Department of Mathematics
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SuSe 2023

Mathematics III Exam
(Module: Differential Equations I)

05.09.2023

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Exercise	Points	Evaluator
1		
2		
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4		
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Exercise 1 (6 points)

Determine the general solution of the differential equation

$$y'(t) + 2y(t) - t\,y(t)^4 = 0.$$

Solution:

The differential equation is a Bernoulli equation.

With $\alpha = 4$, $a = 2$ and $b = t$ and $u = y^{1-\alpha} = y^{-3}$ one obtains the linear differential equation in u

$$u'(t) - 6u(t) = -3t. \qquad \textbf{(1 point)}$$

$u_h' = 6u_h \implies \frac{du_h}{dt} = 6u_h \implies \frac{du_h}{u_h} = 6dt \implies \ln(|u_h|) = 6t + k$

$\implies u_h(t) = Ce^{6t}.$ **(2 points)**

Ansatz for a particular solution:

Version 1) Special ansatz

$u_p(t) = k_1 + k_2 t \overset{\text{ODE}}{\longrightarrow} k_2 - 6k_1 - 6k_2 t \overset{!}{=} -3t$

Comparison of coefficients returns $k_2 = \frac{1}{2}$ and $k_1 = \frac{1}{12}$.

Version 2) Variation of constants

$u_p(t) = C(t)e^{6t} \overset{\text{ODE}}{\longrightarrow} \dot{C}(t)e^{6t} \overset{!}{=} -3t$

$$C(t) = \int -3te^{-6t}dt = \left[-3t\frac{e^{-6t}}{-6}\right] - \int -3\frac{e^{-6t}}{-6}dt = \frac{t}{2}e^{-6t} - \frac{1}{2}\int e^{-6t}dt$$

$$= \left(\frac{t}{2} + \frac{1}{12}\right)e^{-6t} + K.$$

Thus for example with $K = 0$

$u_p(t) = C(t)e^{6t} = \left(\frac{t}{2} + \frac{1}{12}\right)e^{-6t} \cdot e^{6t}$.

$\implies u_p(t) = \frac{t}{2} + \frac{1}{12}.$ **(2 points)**

Hence altogether

$$u(t) = Ce^{6t} + \frac{t}{2} + \frac{1}{12}$$

and

$$y(t) = \left(\frac{1}{u}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1}{u}} \qquad \textbf{(1 point)}$$

Exercise 2 (3 points)

Rewrite the following initial value problem as equivalent initial value problem for a system of first-order differential equations

$$y'''(x) - y''(x) + 2y'(x) - 3y(x) = 0\,, \qquad y(1) = 1,\, y'(1) = 4,\, y''(1) = 9.$$

Solution: (2 points)

Rearranging the differential equation it results

$$y'''(x) = y''(x) - 2y'(x) + 3y(x).$$

We now define

$$\boldsymbol{y}\,(t) := \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} := \begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \end{pmatrix} \qquad \text{and from this} \qquad \boldsymbol{y}\,' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix}$$

hence

$$y_1' = y_2, \qquad y_2' = y_3, \qquad y_3' = y''' = y_3 - 2y_2 + 3y_1\,.$$

The equivalent initial value problem for a system of first order is thus

$$\boldsymbol{y}\,' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad \boldsymbol{y}\,(1) = \begin{pmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}. \qquad \textbf{(3 points)}$$

Exercise 3: (5 points)

Consider the boundary value problem

$$y'' - 4y' + 4y = h(x) \qquad x \in]0,1[$$
$$\alpha y(0) - y'(0) = \gamma_1$$
$$y(1) = \gamma_2 \qquad \alpha, \gamma_1, \gamma_2 \in \mathbb{R}.$$

For which values of α is the boundary problem uniquely solvable for any $\gamma_1, \gamma_2 \in \mathbb{R}$ and any continuous function $h(x)$ on the interval $[0,1]$?

Solution:

Computation of roots of the characteristic polynomial.

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \iff \lambda = 2.$$

The functions

$$y_1(x) = e^{2x}, \qquad y_2(x) = xe^{2x} \qquad \textbf{(2 points)}$$

build a fundamental system of the corresponding homogeneous differential equation.

It holds $y_1'(x) = 2e^{2x}$, $y_2'(x) = (1+2x)e^{2x}$ and

$$\begin{aligned} R_1(y_1) &= \alpha y_1(0) - y_1{}'(0) = \alpha - 2, \\ R_1(y_2) &= \alpha y_2(0) - y_2{}'(0) = -1, \\ R_2(y_1) &= y_1(1) = e^2, \\ R_2(y_2) &= y_2(1) = e^2. \end{aligned}$$

The boundary problem is uniquely solvable for any $\gamma_1, \gamma_2 \in \mathbb{R}$ and any h continuous if and only if the matrix

$$\mathbf{R} := \begin{pmatrix} R_1(y_1) & R_1(y_2) \\ R_2(y_1) & R_2(y_2) \end{pmatrix} = \begin{pmatrix} \alpha - 2 & -1 \\ e^2 & e^2 \end{pmatrix}$$

is invertible. Thus, if and only if

$$(\alpha - 2)e^2 + e^2 = e^2(\alpha - 1) \neq 0 \iff \alpha \neq 1. \qquad \textbf{(3 points)}$$

Exercise 4: (2 points)

Consider the system of differential equations

$$\dot{\boldsymbol{y}}\,(t) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{t^2} & \frac{3}{2t} \end{pmatrix} \boldsymbol{y}\,(t) \,+\, \begin{pmatrix} t^3 \\ 2t^2 \end{pmatrix}, \qquad\qquad t \geq 1.$$

The functions

$$\boldsymbol{y}^{[1]}(t) \;=\; \begin{pmatrix} 2\sqrt{t} \\ \frac{1}{\sqrt{t}} \end{pmatrix} \text{ and } \boldsymbol{y}^{[2]}(t) \;=\; \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

are solutions of the corresponding homogeneous system of differential equations.

Do $\boldsymbol{y}^{[1]}$ and $\boldsymbol{y}^{[2]}$ build a fundamental system for the space of solutions of the corresponding homogeneous system of differential equations?

Solution:

We compute the Wronskian

$$W(t) = \det \boldsymbol{Y}\,(t) \;=\; \det \begin{pmatrix} 2\sqrt{t} & t^2 \\ \frac{1}{\sqrt{t}} & 2t \end{pmatrix}$$

for example at point $t = 1$.

$$W(1) \;=\; \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \;=\; 4 - 1 \neq\; 0\,.$$

It is therefore a fundamental system. **(2 points**

Exercise 5: (4 points)

Consider the initial value problem

$$y''(t) + 4y'(t) + 3y(t) = 2\cos(t) + t^2 e^{-2t}, \text{ for } t > 0, \qquad y(0) = 0, \, y'(0) = 5.$$

Into which algebraic equation can the initial value problem be transformed by Laplace transformation?

Solution:

Let Y be the image of y under the Laplace transformation. Then it holds

$y \circ\!\!-\!\!\bullet Y, \qquad y' \circ\!\!-\!\!\bullet sY - y(0) = sY,$

$y'' \circ\!\!-\!\!\bullet s^2Y - sy(0) - y'(0) = s^2Y - 5,$ **[1 point]**

$\cos(t) \circ\!\!-\!\!\bullet \dfrac{s}{s^2+1}, \qquad t^2 \circ\!\!-\!\!\bullet \dfrac{2!}{s^{2+1}}, \qquad e^{-2t}t^2 \circ\!\!-\!\!\bullet \dfrac{2}{(s+2)^3}.$ **[2 points]**

The initial value problem is transformed into

$$(s^2 + 4s + 3)Y - 5 = \frac{2s}{s^2+1} + \frac{2}{(s+2)^3}. \qquad \textbf{[1 point]}$$