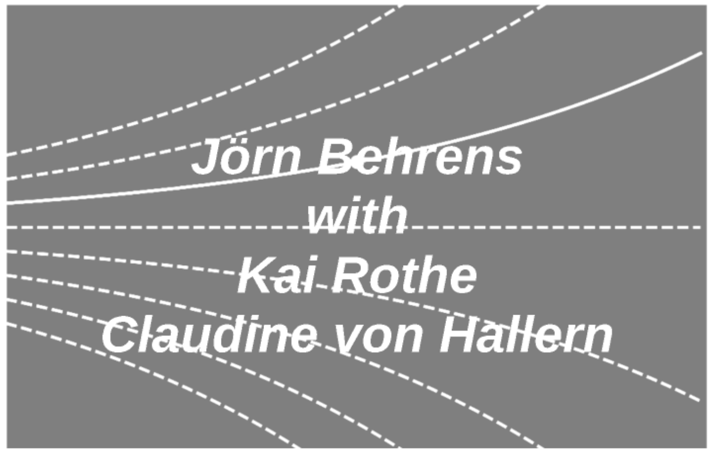
Differential Equations I



Laplace Transformation

Chapters 11.6-11.9

Motivation

Idea: Consider the initial value problem of n^{th} order

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(t), \quad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

with suitable right hand side r.

Question: Can we find a transformation $Y(z) = \mathcal{T}[y(t)]$ resp. $R(z) = \mathcal{T}[r(t)]$, for which the inverse $y(t) = \tilde{\mathcal{T}}[Y(z)]$ resp. $r(t) = \tilde{\mathcal{T}}[R(z)]$ exists, such that

$$Y(z) = F[R(z)], \quad F \text{ suitable functional,}$$

is easily solvable? Then the solution $y(t) = \tilde{\mathcal{T}}[Y(z)]$ could be obtained easily.

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Definition
Laplace Transformation

Definition: (Laplace Transformation)
Let $f : [0, \infty[\to \mathbb{R}$ be a function. Then

$$F(z) = \int_0^\infty e^{-zt} f(t)\, dt$$

with $z \in \mathbb{C}$ defines a function F called Laplace Transform of f. The mapping of f onto F is called Laplace Transformation. We also write $\mathcal{L}[f(t)]$.

Definition: (exponential order)
A function $f : [0, \infty[\to \mathbb{R}$ is of exponential order γ, if constants $M > 0$ and $\gamma \in \mathbb{R}$ exist, such that for all $0 \le t < \infty$

$$|f(t)| \le M e^{\gamma t}.$$

Proposition: (Existence of Laplace Transform)
Let f be piecewise continuous in $[0, \infty[$ and of exponential order γ. Then the Laplace transform $F(z)$ exists for all $z \in \mathbb{C}$ with $\mathrm{Re} z > \gamma$.

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Questions:

- For which f is the Laplace Transform usefully defined?
- Under which conditions does the improper integral exist?

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Remarks:

- Polynomials are of exp. order.
- $\sin$ and $\cos$ are of exp. order.

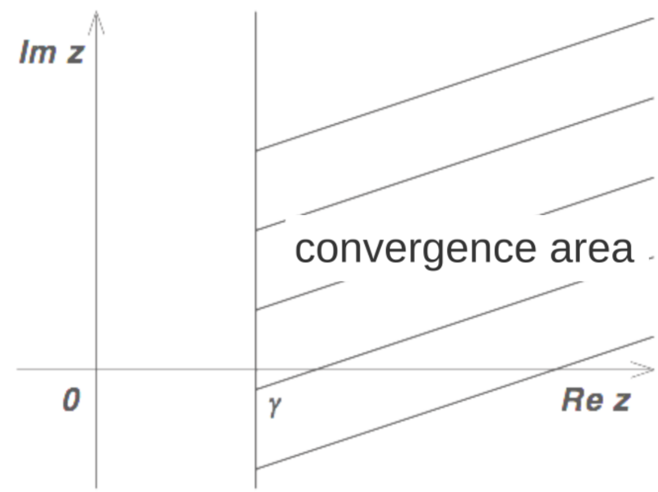
Example: Employ Taylor Series for $t \geq 0$:

$$|t^3| = t^3 \leq 6e^t = 6 + 6t + 3t^2 + t^3 + \cdots$$

Proposition: (Existence of Laplace Transform)
Let f be piecewise continuous in $[0, \infty[$ and of exponential order γ. Then the Laplace transform $F(z)$ exists for all $z \in \mathbb{C}$ with $\mathrm{Re} z > \gamma$.

Observations:

- The integral $\int_0^\infty e^{-zt} f(t)\, dt$ exists in a right half space of the Gaussian plane.
- The weaker $f(t)$ grows for $t \to \infty$, the further the convergence area reaches to the left.



Inverse Laplace Transformation

Proposition: (Inverse of the Laplace Transformation)
Let f be of exponential order γ, f vanishes for $t < 0$ and is piecewise continuous in $\mathbb{R}$. Then for all $x = \mathrm{Re} z > \gamma$

$$\frac{1}{2\pi i} \lim_{A \to \infty} \int_{x-iA}^{x+iA} F(z) e^{zt} \, dz =$$

$$\frac{1}{2\pi} \lim_{A \to \infty} \int_{-A}^{A} F(x+is) e^{(x+is)t} \, ds = \begin{cases} \frac{f(t+0)+f(t-0)}{2} & (t > 0), \\ \frac{f(0+0)}{2} & (t = 0), \\ 0 & (t < 0). \end{cases}$$

In particular, in each point t of continuity of f

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \lim_{A \to \infty} \int_{x-iA}^{x+iA} F(z) e^{zt} \, dz \\ &= \frac{1}{2\pi} \lim_{A \to \infty} \int_{-A}^{A} F(x+is) e^{(x+is)t} \, ds \end{aligned}$$

for $x > \gamma$.

Proposition: (Uniqueness of Laplace Transformation)
Let f_1 and f_2 be of exponential order γ, f_1, f_2 vanish for $t < 0$ and piecewise continuous in $\mathbb{R}$. Further assume for the Laplace transforms $F_1(x) = F_2(x)$ for $\mathrm{Re} z > \gamma$.
Then for each t, at which f_1 and f_2 are continuous, it holds:

$$f_1(t) = f_2(t).$$

Remarks:

- The two propositions allow computation of $f(t)$ from a Laplace transform $F(z)$ by a line integral in the Gaussian plane.
- The Laplace transformation $f(t) \to F(z)$ is a bijective mapping.

Example:

$$f(t) = e^{4t}$$

has the unique Laplace transform

$$F(z) = \frac{1}{z-4}.$$

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Computing Rules for Laplace Transformation

Convention: We denote by $F(z) = \mathcal{L}[f(t)]$ the Laplace transform of a function with exponential order γ that is piecewise constant in $[0, \infty[$.

Proposition: (Linearity)
Let f and g piecewise continuous in $[0, \infty[$ and of exponential order γ.
Then for $a, b \in \mathbb{R}$ it holds

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + g\mathcal{L}[g(t)].$$

Proof follows immediately from linearity of integrals.

Proposition: (Transformation of Derivatives and Integrals)

1. Let f be as in the previous proposition. Then for $\mathrm{Re}\, z > \gamma$ it holds
$$\mathcal{L}[f'(t)] = z\mathcal{L}[f(t)] - f(0).$$
2. Let f be $(k-1)$-times cont. differentiable and $f^{(k-1)}$ piecewise cont. Let $f, f', \dots, f^{(k-1)}$ be of exponential order γ. Then for $\mathrm{Re}\, z > \gamma$ it holds
$$\mathcal{L}[f^{(k)}(t)] = z^k \mathcal{L}[f(t)] - z^{k-1} f(0) - \dots - f^{(k-1)}(0).$$
3. Let f be like in 1. Then for $\mathrm{Re}\, z > \gamma$ it holds
$$\mathcal{L}[\int_0^t f(\tau)\, d\tau] = \frac{1}{z}\mathcal{L}[f(t)].$$

Proposition: (Laplace Transformation of Product with Power Function)
Let $g(t) = (-1)^n t^n f(t)$ and f Laplace transformable with Laplace transform $F(z) = \mathcal{L}[f(t)]$. Then

$$\mathcal{L}[g(t)] = \mathcal{L}[(-1)^n t^n f(t)] = F^{(n)}(z).$$

Proposition: (Transformation of Derivative of Discontinuous Function) Let f be again a function of exponential order γ with the other prerequisites of the prop., and assume f to have a jump discontinuity at $t = 1 > 0$. Then it holds

$$\mathcal{L}[f'(t)] = z\mathcal{L}[f(t)] - f(0) - [f(a+0) - f(a-0)]e^{-az}.$$

Sketch of proof: split the integral

$$\int_0^\infty e^{-z} f'(t)\, dt$$

into $\int_0^{a-0}(\cdot) + \int_{a+0}^\infty(\cdot)$, apply arguments analogous to proposition above.

Proposition: (Laplace Transformation of T-periodic Functions)
Let f be T-periodic (i.e., $f(t-T) = f(t)$), piecewise cont. and bounded.
Then it holds for $\mathrm{Re}\, z > 0$

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-Tz}} \int_0^T e^{-zu} f(u)\, du.$$

Satz: (Damping-Translation, Stretching) Let f be a function of exponential order, γ with the prerequisites of the propositions above, $F(z) = \mathcal{L}[f(t)] = \int_0^\infty e^{-z} f(t)\, dt$, $\mathrm{Re}\, z > \gamma$.

1. A damping factor e^{-at} in the origin domain results in a translation in the image domain:
$$\mathcal{L}[e^{-at} f(t)] = F(z+a) \quad \text{für } \mathrm{Re}\, z > \gamma - a.$$
2. For $a > 0$ it holds
$$\mathcal{L}[f(at)] = \frac{1}{a} F(\frac{z}{a}) \quad \text{für } \mathrm{Re}\, z > a \cdot \gamma.$$

Proposition: (Convolution Rule)
Let f and g be functions of exponential order γ with $f(t) = g(t) = 0$ for $t < 0$.
Let f be continuous and g piecewise continuous in $\mathbb{R}$. Then the Laplace transform of the convolution $f * g$ exists for $\mathrm{Re}\, z > \gamma$ and it holds

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

Definition: (Convolution)
Let f and g be functions. The convolution (product) of f and g is defined in general as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau, \quad t \in \mathbb{R}.$$

Remarks:

- We assume in each case that the improper integral exists.
- For f and g functions as in the Laplace transformation it holds $f(t) = g(t) = 0$ for $t < 0$. Therefore, we have
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Proof:

1. Follows from definition of Laplace transform by partial integration

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^\infty e^{-zt} f'(t)\, dt \\ &= \lim_{A\to\infty} e^{-zt} f(t)|_{t=0}^{t=A} - \int_0^\infty (-z) e^{-zt} f(t)\, dt \\ &= -f(0) + z \int_0^\infty e^{-zt} f(t)\, dt = z\mathcal{L}[f(t)] - f(0).\end{aligned}$$

2. Apply above partial integration k-times.

3. Apply 1. to the function $h(t) = \int_0^t f(\tau)\, d\tau$.

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Solution of ODEs by Laplace Transformation

Idea: Let the IVP of n^{th} order be given

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with r piecewise continuous function of exponential order.
Set

$$Y(z) = \mathcal{L}[y(t)], \quad \text{and} \quad R(z) = \mathcal{L}[r(t)].$$

The Laplace transformation of the IVP is derived from the rules above:

$$\begin{aligned} (z^n + a_{n-1}z^{n-1} + \cdots + a_0)Y(z) &= R(z) \\ \Rightarrow \quad Y(z) = (z^n + a_{n-1}z^{n-1} + \cdots + a_0)^{-1}R(z) &=: G(z)R(z). \end{aligned}$$

If one finds a function $g(t)$ with $\mathcal{L}[g(t)] = G(t)$, then

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The function $K(t, \tau) := g(t - \tau)$ is called Green's Function.

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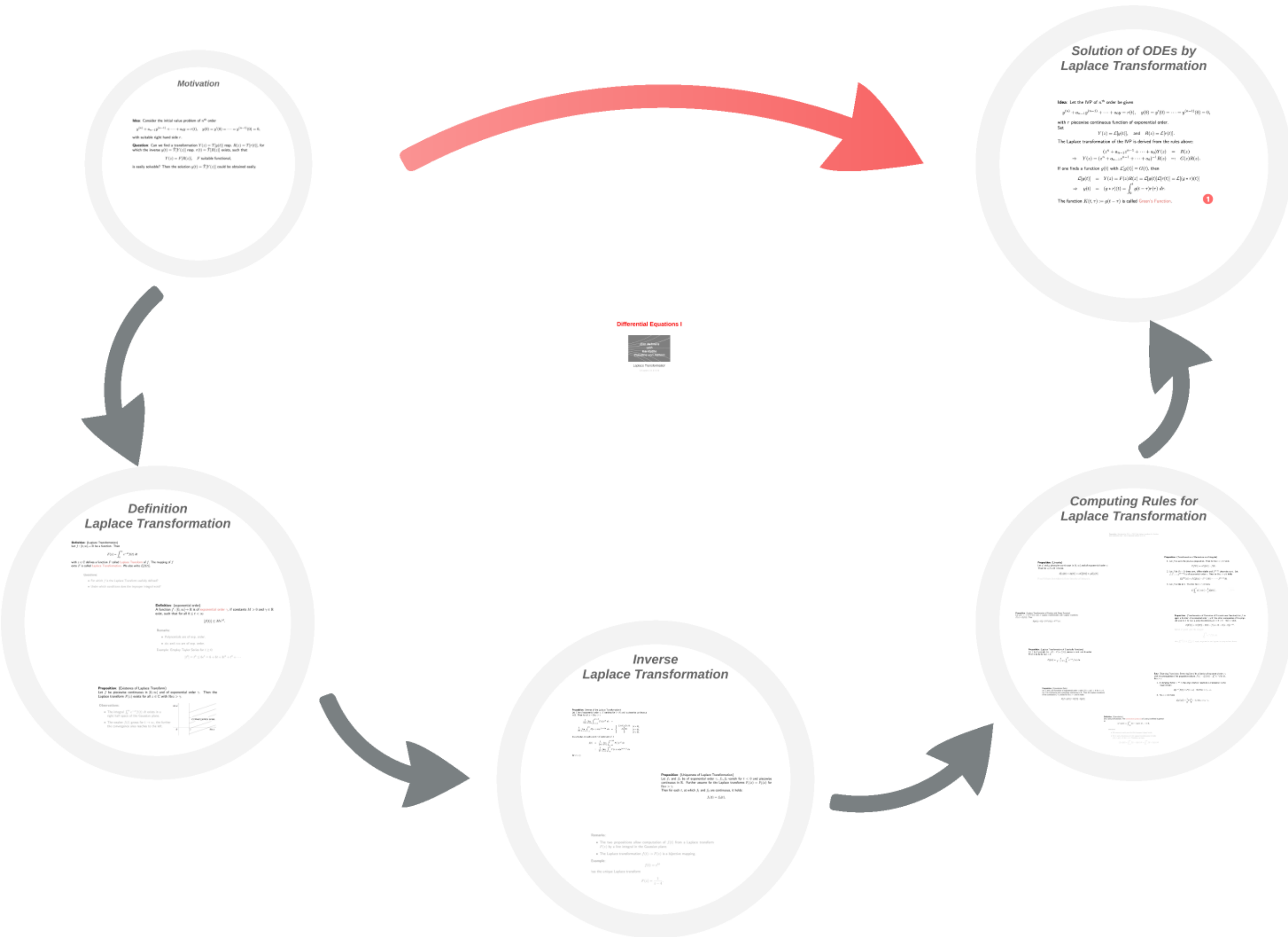
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Differential Equations I
Definition
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