Differential Equations I

week 04 / J. Behrens

TUHH Technische Universität Hamburg

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PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

- VOLLSTÄNDIG GEIMPFTE
- GENESENE
- GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen. Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis. Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

- FULLY VACCINATED
- RECOVERED
- TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now. Otherwise you could be banned from the room!

Thank you for your understanding. Protect yourself and others!

1 . **Proposition**: (Holonomic Solution)

Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be a fundamental system on $]a, b[$ of

$$\mathbf{y}' = A(x)\mathbf{y}.$$

Then any solution $\mathbf{y}$ on $]a, b[$ can be written in the form

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{y}_i, \quad \text{const.} \equiv c_i \in \mathbb{R} \text{ or } \mathbb{C}.$$

This above $\mathbf{y}$ is called holonomic solution of the homogeneous system of differential equations.

Remark: The linear combinations are solutions of $\mathbf{y}' = A(x)\mathbf{y}$, since $\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{y}_i$ yields:

$$\mathbf{y}' = \sum_{i=1}^{n} c_i \mathbf{y}_i' = \sum_{i=1}^{n} c_i A(x)\mathbf{y}_i = A(x) \sum_{i=1}^{n} c_i \mathbf{y}_i = A(x)\mathbf{y}.$$

Observations:

a) Solutions of the lin. syst. of ODES $\vec{y}' = A(x)\vec{y}$ form a vector space over the space of c_i

b) Since we have n lin. independent solutions (fundamental system)
$\Longrightarrow$ Vector space has dimension n.

Special case: $n=1$ or only one ODE $y' = p(x)\,y$

$\Longrightarrow$ Solution $y = C y_1 = C \cdot e^{-P(x)}$ is general solution

Question: How to find fundamental system?
Only solvable if $A(x)$ has only constant entries
otherwise we may find solutions for special cases
or by chance
or numerically

Aim: Find solutions if $A(x)$ has entries $a_{ij} \equiv$ constants.

Then: Let $\vec{v}$ Eigenvector (EVc) of the Eigenvalue (EVa) λ
then $\vec{y}\,' = \lambda e^{\lambda x} \vec{v} = \lambda \vec{y} = A \vec{y}$

So, with λ EVa and $\vec{v}$ an EVc we can construct a solution to $\vec{y}\,' = A\vec{y}$.

Example: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

One finds $\lambda_1 = 1$, $\lambda_2 = 3$ are EVa with EVc

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

With the construction idea we obtain

$$\vec{y}_1 = e^{\lambda_1 x} \vec{v}_1 \quad , \quad \vec{y}_2 = e^{\lambda_2 x} \vec{v}_2$$

Show $[y_1, y_2]$ form a fundamental system $\to$ Wronski-Test

$$W(x) = \det \begin{pmatrix} e^x & e^{3x} \\ e^x & -e^{3x} \end{pmatrix} = -e^x e^{3x} - e^x e^{3x} = -2e^{4x} \neq 0$$

$\Rightarrow$ y_1, y_2 form a fundamental system.

$\Rightarrow$ $\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 = c_1 e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $c_1, c_2 \in \mathbb{R}$

ist the general solution.

(2) **Proposition**: (Solution of system of ODEs with constant coefficients)
Let $A = (a_{ij})$ a constant $n \times n$-matrix with $a_{ij} \in \mathbb{R}$, λ an eigen value (EVa) of A with corresponding eigen vector (EVc) $\mathbf{v}$.
Then

$$\mathbf{y} = e^{\lambda x}\mathbf{v}$$

is a solution of the homogeneous system of ODEs of 1st order $\mathbf{y}' = A\mathbf{y}$.

If A has n pairwise different EVa $\lambda_1, \dots, \lambda_n$ with corresponding EVc $\mathbf{v}_1, \dots, \mathbf{v}_n$, the solutions

$$\mathbf{y}_i = e^{\lambda_i x}\mathbf{v}_i, \quad i = 1, \dots, n$$

form a fundamental system. By linear combination

$$\mathbf{y} = \sum_{i=1}^{n} c_i e^{\lambda_i x}\mathbf{v}_i$$

all solutions of the homogeneous system of ODEs are given.

Remarks: (Application of Linear Algebra)

- Matrices not always have pairwise different EVa, multiplicity > 1 is possible. Therefore, construction of a fundamental system is only possible, if algebraic and geometric multiplicity correspond.
- If the algebraic multiplicity $\sigma_k < n$ corresponding to EVa λ_k equals the geometric multiplicity, then there exists σ_k linearly independent EVc $\mathbf{v}_{k_1}, \dots, \mathbf{v}_{k_{\sigma_k}}$, and thus σ_k linearly independent solutions

$$\mathbf{y}_{k_1} = e^{\lambda_k x}\mathbf{v}_{k_1}, \ \dots, \ y_{k_{\sigma_k}} = e^{\lambda_k x}\mathbf{v}_{k_{\sigma_k}}.$$

- In this case for m different EVa $\lambda_1, \dots, \lambda_m$ with multiplicities $\sigma_1, \dots, \sigma_m$ there are n linearly independent solutions (fundamental system)

$$\mathbf{y}_{k_1} = e^{\lambda_k x}\mathbf{v}_{k_1}, \ \dots, \ y_{k_{\sigma_k}} = e^{\lambda_k x}\mathbf{v}_{k_{\sigma_k}}, \quad (k = 1, \dots, m),$$

since $\sum_{k=1}^{n} \sigma_k = n$.

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Consider:

$$\left.\begin{aligned} y_1' &= -2y_1 - 8y_2 - 12y_3 \\ y_2' &= y_1 + 4y_2 + 4y_3 \\ y_3' &= y_3 \end{aligned}\right\} \circledast$$

Let: $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\vec{y}' \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix}$, $A = \begin{pmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ (*) can be written $\boxed{\vec{y}' = A\vec{y}}$.

Eigenvalues: Characteristic Polynomial: $\chi_A(\lambda) = (1-\lambda)(\lambda-2)\lambda$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2$

Eivectors: One finds $\vec{v}_1 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

lin. independent.

Eigenvector Matrix: $B = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ is regular and where

$AB = BD$ where $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (since $A\vec{v}_k = \lambda\vec{v}_k$)

$\Rightarrow B^{-1}AB = D$

Auxiliary Vector: Let $\vec{z}$ such that $\vec{y} = B\vec{z}$

$\Rightarrow \vec{y}' = AB\vec{z} \Rightarrow B^{-1}\vec{y}' = \vec{z}' = B^{-1}AB\vec{z}$

$\Rightarrow \vec{z}' = D\vec{z}$ or $\left.\begin{aligned} z_1' &= 0 \\ z_2' &= z_2 \\ z_3' &= 2z_3 \end{aligned}\right\}$ decoupled system of ODEs

$\Rightarrow$ Solutions: $\vec{z}_1 = c_1$, $\vec{z}_2 = c_2 e^{x}$, $z_3 = c_3 e^{2x}$

Back substitution: we have $\vec{y} = B\vec{z}$, so

$$\vec{y} = \begin{pmatrix} 4 & 4 & 2 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^x \\ c_3 e^{2x} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 4 + c_2 4 e^x + c_3 2 e^{2x} \\ -c_1 - c_3 e^{2x} \\ -c_2 e^x \end{pmatrix} \quad c_1, c_2, c_3 \in \mathbb{R}}_{\text{general solution}}$$

Question: What if algebraic mult. $\neq$ geom. mult. ?

Consider: $\vec{y}\,' = A\vec{y} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \vec{y}$ (**)

Find: $\lambda = 3$ is double EVa of $A = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$

EVc corresponding to λ have the form $\vec{v} = \begin{pmatrix} t \\ -t \end{pmatrix}$

so $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not lin indep.

Solution: $\vec{y}_1 = e^{\lambda x} \vec{v}_1 = e^{3x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is one solution.

Fundamental System: we need lin. independent other solution

We know, it does not exist in the form $e^{\lambda x} \vec{v}$

Look for a more general solution of the form

$$\vec{y}_2 = x e^{3x} \vec{w} \quad (\cdot) \qquad \text{with } \vec{w} \text{ const.}$$

Fill in (**) :

$$\Rightarrow \underbrace{3x\,e^{3x}\vec{\omega} + e^{3x}\vec{\omega}}_{\vec{y}\,'} - \underbrace{A x e^{3x}\vec{\omega}}_{A\vec{y}}$$

$$= x e^{3x}(3\vec{\omega} - A\vec{\omega}) + e^{3x}\vec{\omega} \overset{!}{=} 0$$

this holds only if $\vec{\omega} = \vec{0}$

Next more general approach:

$$\vec{y}_2 = e^{3x}\vec{v} + x e^{3x}\vec{\omega} \quad (\cdot\cdot) \qquad \text{with } \vec{v} \text{ and } \vec{\omega} \text{ const.}$$

Fill in (**) :

$$3x e^{3x}\vec{\omega} + e^{3x}(\vec{\omega} + 3\vec{v}) = A(x e^{3x}\vec{\omega} + e^{3x}\vec{v})$$

$$\Rightarrow \vec{0} = x e^{3x}(A - 3E)\vec{\omega} + e^{3x}[(A-3E)\vec{v} + \vec{\omega}]$$

comparing coefficients:

$$\Rightarrow (A - 3E)\vec{\omega} = 0 \qquad \text{and} \qquad (A-3E)\vec{v} = \vec{\omega}$$

Now $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ solves the first eq.

$\vec{\omega} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ solves the second eq.

$\vec{v}$ and $\vec{\omega}$ are lin. indep solutions of $(A-3E)^2 v = 0$

general solution:

$$\vec{y}_2 = e^{3x}\vec{v} + x e^{3x}\vec{\omega}$$

$$\Rightarrow \vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 \qquad , c_1, c_2 \in \mathbb{R}$$