Integrate left hand side to y and right hand side to x.
3. If possible, solve analytically for y:
$$H(y) = G(x) + C.$$
If not possible, the solution $y(x)$ is given in implicit form.
4. $C = C_0 := H(y_0) - G(x_0)$ yields a solution of the IVP $y(x_0) = y_0$.

Example:
Let

$$y' = \sin x \cos y.$$

- Note: $\cos y \neq 0$ for $y \neq (k + \frac{1}{2})\pi$ $(k \in \mathbb{Z})$.
- Obtain:
$$\frac{y'}{\cos y} = \sin x \quad \text{resp.} \quad \int \frac{dy}{\cos y} = \int \sin x\,dx.$$
- Integrate: $\ln|\tan(\frac{y}{2} + \frac{\pi}{4})| = -\cos x + C_0$.
- Solve for y:
$$y(x) = 2\arctan(Ce^{-\cos x}) - \frac{\pi}{2} \quad C \in \mathbb{R}$$
- Constant solutions: $y(x) \equiv (k + \frac{1}{2})\pi$
- Remember: slope feld!

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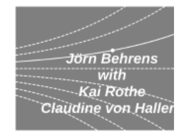

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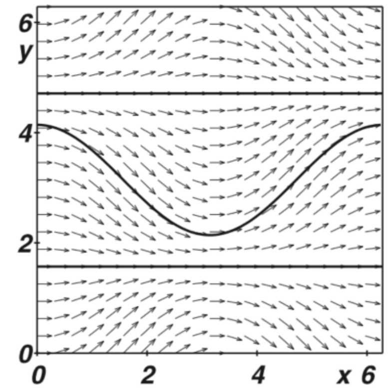

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Linear ODE of order 1

Definition: (Linear differential equation of first order)
Let

$$a(x)y' + b(x)y = c(x).$$

Let the coefficients $(a(x), b(x), c(x))$ be continuous (not necessarily linear) on an interval I and $a(x) \neq 0$. This ODE is called linear ODE of 1st order, if it is linear w.r.t solution $y(x)$, i.e. a linear combination

$$\alpha y_1(x) + \beta y_2(x)$$

of the two solutions y_1 and y_2 is again a solution.

Solution Idea 1:
The homogenous linear ODE $y' + p(x)y = 0$ is a special case of an ODE with separable variables!

For $y > 0$ and $y < 0$ write

$$\frac{dy}{y} = p(x)dx \quad \Rightarrow \quad \ln|y| = -\int p(x)\, dx + C_0$$

with $|y| = e^{C_0} e^{-P(x)}$ resp. $y = Ce^{-P(x)}$ $(C \in \mathbb{R}, C \neq 0)$.
Where $P(x)$ is antiderivative of $p(x)$.

Solution Idea 2: (Variation of Constants)
For a general solution of the homogenous linear ODE $y' + p(x)y = 0$ vary C, i.e. use $C = C(x)$.

- Ansatz:

$$y(x) = C(x)e^{-P(x)}.$$

- Substitute:

$$C'(x)e^{-P(x)} - C(x)p(x)e^{-P(x)} + p(x)C(x)e^{-P(x)} = q(x).$$

- Eliminate and integrate:

$$C'(x)e^{-P(x)} = q(x) \quad \Rightarrow \quad C'(x) = q(x)e^{P(x)}$$
$$\Rightarrow \quad C(x) = \int_{x_0}^{x} q(t)e^{P(t)}\, dt + C_1 \quad C_1 \equiv \text{const.}, C_1 \in \mathbb{R}.$$

- Use the Ansatz:

$$\begin{aligned} y(x) &= e^{-P(x)}\left(C_1 + \int_{x_0}^{x} q(t)e^{P(t)}\, dt\right) \\ &= C_1 e^{-P(x)} + e^{-P(x)} \int_{x_0}^{x} q(t)e^{P(t)}\, dt \\ &= y_{\text{hom}}(x) + y_{\text{inh}}(x). \end{aligned}$$

Example: (Bernoulli's Differential Equation)

$$y' + p(x)y = q(x)y^n.$$

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Observations:

- Differentiation proves:

$$y_{\text{inh}}(x) = e^{-P(x)} \int_{x_0}^{x} q(t)e^{P(t)}\, dt$$

is a particular solution of the inhomogenous ODE.

- Since

$$y_{\text{hom}}(x) = C_1 e^{-P(x)}$$

a general solution of the homogenous ODE, $y(x) = y_{\text{hom}} + y_{\text{inh}}(x)$ is solution to the inhomogenous ODE for each $C_1 \in \mathbb{R}$.

- On the other hand each arbitrary solution $\tilde{y}(x)$ to the inhomogenous ODE is of the above form.

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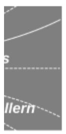

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of the two solutions y_1 and y_2 is again a solution.

Remarks:

- Assuming $a(x) \neq 0$ $(x \in I)$, we have

$$y' + p(x)y = q(x)$$

  with $p(x) = \frac{b(x)}{a(x)}$, $q(x) = \frac{c(x)}{a(x)}$ both continuous.

- Existence and uniqueness are guaranteed (no singular solutions) if $p(x)$ and $q(x)$ are continuous in I.

- If $q(x) = 0$ the ODE is called homogenous, otherwise inhomogenous.

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ODE of order 1 with separable variables

Differential Equations I

Jörn Behrens
with
Kai Rothe
Claudine von Hallern

Separation of Variables
Variation of Constants
Chapters 6.4-6.5

Linear ODE of order 1

Recap