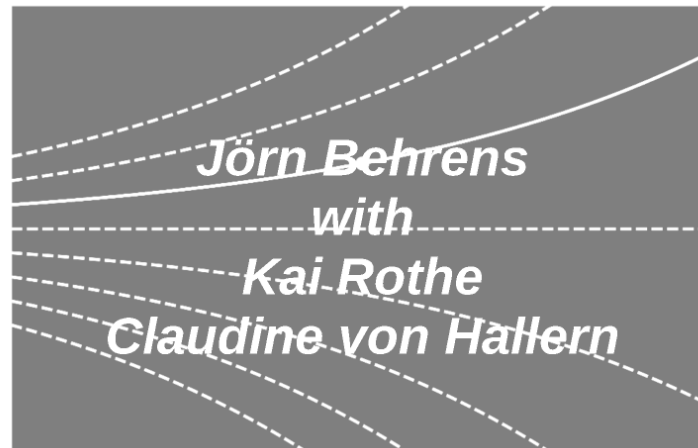


# Differential Equations I



Laplace Transformation

Chapters 11.6-11.9

## Motivation

**Idea:** Consider the initial value problem of  $n^{\text{th}}$  order

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = r(t), \quad y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0,$$

with suitable right hand side  $r$ .

**Question:** Can we find a transformation  $Y(z) = \mathcal{T}[y(t)]$  resp.  $R(z) = \mathcal{T}[r(t)]$ , for which the inverse  $y(t) = \tilde{\mathcal{T}}[Y(z)]$  resp.  $r(t) = \tilde{\mathcal{T}}[R(z)]$  exists, such that

$$Y(z) = F[R(z)], \quad F \text{ suitable functional,}$$

is easily solvable? Then the solution  $y(t) = \tilde{\mathcal{T}}[Y(z)]$  could be obtained easily.

**Idea:** Consider the initial value problem of  $n^{\text{th}}$  order

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = r(t), \quad y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0,$$

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is easily solvable? Then the solution  $y(t) = \tilde{\mathcal{T}}[Y(z)]$  could be obtained easily.

# Definition Laplace Transformation

**Definition:** (Laplace Transformation)  
Let  $f : [0, \infty[ \rightarrow \mathbb{R}$  be a function. Then

$$F(z) = \int_0^{\infty} e^{-zt} f(t) dt$$

with  $z \in \mathbb{C}$  defines a function  $F$  called **Laplace Transform** of  $f$ . The mapping of  $f$  onto  $F$  is called **Laplace Transformation**. We also write  $\mathcal{L}[f(t)]$ .

**Definition:** (exponential order)

A function  $f : [0, \infty[ \rightarrow \mathbb{R}$  is of **exponential order**  $\gamma$ , if constants  $M > 0$  and  $\gamma \in \mathbb{R}$  exist, such that for all  $0 \leq t < \infty$

$$|f(t)| \leq Me^{\gamma t}.$$

**Proposition:** (Existence of Laplace Transform)

Let  $f$  be piecewise continuous in  $[0, \infty[$  and of exponential order  $\gamma$ . Then the Laplace transform  $F(z)$  exists for all  $z \in \mathbb{C}$  with  $\operatorname{Re} z > \gamma$ .



**Definition:** (Laplace Transformation)

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**Questions:**

- For which  $f$  is the Laplace Transform usefully defined?
- Under which conditions does the improper integral exist?

**Definition:** (exponential order)

A function  $f : [0, \infty[ \rightarrow \mathbb{R}$  is of **exponential order**  $\gamma$ , if constants  $M > 0$  and  $\gamma \in \mathbb{R}$  exist, such that for all  $0 \leq t < \infty$

$$|f(t)| \leq Me^{\gamma t}.$$

**Remarks:**

- Polynomials are of exp. order.
- sin and cos are of exp. order.

Example: Employ Taylor Series for  $t \geq 0$ :

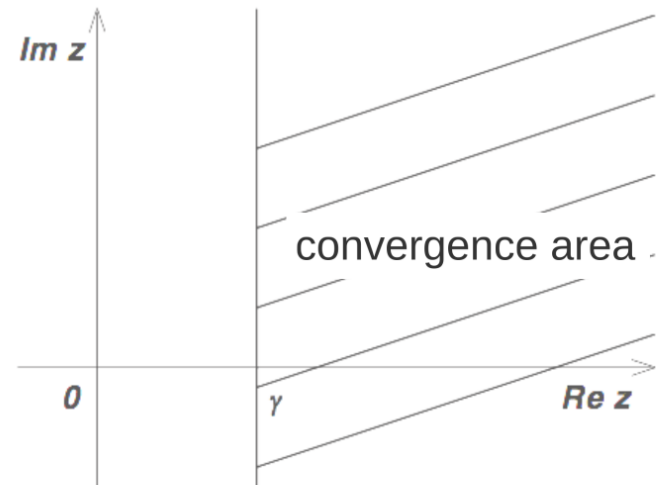
$$|t^3| = t^3 \leq 6e^t = 6 + 6t + 3t^2 + t^3 + \dots$$

**Proposition:** (Existence of Laplace Transform)

Let  $f$  be piecewise continuous in  $[0, \infty[$  and of exponential order  $\gamma$ . Then the Laplace transform  $F(z)$  exists for all  $z \in \mathbb{C}$  with  $\operatorname{Re} z > \gamma$ .

**Observations:**

- The integral  $\int_0^\infty e^{-zt} f(t) dt$  exists in a right half space of the Gaussian plane.
- The weaker  $f(t)$  grows for  $t \rightarrow \infty$ , the further the convergence area reaches to the left.



# Inverse Laplace Transformation

**Proposition: (Inverse of the Laplace Transformation)**  
 Let  $f$  be of exponential order  $\gamma$ ,  $f$  vanishes for  $t < 0$  and is piecewise continuous in  $\mathbb{R}$ . Then for all  $z = \text{Re}z > \gamma$

$$\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{\sigma - i\infty - \epsilon}^{\sigma + i\infty - \epsilon} F(s) e^{st} ds = \begin{cases} f(t) & (t > 0), \\ \frac{f(0^+) + f(0^-)}{2} & (t = 0), \\ 0 & (t < 0). \end{cases}$$

In particular, in each point  $t$  of continuity of  $f$

$$f(t) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{\sigma - i\infty - \epsilon}^{\sigma + i\infty - \epsilon} F(s) e^{st} ds = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s + i\epsilon) e^{(s+i\epsilon)t} ds$$

for  $z > \gamma$ .

**Proposition: (Uniqueness of Laplace Transformation)**  
 Let  $f_1$  and  $f_2$  be of exponential order  $\gamma$ ,  $f_1, f_2$  vanish for  $t < 0$  and piecewise continuous in  $\mathbb{R}$ . Further assume for the Laplace transforms  $F_1(z) = F_2(z)$  for  $\text{Re}z > \gamma$ .

Then for each  $t$ , at which  $f_1$  and  $f_2$  are continuous, it holds:

$$f_1(t) = f_2(t).$$

Remarks:

- The two propositions allow computation of  $f(t)$  from a Laplace transform  $F(z)$  by a line integral in the Gaussian plane.
- The Laplace transformation  $f(t) \rightarrow F(z)$  is a bijective mapping.

Example:

$$f(t) = e^{4t}$$

has the unique Laplace transform

$$F(z) = \frac{1}{z - 4}.$$



**Proposition:** (Inverse of the Laplace Transformation)

Let  $f$  be of exponential order  $\gamma$ ,  $f$  vanishes for  $t < 0$  and is piecewise continuous in  $\mathbb{R}$ . Then for all  $x = \operatorname{Re}z > \gamma$

$$\begin{aligned} \frac{1}{2\pi i} \lim_{A \rightarrow \infty} \int_{x-iA}^{x+iA} F(z)e^{zt} dz &= \\ \frac{1}{2\pi} \lim_{A \rightarrow \infty} \int_{-A}^A F(x+is)e^{(x+is)t} ds &= \begin{cases} \frac{f(t+0)+f(t-0)}{2} & (t > 0), \\ \frac{f(0+0)}{2} & (t = 0), \\ 0 & (t < 0). \end{cases} \end{aligned}$$

In particular, in each point  $t$  of continuity of  $f$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \lim_{A \rightarrow \infty} \int_{x-iA}^{x+iA} F(z)e^{zt} dz \\ &= \frac{1}{2\pi} \lim_{A \rightarrow \infty} \int_{-A}^A F(x+is)e^{(x+is)t} ds \end{aligned}$$

for  $x > \gamma$ .

**Proposition:** (Uniqueness of Laplace Transformation)

Let  $f_1$  and  $f_2$  be of exponential order  $\gamma$ ,  $f_1, f_2$  vanish for  $t < 0$  and piecewise continuous in  $\mathbb{R}$ . Further assume for the Laplace transforms  $F_1(x) = F_2(x)$  for  $\text{Re}z > \gamma$ .

Then for each  $t$ , at which  $f_1$  and  $f_2$  are continuous, it holds:

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# Computing Rules for Laplace Transformation

Copyright: the above by P. J. R. 2011. The Laplace transform of a function and exponential order is that a piecewise constant is B.C.

**Proposition: (Linearity)**  
Let  $f$  and  $g$  piecewise continuous in  $[0, \infty)$  and of exponential order  $\gamma$ .  
Then for  $a, b \in \mathbb{R}$  it holds

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}f(t) + b\mathcal{L}g(t).$$

Proof follows immediately from linearity of integrals.

**Proposition: (Laplace Transformation of Product with Power Function)**  
Let  $g(t) = t^{-\alpha} f(t)$  and  $f$  Laplace transformable with Laplace transform  $F(s) = \mathcal{L}f(t)$ . Then

$$\mathcal{L}g(t) = \mathcal{L}(t^{-\alpha} f(t)) = s^{\alpha} F(s).$$

**Proposition: (Laplace Transformation of T-periodic Functions)**  
Let  $f$  be T-periodic (i.e.  $f(t+T) = f(t)$ ), piecewise cont. and bounded. Then it holds for  $\text{Re } s > 0$

$$\mathcal{L}f(t) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

**Proposition: (Convolution Rule)**  
Let  $f$  and  $g$  be functions of exponential order  $\gamma$  with  $f(t) = g(t) = 0$  for  $t < 0$ . Let  $f$  be continuous and  $g$  piecewise continuous in  $\mathbb{R}$ . Then the Laplace transform of the convolution  $f * g$  holds for  $\text{Re } s > \gamma$  and it holds

$$\mathcal{L}(f * g)(t) = \mathcal{L}f(t) \cdot \mathcal{L}g(t)$$

**Definition: (Convolution)**  
Let  $f$  and  $g$  be functions. The convolution (product) of  $f$  and  $g$  is defined as general as

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau, \quad t \in \mathbb{R}.$$

Remarks:

- In the above it is not clear that the integral exists.
- For  $f$  and  $g$  bounded as in the Laplace transform it holds

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t f(t-\tau) g(\tau) d\tau.$$

**Proposition: (Transformation of Derivatives and Integrals)**

1. Let  $f$  be as in the previous proposition. Then for  $\text{Re } s > \gamma$  it holds

$$\mathcal{L}f'(t) = s\mathcal{L}f(t) - f(0).$$

2. Let  $f$  be  $(k-1)$ -times cont. differentiable and  $f^{(k-1)}$  piecewise cont. Let  $f, f', \dots, f^{(k-1)}$  be of exponential order  $\gamma$ . Then for  $\text{Re } s > \gamma$  it holds

$$\mathcal{L}f^{(k)}(t) = s^k \mathcal{L}f(t) - s^{k-1} f(0) - \dots - f^{(k-1)}(0).$$

3. Let  $f$  be like in 1. Then for  $\text{Re } s > \gamma$  it holds

$$\mathcal{L} \int_0^t f(\tau) d\tau = \frac{1}{s} \mathcal{L}f(t).$$

**Proposition: (Transformation of Derivative of Discontinuous Function)** Let  $f$  be again a function of exponential order  $\gamma$  with the other properties of the prop. and assume  $f$  to have a jump discontinuity at  $t = 1 > 0$ . Then it holds

$$\mathcal{L}f'(t) = s\mathcal{L}f(t) - f(0) - f(1+0) - f(1-0)e^{-s}.$$

Sketch of proof: split the integral

$$\int_0^{\infty} e^{-st} f'(t) dt$$

with  $\int_0^{\infty} e^{-st} f(t) dt$ , apply integration by parts to the first integral.

**Satz: (Damping Translation, Shifting)** Let  $f$  be a function of exponential order  $\gamma$  with the properties of the propositions above.  $F(s) = \mathcal{L}f(t) = \int_0^{\infty} e^{-st} f(t) dt$ ,  $\text{Re } s > \gamma$ .

1. A damping factor  $e^{-at}$  in the origin domain results in a translation in the image domain:

$$\mathcal{L}(e^{-at} f(t)) = F(s+a) \quad \text{for } \text{Re } s > \gamma - a.$$

2. For  $a > 0$  it holds

$$\mathcal{L}f(t) = \int_0^{\infty} F(s-a) ds \quad \text{for } \text{Re } s > \gamma + a.$$



**Convention:** We denote by  $F(z) = \mathcal{L}[f(t)]$  the Laplace transform of a function with exponential order  $\gamma$  that is piecewise constant in  $[0, \infty[$ .

**Proposition:** (Linearity)

Let  $f$  and  $g$  piecewise continuous in  $[0, \infty[$  and of exponential order  $\gamma$ .  
Then for  $a, b \in \mathbb{R}$  it holds

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$

Proof follows immediately from linearity of integrals.

**Proposition:** (Transformation of Derivatives and Integrals)

1. Let  $f$  be as in the previous proposition. Then for  $\operatorname{Re}z > \gamma$  it holds

$$\mathcal{L}[f'(t)] = z\mathcal{L}[f(t)] - f(0).$$

2. Let  $f$  be  $(k - 1)$ -times cont. differentiable and  $f^{(k-1)}$  piecewise cont. Let  $f, f', \dots, f^{(k-1)}$  be of exponential order  $\gamma$ . Then for  $\operatorname{Re}z > \gamma$  it holds

$$\mathcal{L}[f^{(k)}(t)] = z^k \mathcal{L}[f(t)] - z^{k-1} f(0) - \dots - f^{(k-1)}(0).$$

3. Let  $f$  be like in 1. Then for  $\operatorname{Re}z > \gamma$  it holds

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{z} \mathcal{L}[f(t)].$$

Proof:  
1. Using the definition of Laplace transform by partial integration:  
$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-zt} f'(t) dt = \left[ -\frac{1}{z} e^{-zt} f'(t) \right]_0^\infty + \int_0^\infty e^{-zt} f(t) dt = \frac{1}{z} f(0) + \mathcal{L}[f(t)]$$
  
2. Apply the same method repeatedly  $k$  times.  
3. Apply 1. to the function  $f(t) = \int_0^t f(\tau) d\tau$ .

**Proof:**

1. Follows from definition of Laplace transform by partial integration

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^{\infty} e^{-zt} f'(t) dt \\ &= \lim_{A \rightarrow \infty} e^{-zt} f(t) \Big|_{t=0}^{t=A} - \int_0^{\infty} (-z) e^{-zt} f(t) dt \\ &= -f(0) + z \int_0^{\infty} e^{-zt} f(t) dt = z\mathcal{L}[f(t)] - f(0).\end{aligned}$$

2. Apply above partial integration  $k$ -times.
3. Apply 1. to the function  $h(t) = \int_0^t f(\tau) d\tau$ .



**Proposition:** (Transformation of Derivative of Discontinuous Function) Let  $f$  be again a function of exponential order  $\gamma$  with the other prerequisites of the prop., and assume  $f$  to have a jump discontinuity at  $t = a > 0$ . Then it holds

$$\mathcal{L}[f'(t)] = z\mathcal{L}[f(t)] - f(0) - [f(a+0) - f(a-0)]e^{-az}.$$

Sketch of proof: split the integral

$$\int_0^{\infty} e^{-z} f'(t) dt$$

into  $\int_0^{a-0}(\cdot) + \int_{a+0}^{\infty}(\cdot)$ , apply arguments analogous to proposition above.

**Satz:** (Damping-Translation, Stretching) Let  $f$  be a function of exponential order,  $\gamma$  with the prerequisites of the propositions above,  $F(z) = \mathcal{L}[f(t)] = \int_0^\infty e^{-z} f(t) dt$ ,  $\operatorname{Re} z > \gamma$ .

1. A damping factor  $e^{-at}$  in the origin domain results in a translation in the image domain:

$$\mathcal{L}[e^{-at} f(t)] = F(z + a) \quad \text{für } \operatorname{Re} z > \gamma - a.$$

2. For  $a > 0$  it holds

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{z}{a}\right) \quad \text{für } \operatorname{Re} z > a \cdot \gamma.$$

**Definition:** (Convolution)

Let  $f$  and  $g$  be functions. The **convolution (product)** of  $f$  and  $g$  is defined in general as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t - \tau)g(\tau) d\tau, \quad t \in \mathbb{R}.$$

**Remarks:**

- We assume in each case that the improper integral exists.
- For  $f$  and  $g$  functions as in the Laplace transformation it holds  $f(t) = g(t) = 0$  for  $t < 0$ . Therefore, we have

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - \tau)g(\tau) d\tau = \int_0^{\infty} f(t - \tau)g(\tau) d\tau$$

**Proposition:** (Convolution Rule)

Let  $f$  and  $g$  be functions of exponential order  $\gamma$  with  $f(t) = g(t) = 0$  for  $t < 0$ . Let  $f$  be continuous and  $g$  piecewise continuous in  $\mathbb{R}$ . Then the Laplace transform of the convolution  $f * g$  exists for  $\operatorname{Re}z > \gamma$  and it holds

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

**Proposition:** (Laplace Transformation of  $T$ -periodic Functions)

Let  $f$  be  $T$ -periodic (i.e.,  $f(t - T) = f(t)$ ), piecewise cont. and bounded.  
Then it holds for  $\operatorname{Re}z > 0$

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-Tz}} \int_0^T e^{-zu} f(u) du.$$

**Proposition:** (Laplace Transformation of Product with Power Function)

Let  $g(t) = (-1)^n t^n f(t)$  and  $f$  Laplace transformable with Laplace transform  $F(z) = \mathcal{L}[f(t)]$ . Then

$$\mathcal{L}[g(t)] = \mathcal{L}[(-1)^n t^n f(t)] = F^{(n)}(z).$$

## Solution of ODEs by Laplace Transformation

**Idea:** Let the IVP of  $n^{\text{th}}$  order be given

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = r(t), \quad y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0,$$

with  $r$  piecewise continuous function of exponential order.

Set

$$Y(z) = \mathcal{L}[y(t)], \quad \text{and} \quad R(z) = \mathcal{L}[r(t)].$$

The Laplace transformation of the IVP is derived from the rules above:

$$\begin{aligned} (z^n + a_{n-1}z^{n-1} + \dots + a_0)Y(z) &= R(z) \\ \Rightarrow Y(z) &= (z^n + a_{n-1}z^{n-1} + \dots + a_0)^{-1}R(z) =: G(z)R(z). \end{aligned}$$

If one finds a function  $g(t)$  with  $\mathcal{L}[g(t)] = G(z)$ , then

$$\begin{aligned} \mathcal{L}[y(t)] &= Y(z) = F(z)R(z) = \mathcal{L}[g(t)]\mathcal{L}[r(t)] = \mathcal{L}[(g * r)(t)] \\ \Rightarrow y(t) &= (g * r)(t) = \int_0^t g(t - \tau)r(\tau) d\tau. \end{aligned}$$

The function  $K(t, \tau) := g(t - \tau)$  is called **Green's Function**.

1

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$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = r(t), \quad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

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The Laplace transformation of the IVP is derived from the rules above:

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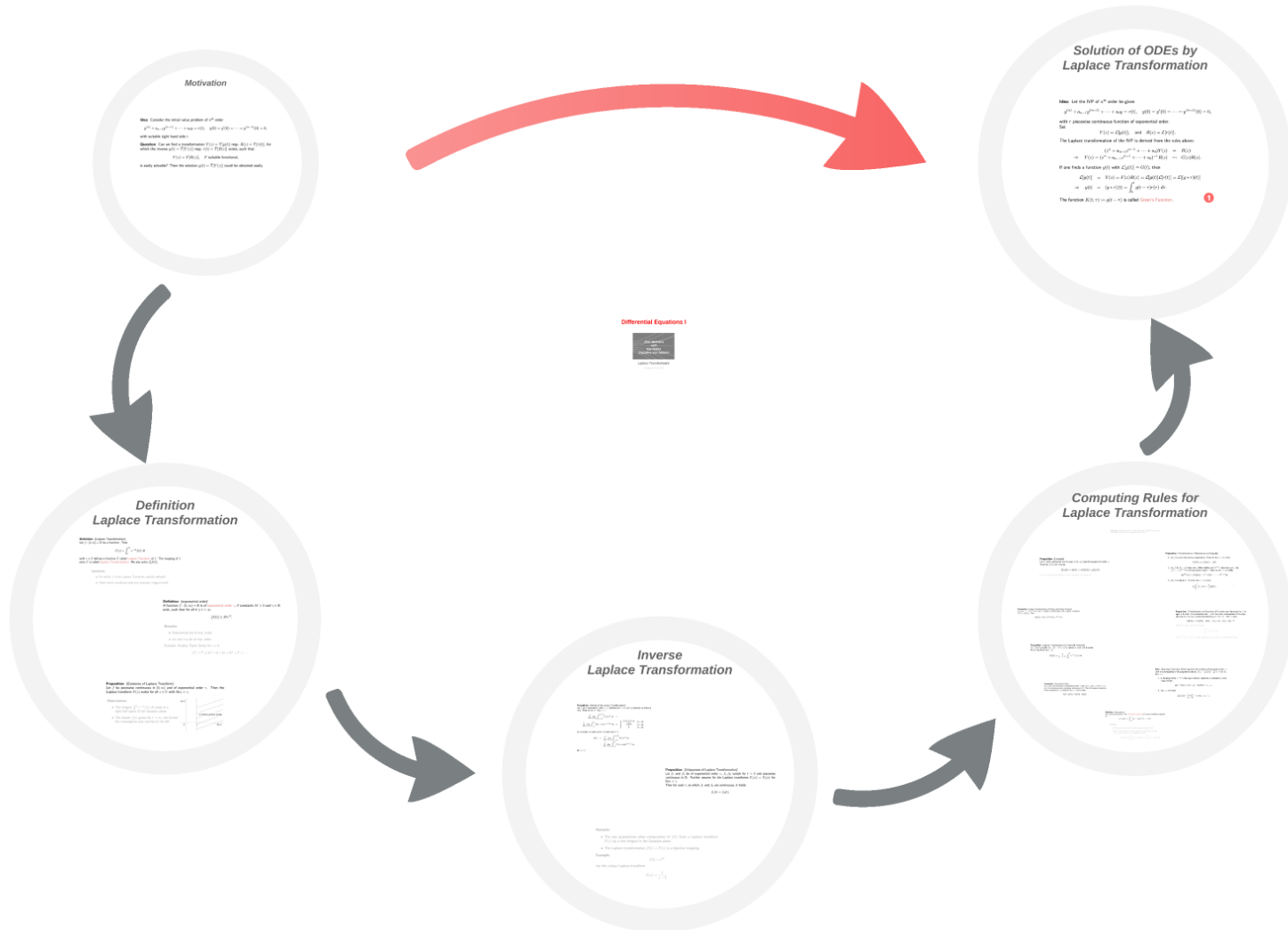
If one finds a function  $g(t)$  with  $\mathcal{L}[g(t)] = G(z)$ , then

$$\begin{aligned} \mathcal{L}[y(t)] &= Y(z) = F(z)R(z) = \mathcal{L}[g(t)]\mathcal{L}[r(t)] = \mathcal{L}[(g * r)(t)] \\ \Rightarrow y(t) &= (g * r)(t) = \int_0^t g(t - \tau)r(\tau) d\tau. \end{aligned}$$

The function  $K(t, \tau) := g(t - \tau)$  is called **Green's Function**.

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**Motivation**

Now consider the initial value problem of  $y'' + ay' + by = f(x)$  with  $y(0) = y_0$  and  $y'(0) = y_0'$ .  
 We assume  $f(x) = e^{-\alpha x}$  with  $\alpha > 0$ .  
 The Laplace transform of the left-hand side is  $(s^2 + as + b)Y(s) - sy_0 - y_0' + a(sy_0 - y_0) + by_0 = (s^2 + as + b)Y(s) - sy_0 - y_0' + a(sy_0 - y_0) + by_0$ .  
 The Laplace transform of the right-hand side is  $\frac{1}{s + \alpha}$ .  
 It is easy to check that the unique solution  $y(x) = \int_0^x f(x-t)g(t)dt$  is given by

**Solution of ODEs by Laplace Transformation**

Now let the ODE  $y'' + ay' + by = f(x)$  be given with  $y(0) = y_0$  and  $y'(0) = y_0'$ .  
 We assume  $f(x) = e^{-\alpha x}$  with  $\alpha > 0$ .  
 The Laplace transform of the left-hand side is  $(s^2 + as + b)Y(s) - sy_0 - y_0' + a(sy_0 - y_0) + by_0 = (s^2 + as + b)Y(s) - sy_0 - y_0' + a(sy_0 - y_0) + by_0$ .  
 The Laplace transform of the right-hand side is  $\frac{1}{s + \alpha}$ .  
 It is easy to check that the unique solution  $y(x) = \int_0^x f(x-t)g(t)dt$  is given by

**Definition Laplace Transformation**

**Definition (Laplace Transformation)**  
 Let  $f: [0, \infty) \rightarrow \mathbb{C}$  be a function. The Laplace transform of  $f$  is the function  $F: \mathbb{C} \rightarrow \mathbb{C}$  defined by  $F(s) = \int_0^{\infty} e^{-st} f(t) dt$  for  $s \in \mathbb{C}$  with  $\text{Re}(s) > \sigma$ .  
 The function  $F$  is called the Laplace transform of  $f$ .  
 The function  $f$  is called the inverse Laplace transform of  $F$ .  
 The function  $f$  is called the Laplace transform of  $F$ .

**Inverse Laplace Transformation**

**Definition (Inverse Laplace Transformation)**  
 Let  $F: \mathbb{C} \rightarrow \mathbb{C}$  be a function. The inverse Laplace transform of  $F$  is the function  $f: [0, \infty) \rightarrow \mathbb{C}$  defined by  $f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} F(s) ds$  for  $x > 0$  and  $\gamma > \sigma$ .  
 The function  $f$  is called the inverse Laplace transform of  $F$ .  
 The function  $F$  is called the Laplace transform of  $f$ .

**Computing Rules for Laplace Transformation**

**Linearity**  
 Let  $f, g: [0, \infty) \rightarrow \mathbb{C}$  be functions and  $\alpha, \beta \in \mathbb{C}$ . Then  $\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g)$ .  
**Derivatives**  
 Let  $f: [0, \infty) \rightarrow \mathbb{C}$  be a function and  $f'$  its derivative. Then  $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ .  
**Integration**  
 Let  $f: [0, \infty) \rightarrow \mathbb{C}$  be a function and  $F$  its antiderivative. Then  $\mathcal{L}(F) = \frac{1}{s}\mathcal{L}(f) + \frac{f(0)}{s}$ .