Differential Equations I

Week 10 / J. Behrens

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Zutritt zur Lehrveranstaltung haben nur:

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–GENESENE
–GETESTETE
(negatives Testergebnis ist max. 24 Std. gültig)

Admission to the course is restricted to persons who are:

–FULLY VACCINATED
–RECOVERED
–TESTED
(negative test result is valid for max. 24 hours)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

If you cannot prove this, please leave the room now.
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Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Thank you for your understanding.
Protect yourself and others!

1 **Definition**: (Differential Operator of 2nd Order)
Let $I \subset \mathbb{R}$ be a closed intervall and $a_0(x) \neq 0$, $a_1(x)$, $a_2(x)$, $r(x)$ continuous functions. Then

$$D[y] := a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$$

defines a differential operator that maps twice differentiable functions $y(x)$ on I into continuous functions $D[y]$.

Remarks:

- Consider the ODE $D[y] = r(x)$.
- With initial conditions

  $$y(\xi) = \eta_a, \quad y'(\xi) = \gamma_a, \quad \xi \in I, \ \eta_a, \gamma_a \in \mathbb{R},$$

  there is a unique solution on I according to the proposition.
- Question: What if apart from position ξ conditions at other positions are required?

- Example: Bending of a beam

- Equation: $y'' = \underbrace{-C\left(1-\left(\frac{x}{l}\right)^2\right)x}_{v(x)}$

$$C \neq 0 \quad , \quad 0 \leq x \leq l$$

- Boundary Conditions: $y(0) = 0 = y(l)$

- General Solution: $y(x) = -C\left(\frac{x^3}{6} - \frac{x^5}{20l^2}\right) + C_1 x + C_2 \quad , \quad C_1, C_2 \in \mathbb{R}$

- Evaluation of BCs:

$$0 = y(0) = C_2 \qquad \Rightarrow \quad C_2 = 0$$

$$0 = y(l) = -C\left(\frac{l^3}{6} - \frac{l^3}{20}\right) + C_1 l \quad \Rightarrow \quad C_1 = C\frac{7}{60}l^2$$

- With this: $y(x) = C\left[\frac{7}{60}l^2 x - \frac{x^3}{6} + \frac{x^5}{20l^2}\right]$ is Solution of the boundary value problem.

- Variation of BCs:

a) $y'(0) = 0 \quad , \quad y(l) = 0$

$$y'(x) = -C\left[\frac{x^2}{2} - \frac{x^4}{4l^2}\right] + C_1$$

$$\Rightarrow \quad y(x) = C\left[\frac{7l^2}{60}x - \frac{x^3}{6} + \frac{x^5}{20l^2}\right]$$

b) $y'(0) = y'(l) = 0$

$$0 = y'(0) = C_1 \qquad \Rightarrow \quad C_1 = 0$$

$$0 = y'(l) = -C\frac{l^2}{4} \quad ↯$$

Conclusion: $\nexists$ constants C_1, C_2, such that the ODE is solvable with these given BCs.

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Remark: (Linear System of Equations)
Question: Solvability of the ODE with boundary conditions.

- General Solution: $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$, where $c_1, c_2 \in \mathbb{R}$, $\{y_1, y_2\}$ fundamental system of the homogeneous ODE $D[y] = 0$ and y_p particular solution of the inhomogeneous ODE $D[y] = r(x)$.
- Derivative: $y'(x) = c_1 y_1'(x) + c_2 y_2'(x) + y_p'(x)$.
- This yields for the boundary conditions:

$$\begin{aligned}\alpha_1[c_1 y_1(a) + c_2 y_2(a) + y_p(a)] + \beta_1[c_1 y_1'(a) + c_2 y_2'(a) + y_p'(a)] &= \gamma_1 \\ \alpha_2[c_1 y_1(b) + c_2 y_2(b) + y_p(b)] + \beta_2[c_1 y_1'(b) + c_2 y_2'(b) + y_p'(b)] &= \gamma_2\end{aligned}$$

- Reformulation:

$$\begin{aligned}(\alpha_1 y_1(a) + \beta_1 y_1'(a))c_1 + (\alpha_1 y_2(a) + \beta_1 y_2'(a))c_2 &= \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a) \\ (\alpha_2 y_1(b) + \beta_2 y_1'(b))c_1 + (\alpha_2 y_2(b) + \beta_2 y_2'(b))c_2 &= \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b).\end{aligned}$$

- Use definitions for R_1, R_2 and

$$r_1 = \gamma_1 - \alpha_1 y_p(a) - \beta_1 y_p'(a), \quad r_2 = \gamma_2 - \alpha_2 y_p(b) - \beta_2 y_p'(b)$$

obtain linear system of equations

$$\begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

- If the linear system is solvable, then the ODE with boundary conditions is solvable. Thus,

$$\det \begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} \overset{!}{\neq} 0$$

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- Example: $y'' = e^{2x}$, $y(0) = 1$, $y(1) = 3$
- General Solution: $y(x) = c_1 x + c_2 + \frac{1}{4} e^{2x}$
$= c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- With: $y_1(x) = x$, $y_2(x) \equiv 1$, $\left.\begin{aligned}\alpha_k &= 1\\ \beta_k &= 0\end{aligned}\right\} k=1,2$, $\gamma_1 = 1$, $\gamma_2 = 3$

$\Rightarrow$ $R_1(y_1) = y_1(0) = 0$, $r_1 = 1 - y_p(0) = 1 - \frac{1}{4} = \frac{3}{4}$

$R_1(y_2) = y_2(0) = 1$, $r_2 = 3 - y_p(1) = 3 - \frac{1}{4}e^2$

$R_2(y_1) = y_1(1) = 1$

$R_2(y_2) = y_2(1) = 1$

- Lin. System of Eq.:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 3-\frac{1}{4}e^2 \end{pmatrix}$$

- Solution: $c_2 = \frac{3}{4}$, $c_1 = \frac{1}{4}(9-e^2)$

- Solution of the ODE:

$$y(x) = \frac{1}{4}(9-e^2)x + \frac{3}{4} + \frac{1}{4}e^{2x}$$

3 **Example**: A self adjoint differential operator for $n = 2$:
We already computed:

$$\begin{aligned} D[y] &= a_0(x)y'' + a_1(x)y' + a_2(x)y \\ D^*[y] &= (a_0(x)y)'' - (a_1(x)y)' + a_2(x)y \\ &= a_0(x)y'' + (2a_0'(x) - a_1(x))y' + (a_0''(x) - a_1'(x) + a_2(x))y. \end{aligned}$$

With $D^*[y] = D[y]$ it follows:

$$\begin{aligned} 2a_0' - a1 = a_1 \quad &\Rightarrow \quad a_0' = a_1 \\ a_0'' - a_1' + a_2 = a_2 \quad &\Rightarrow \quad a_0'' = a_1'. \end{aligned}$$

One obtains

$$D[y] = (a_0(x)y')' + a_2(x)y.$$

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- Question: Let $D[y] = r(x)$ be given. Is there an equivalent ODE with a self adjoint operator?

- Observation: Multiplication with $e^{S(x)}$ ($S(x)$ is arbitrary, differentiable) does not change the solution set.

- So:

$$\begin{aligned} & D[y] = a_0 y'' + a_1 y' + a_2 y = r(x) \\ \Rightarrow \quad & e^{S(x)}(a_0 y'' + a_1 y' + a_2 y) = e^{S(x)} r(x) \\ \Rightarrow \quad & (e^{S(x)} a_0 y')' + e^{S(x)}(a_1 - a_0' - S' a_0) y' + e^{S(x)} a_2 y = e^{S(x)} r(x) \end{aligned}$$

- Idea: Select s, such that $(a_1 - a_0' - s'a_0) = 0$

$$\Rightarrow L[y] := (e^{s(x)} a_0 y')' + e^{s(x)} a_2 y = e^{s(x)} v(x) := z(x)$$

is equivalent to $D[y] = v(x)$ and self adjoint.

- Set: $p(x) = e^{s(x)} a_0(x)$, $q(x) = e^{s(x)} a_2(x)$

$$\Rightarrow L[y] = (p(x) y')' + q(x) y$$

We have $p(x) \neq 0$ since $a_0(x) \neq 0$, w.o.l.g. assume $p(x) > 0$

- Find s, such that $(a_1 - a_0' - s'a_0) = 0$

select $s' = \frac{a_1 - a_0'}{a_0} \Rightarrow s(x) = \int \frac{a_1 - a_0'}{a_0} \, dx$,

- Conclusion: With this choice of s it is always possible to find an equivalent ODE corresponding to $D[y] = v(x)$, i.e.

$$L[y] = z(x)$$

with L self adjoint.