Differential Equations I

Week 06 / J. Behrens

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BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

–VOLLSTÄNDIG GEIMPFTE
–GENESENE
–GETESTETE
(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

–FULLY VACCINATED
–RECOVERED
–TESTED
(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

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- Consider the homogeneous ODE of order n.
- Ansatz:

$$y(x) = e^{\lambda x}.$$

- We have: $y^{(k)} = \frac{d^k}{dx^k} e^{\lambda x} = \lambda^k e^{\lambda x}$ and $y = e^{\lambda x} \neq 0$ for $x \in \mathbb{R}$.
- Therefore $y = e^{\lambda x}$ $(g = 0)$ is solution, iff λ is a root of

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0.$$

Solution Approach:
Ivestigating the roos of $P(\lambda)$ yields the following cases:

1. $P(\lambda)$ has n different real roots $\lambda_1, \ldots, \lambda_n$.
2. $P(\lambda)$ has a complex root λ_k.
3. $P(\lambda)$ has a (real or complex) r-multiple root λ_1 $(r \geq 2)$.

Roots of $P(\lambda)$.

Case 1: $P(\lambda)$ has n different roots $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

$\Rightarrow$ The homogeneous ODE $L[y] = 0$ has n solutions

$$e^{\lambda_1 x}, \dots, e^{\lambda_n x}$$

Case 2: $P(\lambda)$ has a complex root, $\lambda_k \in \mathbb{C}$.

- $e^{\lambda x}$ is sensibly defined for $\lambda \in \mathbb{C}$ and $\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}$, $\lambda \in \mathbb{C}$

  $\Rightarrow$ $e^{\lambda x}$ solves the homog. ODE for $\lambda_k \in \mathbb{C}$.
- If $a_0, \dots, a_{n-1} \in \mathbb{R}$, then there exist for $e^{\lambda_k x}$ two real-valued solutions
- Let $y_1(x), y_2(x)$ $(x \in \mathbb{R})$ are real-valued functions

  $y(x) = y_1(x) + i\, y_2(x)$ is complex-valued

$\Rightarrow$ $y'(x) = y_1'(x) + i\, y_2'(x)$ or $y^{(l)}(x) = y_1^{(l)}(x) + i\, y_2^{(l)}(x)$ $l \in \mathbb{N}$

$\Rightarrow$ $y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_0 y(x)$

$$= \underbrace{[y_1^{(n)} + a_{n-1} y_1^{(n-1)}(x) + \dots a_0 y_1(x)]}_{Re} + i \underbrace{[y_2^{(n)}(x) + a_{n-1} y_2^{(n-1)}(x) + \dots + a_0 y(x)]}_{Im} = 0$$

- We have that Re and Im both need to vanish
- $y(x)$ is solution to $L[y] = 0$

  $\Leftrightarrow$ $y_1 = Re(y)$ and $y_2 = Im(y)$ are solutions
- Use Euler's Formula $e^{i\phi} = \cos\phi + i \sin\phi$ $\phi \in \mathbb{R}$
  and addition rules $e^{(a+ib)} = e^a e^{ib}$ $a, b \in \mathbb{R}$
  write $\lambda_k = \sigma_k + i\, \tau_k$

$\Rightarrow \quad y_k(x) = e^{\lambda_k x} = e^{\sigma_k x}(\cos \tau_k x + i \sin \tau_k x)$

$\Rightarrow$ we obtain the two solutions

$$e^{\sigma_k x} \cos \tau_k x \quad \text{and} \quad e^{\sigma_k x} \sin \tau_k x$$

- Since with λ_k also $\bar{\lambda}_k$ is a root of $P(\lambda)$

$$\Rightarrow \quad e^{\sigma_k x} \cos(-\tau_k x) = e^{\sigma_k x} \cos(\tau_k x)$$
$$e^{\sigma_k x} \sin(-\tau_k x) = -e^{\sigma_k x} \sin(\tau_k x)$$

these are (up to the sign) exactly the same solutions

Case 3: $P(\lambda)$ has a root λ_1 of multiplicity r $(r \geq 2)$
$\lambda \in \mathbb{R}$ or $\mathbb{C}$.

Then $y(x) = e^{\lambda_1 x}$ is a solution.

- We have that using Schwarz' theorem and continuity of $f(\lambda, x) = e^{\lambda x}$

$$L[e^{\lambda x}] = e^{\lambda x} P(\lambda) = e^{\lambda x} (\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \cdots (\lambda - \lambda_n)$$
$$= e^{\lambda x} (\lambda - \lambda_1)^r Q(\lambda)$$

- Differentiation

$$L[x e^{\lambda x}] = e^{\lambda x} [x (\lambda - \lambda_1)^r Q(\lambda) + r (\lambda - \lambda_1)^{r-1} Q(\lambda) + (\lambda - \lambda_1)^r Q'(\lambda)]$$

- Since $r \geq 2$ the right hand side vanishes for $\lambda = \lambda_1$, i.e

$\underline{y(x) = x e^{\lambda_1 x}}$ is also solution to $L[y] = 0$

- Repeat this procedure $r-1$ times:

$e^{\lambda_1 x}, x e^{\lambda_1 x}, x^2 e^{\lambda_1 x}, \ldots, x^{r-1} e^{\lambda_1 x}$ are all solutions.

② Example:

- consider $y'' - 4y = 0$
- Charact. Polynomial : $\lambda^2 - 4 = P(\lambda)$
- Roots : $\lambda_1 = 2$, $\lambda_2 = -2$
- Fundamental system : $y_1(x) = e^{2x}$, $y_2(x) = e^{-2x}$
- General Solution : $y(x) = C_1 e^{2x} + C_2 e^{-2x}$

③ **Preliminary Remarks**:

- As an example, consider

$$y'' + a(x)y' + b(x)y = g(x).$$

- Let $y_1(x)$ and $y_2(x)$ be lin. independent solutions of the homogeneous equation ($g(x) = 0$).
- It holds

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

- The solution of the homogeneous equation is given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

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- $y(x) = C_1(x)\, y_1(x) + C_2(x)\, y_2(x)$
- $y'(x) = C_1(x)\, y_1'(x) + C_2(x)\, y_2'(x) + \underbrace{C_1'(x)\, y_1(x) + C_2'(x)\, y_2(x)}_{=!\,0}$

Assume $C_1(x)$ and $C_2(x)$ fulfill

$C_1'(x)\, y_1(x) + C_2'(x)\, y_2(x) = 0$ ⊛

- $y''(x) = C_1(x)\, y_1''(x) + C_2(x)\, y_2''(x) + C_1'(x)\, y_1'(x) + C_2'(x)\, y_2'(x)$

- Substitute into $y'' + ay' + by = g$

$$c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x) +$$
$$a(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] + b(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right] = g(x)$$

- Reordering:

$$c_1(x)\underbrace{\left[y_1''(x) + a(x)y_1'(x) + b(x)y_1(x)\right]}_{=0} + c_2(x)\underbrace{\left[y_2''(x) + a(x)y_2'(x) + b(x)y_1(x)\right]}_{=0 \text{ homog. solutions}}$$
$$+ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x)$$

$$\Rightarrow c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x) \quad (**)$$

- With $(*)$ and $(**)$ we obtain:

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$

- Since y_1, y_2 form a fundamental system: $W(x) = y_1 y_2' - y_1' y_2 \neq 0$

- Solve by Cramer's rule:

$$c_1'(x) = -\frac{y_2(x)\,g(x)}{W(x)} \qquad c_2'(x) = \frac{y_1(x)\,g(x)}{W(x)}$$

- Integration:

$$c_1(x) = -\int \frac{y_2(x)\,g(x)}{W(x)}\,dx + C_3$$

$$c_2(x) = \int \frac{y_1(x)\,g(x)}{W(x)}\,dx + C_4$$

- Solution:

$$y(x) = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$$