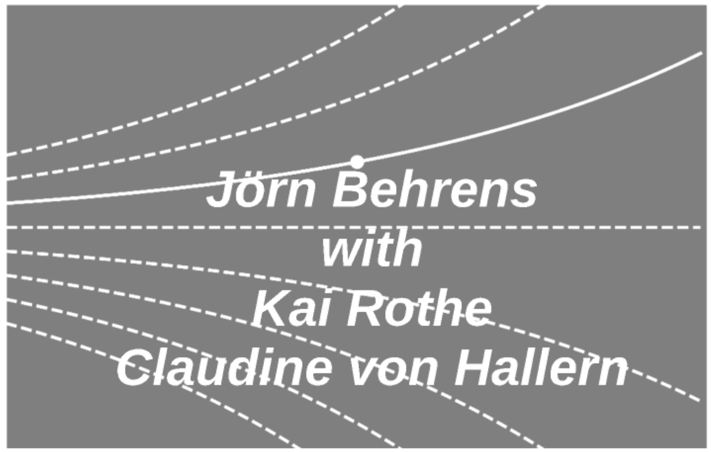
Differential Equations I

Winter 2021/22



Solution of ODEs by Transformation
Systems of 1st Order

Chapters 6.6-6.7

Recap:
Separation of Variables
Variation of Constants

Solution scheme:
Let an ODE of the form

$$y' = \frac{g(x)}{h(y)},$$

be given and let $g(x)$ and $h(y)$ for $(x, y) \in D_f$ continuous, $h(y) \neq 0$, $G(x)$, $H(y)$ as before.

1. Write the ODE in form $h(y)y' = g(x)$ resp. $h(y)dy = g(x)dx$.
2. Integrate left hand side to y and right hand side to x.
3. If possible, solve analytically for y:

   $$H(y) = G(x) + C.$$

   If not possible, the solution $y(x)$ is given in implicit form.
4. $C = C_0 := H(y_0) - G(x_0)$ yields a solution of the IVP $y(x_0) = y_0$.

Solution Idea 2: (Variation of Constants)
For a general solution of the homogenous linear ODE $y' + p(x)y = 0$ vary C, i.e. use $C = C(x)$.

- Ansatz:

  $$y(x) = C(x)e^{-P(x)}.$$

- Substitute:

  $$C'(x)e^{-P(x)} - C(x)p(x)e^{-P(x)} + p(x)C(x)e^{-P(x)} = q(x).$$

- Eliminate and integrate:

  $$C'(x)e^{-P(x)} = q(x) \quad \Rightarrow \quad C'(x) = q(x)e^{P(x)}$$
  $$\Rightarrow \quad C(x) = \int_{x_0}^{x} q(t)e^{P(t)}\,dt + C_1 \quad C_1 \equiv \text{const.}, C_1 \in \mathbb{R}.$$

- Use the Ansatz:

  $$\begin{aligned} y(x) &= e^{-P(x)}\left(C_1 + \int_{x_0}^{x} q(t)e^{P(t)}\,dt\right) \\ &= C_1e^{-P(x)} + e^{-P(x)}\int_{x_0}^{x} q(t)e^{P(t)}\,dt \\ &= y_{\text{hom}}(x) + y_{\text{inh}}(x). \end{aligned}$$

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Transformation

Preliminary Remarks:

- **Goal**: Solution of diverse ODEs of 1st and 2nd order
- **Type**: Consider ODE of the form

$$F(x, y', y'') = 0$$

Remark:
Let the 2nd order ODE be given (note that x does not appear explicitly):

$$F(y, y', y'') = 0$$

Consider:
ODE of the form $y' = \phi(ax + by + c)$, $b \neq 0$. Let ϕ be continuous.

Solution Idea:

- **Substitution**: $z = ax + by + c$ and $z' = a + by'$ yields:

$$y' = \frac{z' - a}{b} = \phi(z)$$

  Therefore

$$z' = a + b\phi(z).$$

- **Separation of Variables**: Obtain solution

$$\frac{dz}{a + b\phi(z)} = dx \quad \Rightarrow \quad \int \frac{dz}{a + b\phi(z)} = \int dx + C = x + C.$$

Consider:
ODE of the form $y' = \phi(\frac{y}{x})$, with $x \neq 0$ and ϕ continuous.

Preliminary Remarks:

- **Goal**: Solution of diverse ODEs of 1^{st} and 2^{nd} order
- **Type**: Consider ODE of the form

$$F(x, y', y'') = 0$$

Idea:

- **Substitution**: using $v := y'$ we obtain ODE of 1^{st} order:

$$F(x, v, v') = 0$$

- **Integration**: If $v = \Psi(x, C)$ is a general solution to the 1^{st} order ODE, then

$$y(x) = \int \Psi(\zeta, C)\, d\zeta + C_1, \quad C, C_1 \in \mathbb{R}$$

is a general solution to the 2^{nd} order ODE. 1

Remark:
Let the 2nd order ODE be given (note that x does not appear explicitly):

$$F(y, y', y'') = 0$$

Solution Idea:

- **Substitution**: with $v(y) := y'$ and the chain rule we obtain:

$$y'' = \frac{d}{dx}v(y) = \frac{dv}{dy}\frac{dy}{dx} = v'(y)y' = v'(y)v(y)$$

  This yields a 1st order ODE for v: $F(y, v, v'v) = 0$.

- **Integration**: If $v = \Psi(x, C)$ is general solution of 1st order ODE, then with $v(y) = y'$ we obtain

$$y' = \Psi(y, C)$$

  an ODE with separable variables for y, with general implicit solution

$$\int_{y_0}^{y} \frac{d\zeta}{\Psi(\zeta, C)} = x + C_1, \quad C, C_1 \in \mathbb{R}.$$

2

Consider:
ODE of the form $y' = \phi(\frac{y}{x})$, with $x \neq 0$ and ϕ continuous.

Solution Idea:

- **Substitution**: $u = \frac{y}{x}$ yields:

$$y = xu \quad \Rightarrow \quad y' = u + xu' = \phi(u)$$

Therefore

$$xh' = \phi(u) - u \quad \Rightarrow \quad u' = \frac{\phi(u) - u}{x}.$$

- **Separation of Variables**: We obtain as solution

$$\frac{du}{\phi(u) - u} = \frac{dx}{x} \quad \Rightarrow \quad \int \frac{du}{\phi(u) - u} = \ln|x| + C.$$

3

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Euler's ODE

Definition:
Differential equations of the form

$$\sum_{j=0}^{k} a_j x^j y^{(j)}(x) = f(x),$$

with $a_j \in \mathbb{R}$ $(j = 0, \dots, k)$ constant, $a_k \neq 0$, $x > 0$, are called Euler's Differential Equations of k^{th} order.

Solution Approach:
The ansatz $y(x) = x^r$ for the homogenous equation, i.e. $f(x) \equiv 0$, yields:

$$\sum_{j=0}^{k} a_j r(r-1)\cdots(r-j+1) = 0.$$

We obtain: Solution of this equation are roots of a polynomial in r of degree k.

Computation for case $k = 2$:

- Euler's ODE (homogenous): $a_0 y + a_1 x y' + a_2 x^2 y'' = 0$.
- Substitution yields: $a_0 + a_1 r + a_2 r(r-1) = 0$, quadratic polynomial.
- Differentiation proves: $y = x^r$ is solution of homogenous Euler's ODE, if r root of polynomial.
- If $r_1 \neq r_2$ are real roots of polynomial, then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ are solutions of ODE.
- If $r_1, r_2 \in \mathbb{C}$ are complex roots, then if $r_1 = a + ib$ is root, so is $r_2 = \bar{r}_1 = a - ib$.
- Complex solution for $y = x^r$:

  $$x^{a+ib} = e^{\ln x^{a+ib}} = e^{(a+ib)\ln x} = e^{a \ln x} e^{ib \ln x} = x^a[\cos(b \ln x) + i \sin(b \ln x)]$$

- For complex solutions of the problem one finds

  $$y_1(x) = x^a \cos(b \ln x) \quad \text{and} \quad y_2(x) = x^a \sin(b \ln x)$$

  two solutions of the homogenous Euler's ODE.
- General solution: due to linearity the general solution is

  $$y(x) = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \ln x).$$

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Transformation

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Euler's ODE