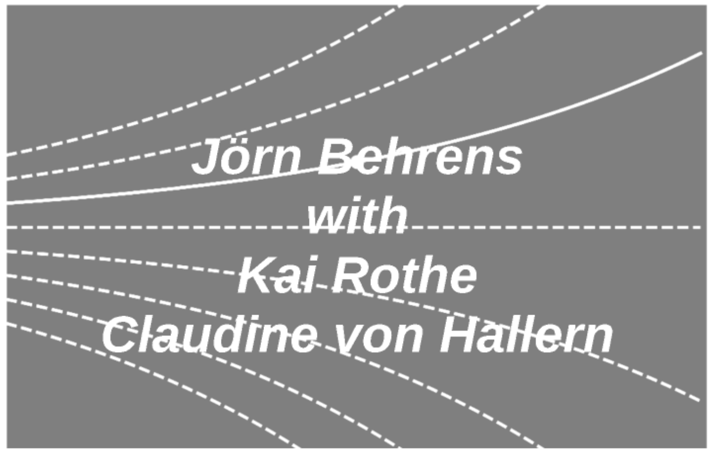
Differential Equations I



Eigenvalue Problems

Chapter 6.13

Recap: Self-adjoint Differential Operators

Consider: (Boundary Value Problem)
Let us seek the solution on interval $I = [a, b]$ of

$$\begin{aligned} -L[y] &= \lambda w(x) y, \\ R_1(y) &= \alpha_1 y(a) + \beta_1 y'(a) = 0, \\ R_2(y) &= \alpha_2 y(b) + \beta_2 y'(b) = 0; \end{aligned}$$

with L Sturm-Liouville differential operator, $\lambda \in \mathbb{R}$ a parameter, $\alpha_k, \beta_k \in \mathbb{R}$ with $\alpha_k^2 + \beta_k^2 > 0$ $(k = 1, 2)$, $w(x)$ a positive continuous function on I.

Assume $C^2([a, b], \mathbb{R})$ as domain of L, more precisely the subset $M \subset C^2([a, b], \mathbb{R})$ of functions fulfilling the boundary conditions!. The elements in M are called test functions.

Proposition: (Self Adjoint Sturm-Liouville Eigen Value Problem)
Let $L[y] = (p(x)y')' + q(x)y$ be the Sturm-Liouville differential operator for $x \in [a, b]$ with cont. diff. function $p(x) > 0$, cont. diff. function $q(x)$ and cont. function $w(x) > 0$, $\lambda \in \mathbb{R}$ a parameter and $\alpha_k, \beta_k \in \mathbb{R}$ with $\alpha_k^2 + \beta_k^2 > 0$ $(k = 1, 2)$.
Then the Sturm-Liouville eigen value problem

$$L[y] + \lambda w(x) y = 0, \quad \alpha_1 y(a) + \beta_1 y'(a) = 0, \; \alpha_2 y(b) + \beta_2 y'(b) = 0$$

is self adjoint.

Non-trivial solutions $y_\lambda(x)$ corresponding to given parameters λ are called eigenfunctions (if they exist). The corresponding parameters λ are called eigenvalues of the Sturm-Liouville eigen value problem.

Definition: (General Self Adjoint Differential Operator)
Let L be a self adjoint differential operator of 2^{nd} order on $I = [a, b]$, and $M \subset C^2([a, b], \mathbb{R})$ the set of all functions fulfilling given boundary conditions $x = a$ and $x = b$ (test functions).
If for all $u, v \in M$ it holds that

$$(L[u], v) = (u, L[v]),$$

we call L (general) self adjoint differential operator on M. The corresponding boundary value problem is also called self adjoint.

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Orthogonality

Definitions:

- Introduce a scalar product on the vector space $C^2([a,b],\mathbb{R})$:

$$\langle u, v\rangle := \int_a^b u(x)v(x)w(x)\, dx.$$

- With $w : [a,b] \to \mathbb{R}$ an integrable and in $]a,b[$ positive weight function.
- Two elements $u, v \in C^2([a,b],\mathbb{R})$ are called orthogonal, if $\langle u, v\rangle = 0$.

Recall: (Orthogonality in $\mathbb{R}^n$)

- If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ ($(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}$), then each vector $\mathbf{x} \in \mathbb{R}^n$ can be written:

$$\mathbf{x} = \sum_{k=1}^{n} c_k \mathbf{e}_k.$$

- For the coefficients it holds: $c_j = (\mathbf{x}, \mathbf{e}_j)$, $j = 1, \ldots, n$.

Proposition: (Orthogonality in Sturm-Liouville Eigenvalue Problems)
For the coefficient functions of the homogeneous Sturm-Liouville differential equation

$$L[y] + \lambda w y = (p(x)y')' + q(x)y + \lambda w y = 0$$

with $\lambda \in \mathbb{R}$ a parameter, assume:

- For $x \in [a,b]$ let $p(x)$ be continuously differentiable,
- let $q(x), w(x)$ be continuous.
- For $x \in]a,b[$ let $p(x) > 0$ and $w(x) > 0$.

Then two non-trivial solutions $y_1(x), y_2(x) \in C^2([a,b],\mathbb{R})$ corresponding to two different parameter values $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are orthogonal i.e.,

$$\langle y_1, y_2\rangle = \int_a^b y_1(x)y_2(x)w(x)\, dx = 0,$$

if

1. y_1 and y_2 satisfy the homogeneous boundary conditions $R_1(y) = 0 = R_2(y)$ i.e., λ_1, λ_2 are eigenvalues corresponding to eigenfunctions y_1, y_2 of the Sturm-Liouville eigenvalue problem, or
2. the coefficient function $p(x)$ fulfills the condition $p(a) = p(b) = 0$.

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Expansion with Eigenfunctions

Proposition: (Sequence of Eigenvalues and Oscillation of Eigenfunctions)
Let a Sturm-Liouville eigenvalue problem with boundary conditions be given:

$$L[y] + \lambda w y = 0, R_1(y) = \alpha_1 y(a) + \beta_1 y'(a) = 0 = \alpha_1 y(b) + \beta_2 y'(b) = R_2(y),$$

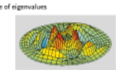
with $p(x) > 0$ and $w(x) > 0$. Then the eigenvalues of this eigenvalue problem are easily computed and form an infinite sequence of real values $\lambda_1 < \lambda_2 < \cdots$, tending towards ∞. Each eigenfunction corresponding to λ_n has exactly n roots in $]a, b[$.

Motivation: (Clamped Membrane)
Bessel's differential equations

$$-L[y] = -(\rho y')' + \frac{n^2}{\rho} y = \omega^2 \rho y, \quad y(a) = y(b) = 0$$

represents the oscillation of a (ring-shaped) membrane, fixed (clamped) at the boundary, where a is the inner radius and b the outer radius and ρ a material property.

- According to the Proposition there is for each $n \in \mathbb{N}$ a sequence of eigenvalues $\omega_0^2 < \omega_1^2 < \cdot$ with $\omega_k^2 \to \infty$ $(k \to \infty)$.
- ω_k are the eigen frequencies of the membrane.
- k is the number of the wave maxima in radial direction.



Proposition: (Expansion)
Let $(y_n(x))$ be a sequence of normalized eigenfunctions, corresponding to eigenvalues λ_n of the eigenvalue problem

$$-L[y] = \omega w y, R_1(y) = 0 = R_2(y)$$

with coefficient function $p(x) > 0$ and weight function $w(x) > 0$ on $[a, b]$. Thus, it holds:

$$\langle y_k, y_j \rangle = \delta_{kj}.$$

Then each continuously differentiable function f, satisfying the boundary conditions of the eigenvalue problem, can be represented as function series

$$f(x) = \sum_{n=1}^{\infty} \langle f, y_n \rangle y_n(x).$$

The series converges in $[a, b]$ uniformly and absolutely.

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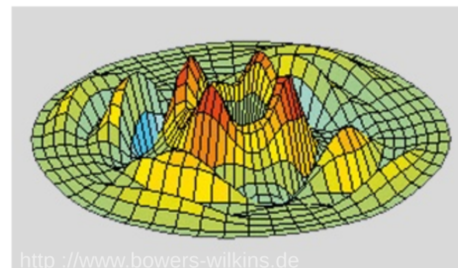

Idea: (Expansion by Eigenfunctions)

- These states of oscillation are to be represented by the (dominant) frequencies (eigenfunctions).
- Due to the orthogonality relation of eigenfunctions the (solution) functions can be represented by eigenfunction series with suitable boundary conditions!

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2

The series converges in $[a, b]$ uniformly and absolutely.

Non-Linear ODEs

Motivation: (Pendulum)

- The equation describing the motion of a pendulum is given by

$$\ddot{\varphi} + k \sin \varphi = 0.$$

- Observation: this equation is non-linear!
- For small displacements φ it holds $\sin \varphi \approx \varphi$.
- One obtains an approximate linear ODE

$$\ddot{\varphi} + k\varphi = 0.$$

Definition: (Dynamical System)
Consider the mapping

$$\mathbf{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n \quad \text{und} \quad \mathbf{x} : \mathbb{R} \to \mathbb{R}^n,$$

$\mathbf{x}$ differenciable. The system of differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t),$$

with $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^\top$ and $\mathbf{F}(\mathbf{x}, t) = (F_1(\mathbf{x}, t), \ldots, F_n(\mathbf{x}, t))^\top$, is called dynamical System.
The space of solution curves $\mathbf{x}(t)$ is called phase space and the solution curves phase curves.

Remark: (System of first Order)
In analogy to the linear case, an ODE of n^{th} order can be reformulated as a system of n equations of first order:

- Let: $y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)}, t)$.
- Introduce: $x_1(t) = y(t)$, $x_2(t) = y'(t)$, …, $x_n(t) = y^{(n-1)}(t)$.
- The dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ with

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(x_1, x_2, \ldots, x_n, t) \end{pmatrix} =: \mathbf{F}(x_1, \ldots, x_n, t)$$

is equivalent to the ODE of n^{th} order above.

Remark: (Initial Value Problem)
For a dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ let the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

be given. We obtain an initial value problem (IVP).

Proposition: (Existence and Uniqueness of Solution of an Initial Value Problem)
Let:

- Functions $F_1, \ldots, F_n$ be partially integrable for $x_1, \ldots, x_n$.
- Partial Derivatives be continuous on a rectangular domain $B \subset \mathbb{R}^{n+1}$.
- Point $(\mathbf{x}_0, t_0)$ be located in the interior of B.

Then there is an interval $]t_0 - h, t_0 + h[$, in which a unique solution $\mathbf{x}(t)$ of the dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ exists.

Definition: (Autonomous System)
If the mapping $\mathbf{F}$ of the dynamical system does not depend on t i.e.,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

with $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, then the system is called autonomous system.

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  is equivalent to the ODE of n^{th} order above.

Example: The system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -k \sin x_1 \end{pmatrix}$$

is equivalent to the ODE of 2^{nd} order

$$\ddot{\varphi} + k \sin \varphi = 0.$$

Remark: (Initial Value Problem)
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