Mathematics Department
Prof. Dr. T. Schmidt

SuSe 2024

Mathematics III Exam
(Module: Differential Equations I)

26 August 2024

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Assessment according to examin. reg: with Analysis III ☐ single scoring ☐

I was instructed about the fact that the exam performance will only be assessed if the Central Examination Office of TUHH verifies my official admission before the exam's beginning in retrospect.

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Exercise	Points	Evaluator
1		
2		
3		
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$\sum$ =

Exercise 1) (3 points)

Compute the general solution of the following differential equation

$$y'(t) = \cos(t) \cdot \frac{1}{4y^2(t)}.$$

Solution

The differential equation is separable.

$\frac{dy}{dt} = \cos(t) \cdot \frac{1}{4y^2(t)} \overset{(*)}{\Longleftrightarrow} y^2 dy = \frac{\cos(t)}{4} dt.$ **(1 point)**

Therefore

$\int y^2 dy = \int \frac{\cos(t)}{4} dt \Longleftrightarrow \frac{y^3(t)}{3} = \frac{\sin(t)}{4} + \tilde{C}$ **(1 point)**

$\Longleftrightarrow y^3(t) = \frac{3}{4}\sin(t) + 3\tilde{C} = \frac{3}{4}\sin(t) + C$

and we get $y(t) = \sqrt[3]{\frac{3}{4}\sin(t) + C}.$ **(1 point)**

Exercise 2) (5 points)

a) Which of the following differential equations for $u(t)$ is exact?

(i) $u + u^3 + 3u^2 u' = 0\,.$

(ii) $u^5 + \sin(t) + 5tu^4 u' = 0\,.$

(iii) $ut^2 - tu^2 u' = 0\,.$

Justify your answers.

b) Determine the corresponding potential and the general solution for an exact differential equation in part a).

Solution:

a) (i) $u + u^3 + 3u^2 u' = 0$. For $f(t,u) = u + u^3$ and $g(t,u) = 3u^2$, we get $f_u = 1 + 3u^2 \neq g_t = 0\,.$
Therefore, the differential equation is not exact.

(ii) $u^5 + \sin(t) + 5tu^4 u' = 0$. It holds:
$f_u(t,u) = 5u^4 = g_t(t,u) = (5tu^4)_t = 5u^4$. The differential equation is exact.

(iii) $ut^2 - tu^2 u' = 0$. For $f_u(t,u) = t^2$ and $g_t(t,u) = -u^2$ the condition $f_u = g_t$ can only be fulfilled for $t = u = 0$.
The differential equation is not exact. **(2,5 points)**

b) We determine a potential Ψ for the differential equation from part a)ii).
$u^5 + \sin(t) + 5tu^4 u' = 0.$

$$f(t,u) = u^5 + \sin(t),\; g(t,u) = 5tu^4\,,$$

$$\Psi_t(t,u) = u^5 + \sin(t) \implies \Psi(t,u) = u^5 t - \cos(t) + c(u) \Rightarrow$$

$$\Psi_u(t,u) = 5tu^4 + 0 + c'(u) \overset{!}{=} g(t,u) = 5tu^4$$

$$\implies c'(u) = 0 \iff c(u) = k \iff \Psi(t,u) = u^5 t - \cos(t) + k\,.$$ **(1,5 points)**

Solutions of the differential equation fulfill:

$$\Psi(t,u) = u^5 t - \cos(t) + k = \tilde{K} \iff u^5 t - \cos(t) = K\,.$$ **(1 point)**

General solution: $u(t) = \sqrt[5]{\frac{K+\cos(t)}{t}}$ for $t \neq 0$.

Exercise 3) (6 points)

Determine the general solution of the following differential equation

$$u'''(t) + 4u''(t) - 5u'(t) = -1 - 5t\,.$$

Solution:

Characteristic polynomial:

$P(\lambda) = \lambda^3 + 4\lambda^2 - 5\lambda = \lambda(\lambda^2 + 4\lambda - 5\,)\,.$ **Ansatz (1 point)**

$\lambda^2 + 4\lambda - 5 = 0 \Longleftrightarrow (\lambda+2)^2 - 9 = 0 \Longleftrightarrow \lambda \in \{-2-3, -2+3\}\,.$

The roots of P are: $\lambda_1 = -5,\ \lambda_2 = 0,\ \lambda_3 = 1\,.$

Fundamental system of the corresponding homogeneous differential equation:

$u_1(t) = e^{-5t},\ u_2(t) = e^0,\ u_3(t) = e^t.$

General solution of the corresponding homogeneous differential equation:

$u_h(t) = c_1 e^{-5t} + c_2 + c_3\, e^t\,.$ **(2 points)**

In order to compute a particular solution for the inhomogeneous solution, we employ a special ansatz.

The inhomogeneity is a polynomial of first order multiplied by $e^{0\cdot t}$, where 0 is a single root of the characteristic polynomial.

Ansatz: $u_p =$ polynomial of first order $\cdot e^{0\cdot t} \cdot t = at + bt^2\,.$ **(1 point)**

It holds $u'(t) = a + 2bt,\ u''(t) = 2b,\ u'''(t) = 0\,.$ We plug this into the differential equation and obtain

$0 + 4\cdot 2b - 5(a + 2bt) = -10bt + 8b - 5a \overset{!}{=} -1 - 5t\,.$

Comparison of coefficients yields $b = \frac{1}{2},\ a = 1\,.$

Therefore $u_p(t) = t + \frac{t^2}{2}\,.$ **(1 point)**

Then, we obtain a representation of the general solution of the inhomogeneous differential equation

$u(t) = u_h(t) + u_p(t) = c_1 e^{-5t} + c_2 + c_3\, e^t + t + \frac{t^2}{2}\,.$ **(1 point)**

Exercise 4) (6 points)

Consider the system of differential equations

$$\boldsymbol{u}'(t) = \boldsymbol{A} \cdot \boldsymbol{u}(t) = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & \beta \\ 2 & -\beta & 1 \end{pmatrix} \cdot \boldsymbol{u}(t)$$

with parameter $\beta \in \mathbb{R}$.

a) Analyse the stability of the stationary point $(0,0,0)^T$ of the system.

b) Let $\beta = 0$. Determine a fundamental system of the system of differential equations.

Solution:

a) Computation of eigenvalues

$$P(\lambda) := \det \begin{pmatrix} -1-\lambda & 0 & 0 \\ 1 & 1-\lambda & \beta \\ 2 & \beta & 1-\lambda \end{pmatrix} = (-1-\lambda) \cdot \det \begin{pmatrix} 1-\lambda & \beta \\ -\beta & 1-\lambda \end{pmatrix}.$$

$$P(\lambda) = (-1-\lambda) \cdot ((1-\lambda)^2 + \beta^2) = 0 \implies \lambda_1 = -1\,,\ \lambda_{2,3} = 1 \pm \sqrt{-\beta^2} = 1 \pm i\beta.$$

There is (at least) one eigenvalue with a positive real part. The zero solution is unstable. **(2,5 points)**

b) $\beta = 0$. Computation of eigenvectors

$$\begin{pmatrix} -1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_1 = -1$ the system of equations yields

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies v_2 = -\frac{1}{2}v_1,\ v_3 = -v_1.$$

We choose, for example, $v^{[1]} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and obtain $\boldsymbol{u}^{[1]}(t) = e^{-t} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$.

For $\lambda_2 = \lambda_3 = 1$ the system of equations yields

$$\begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies v_1 = 0.$$

We choose, for example, $v^{[2]} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $v^{[3]} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and get $\boldsymbol{u}^{[2]}(t) = e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{u}^{[3]}(t) = e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

The general solution is:

$\boldsymbol{u}(t) = c_1\, \boldsymbol{u}^{[1]}(t) + c_2\, \boldsymbol{u}^{[2]}(t) + c_3\, \boldsymbol{u}^{[3]}(t) \qquad c_1, c_2, c_3 \in \mathbb{R}.$ **(3,5 points)**