Differential Equations I

Week 12 / J. Behrens

① Example of autonomous System: Predator-Prey-Model

- Model: let x # Prey animals $x = x(t)$
  y # Predators $y = y(t)$

- Growth of the prey population can be assumed exponential

$$x = x_0 \, e^{ax}$$

This is the solution of a ODE: $\dot{x} = ax$

- The meetup of Predator and Prey is proportional to $x \cdot y$

$$\Rightarrow \boxed{\dot{x} = ax - b\,x \cdot y} \quad \text{Prey equation}$$

- Predator Population decreases if there is no prey: $\dot{y} = -dy$

On the other hand Predator population grows, with the rate that the prey population is diminished, so proportional to $x \cdot y$

$$\Rightarrow \boxed{\dot{y} = c\,x \cdot y - dy} \quad \text{Predator equation}$$

- Stationary Point: If the two populations don't change (with time), then $\dot{x} = \dot{y} = 0$, thus

$$0 = ax - bxy \quad \text{and} \quad 0 = cxy - dy$$

$$\Rightarrow \bar{x} = \frac{d}{c} \quad \text{and} \quad \bar{y} = \frac{a}{b}$$

- Nonstationary Solutions:

Ansatz: $\frac{d}{x} \cdot \dot{x} + \frac{a}{y} \dot{y} = \cancel{ad} - bdy + acx - \cancel{ad}$

$$= acx - cbxy + cbxy - bdy$$
$$= c(ax - bxy) + b(cxy - dy)$$
$$= c\dot{x} + b\dot{y}$$

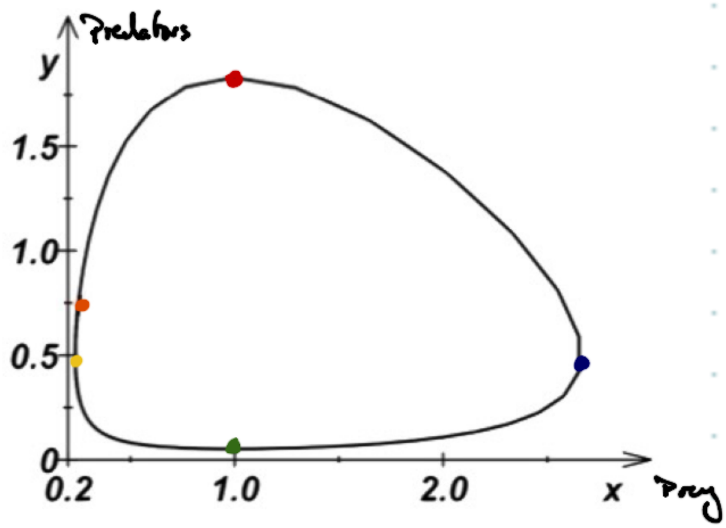
- Integrating: $d \ln x + a \ln y - c - b \overset{!}{=} \text{const.}$

$$\Rightarrow \frac{d}{dt}\big(\underbrace{d \ln x - a \ln y - c - b}_{=:\, E(x,y)}\big) = 0$$

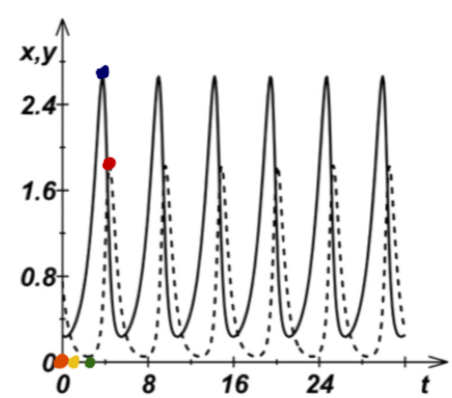
- $E(x,y)$ as defined above is on each solution curve constant

Conservation property of the system

- Each solution of the system is aligned with the contour lines of E
- For $a = 1$, $b = c = d = 2$, $(x_0, y_0) = (0.25, 0.75)$ we obtain



- time-periodic behavior of the populations



② Linear autonomous system with $n=2$: $Ax = \dot{x}$, $A \in \mathbb{R}^{2\times 2}$

- 4 Situations for the eigenvalues [EA] λ_1, λ_2 of A

a) $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$ with eigenvectors [EV] $\vec{e}_1$ and $\vec{e}_2$

$$\Rightarrow \vec{x}(t) = c_1 \exp(\lambda_1 t)\vec{e}_1 + c_2 \exp(\lambda_2 t)\vec{e}_2$$

b) λ has algebraic and geom. multiplicity 2 , $\vec{e}_1 \neq \vec{e}_2$

$$\Rightarrow \vec{x}(t) = c_1 \exp(\lambda t)\vec{e}_1 + c_2 \exp(\lambda t)\vec{e}_2$$

c) λ has algebraic multiplicity 2 but geom. mult. 1 , $\vec{e}_1$ EV and $\vec{e}_2$ gen. EV

$$\Rightarrow \vec{x}(t) = c_1 \exp(\lambda t)\vec{e}_1 + c_2 t \exp(\lambda t)\vec{e}_2$$

d) $\lambda_1 = a + ib \in \mathbb{C}$ $\Rightarrow \lambda_2 = \overline{\lambda_1} \in \mathbb{C}$ with EV $\vec{e}_1, \vec{e}_2$

$$\Rightarrow \vec{x}(t) = c_1 \exp(at)\exp(ibt)\vec{e}_1 + c_2 \exp(at)\exp(-ibt)\vec{e}_2$$

- Estimate the distance of a solution $\vec{x}(t)$ from $\vec{x}_0 = 0$:

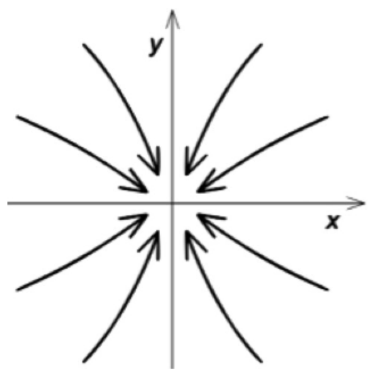
a), b) :
$$|\vec{x}(t) - \vec{x}_0|^2 = |c_1 \exp(\lambda_1 t)\vec{e}_1 + c_2 \exp(\lambda_2 t)\vec{e}_2|^2$$
$$= (\exp(\lambda_1 t)\, c_1 e_{11} + \exp(\lambda_2 t) c_2 e_{21})^2 + (\exp(\lambda_1 t)\, c_1 e_{12} + \exp(\lambda_2 t) c_2 e_{22})^2$$

$\Rightarrow$ $|\vec{x}(t) - \vec{x}_0| \to 0$ for $t \to \infty$ if $\lambda_i < 0$ $(i = 1,2)$
$\to \infty$ for $t \to \infty$ if $\lambda_1 > 0$ or $\lambda_2 > 0$

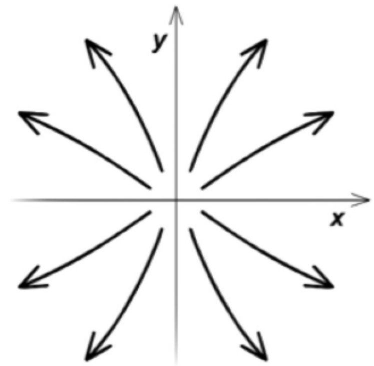
c) $|\vec{x}(t) - \vec{x}_0| \to 0$ if $\lambda < 0$
$\to \infty$ if $\lambda \geq 0$

d) $|\vec{x}(t) - x_0| \to 0$ if $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = a < 0$
$\to \infty$ otherwise

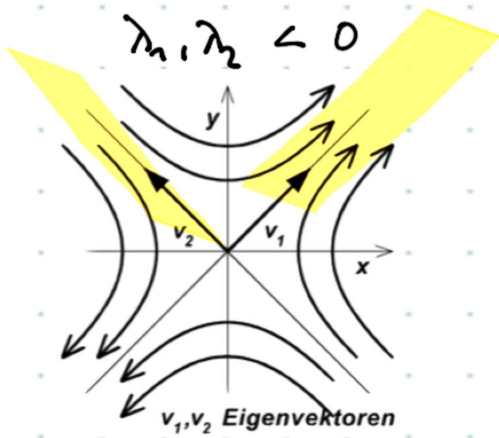
- Phase portraits:



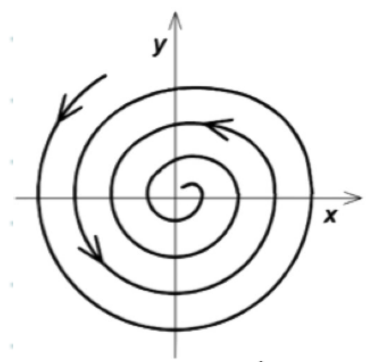
$\lambda_1, \lambda_2 < 0$



$\lambda_1 > 0$ and $\lambda_2 > 0$



$\lambda_1 > 0$ and $\lambda_2 < 0$



$\lambda = a + ib$ with $a < 0$