Differential Equations I

Week 13 / J. Behrens

① Error considerations:

- Recall: Use a general (implicit) form of the calculation rule

$$y_{k+1} = y_k + h\Phi(x_k, y_k, y_{k+1}, h).$$

Definition: (Lokal Discretization Error)
The lokal discretization error at x_{k+1} is defined by

$$d_{k+1} := y(x_{k+1}) - y(x_k) - h\Phi(x_k, y(x_k), y(x_{k+1}), h).$$

Definition: (Global Discretization Error)
The global discretization error at x_k is defined by

$$g_k := y(x_k) - y_k.$$

- Lipschitz. condition:
In order to estimate the error, Φ needs to fulfill certain requirements:

$$|\Phi(x,y,z,h) - \Phi(x,y^*,z,h)| \leq L\,|y-y^*|$$

$$|\Phi(x,y,z,h) - \Phi(x,y,z^*,h)| \leq L\,|z-z^*|$$

with (x,y,z,h), (x,y^*,z,h) (x,y,z^*,h) arbitrary points in domain B
$0 < L < \infty$ Lipschitz constant.

- Remark: If Φ is continuous and partial derivatives Φ_y and Φ_z exist, are continuous and bounded, i.e. $|\Phi_y(x,y,z,h)| < M$ and $|\Phi_z(x,y,z,h)| < M$ then Φ is Lipschitz continuous with $L = M$.

- From local error: $y(x_{k+1}) = y(x_k) + h\,\Phi(x_k, y(x_k), y(x_{k+1}), h) + d_{k+1}$
- Subtract computing rule: $y(x_{k+1}) - y_{k+1} = g_{k+1}$

$$\Rightarrow g_{k+1} = \underbrace{y(x_k) - y_k}_{g_k} + h\left[\Phi(x_k, y(x_k), y(x_{k+1}), h) - \Phi(x_k, y_k, y_{k+1}, h)\right] + d_{k+1}$$

$$= g_k + h\left[\underline{\Phi(x_k, y(x_k), y(x_{k+1}), h) - \Phi(x_k, y_k, y(x_{k+1}), h)} + \underline{\Phi(x_k, y_k, y(x_{k+1}), h) - \Phi(x_k, y_k, y_{k+1}, h)}\right] + d_{k+1}$$

$$\Rightarrow |g_{k+1}| \leq |g_k| + h\left[L\,|\underbrace{y(x_k) - y_k}_{g_k}| + L\,|\underbrace{y(x_{k+1}) - y_{k+1}}_{g_{k+1}}|\right] + |d_{k+1}|$$

$$= (1 + hL)\,|g_k| + hL\,|g_{k+1}| + |d_{k+1}| \quad \circledast$$

- With $hL < 1$:

$$|g_{k+1}| \leq \frac{1 + hL}{1 - hL}\,|g_k| + \frac{|d_{k+1}|}{1 - hL}$$

- For each $h > 0$ ex $K > 0$ const., such that $\frac{1 + hL}{1 - hL} = 1 + hK$

- Explicit one-step method (i.e. independent from y_{k+1} or $y(x_{k+1})$)
  $\rightarrow$ neglect $hL\,|g_{k+1}|$ in $\circledast$

$$\Rightarrow |g_{k+1}| \leq (1 + hL)\,|g_k| + |d_{k+1}|$$

- with $\max_k |d_k| < D$ and appropriate constants a, b:

$$\Rightarrow |g_{k+1}| \leq (1 + a)\,|g_k| + b$$

from this the proposition in the video follows.