Fachbereich Mathematik **SoSe 2022**
Prof. Dr. J. Behrens

Exam Differential Equations II

06. September 2022

Sie haben 60 Minuten Zeit zum Bearbeiten der Klausur. In die Wertung gehen maximal 20 Punkte ein.

Bitte kennzeichnen Sie jedes Blatt mit Ihrem Namen und Ihrer Matrikelnummer.

Tragen Sie bitte zunächst Ihren Namen, Ihren Vornamen und Ihre Matrikelnummer in **DRUCKSCHRIFT** in die folgenden jeweils dafür vorgesehenen Felder ein. Diese Eintragungen werden auf Datenträger gespeichert.

Name:

Vorname:

Matr.-Nr.:

Studiengang: AIW | CI | ET | GES | IIW | MB | MTB | SB |

Ich bin darüber belehrt worden, dass die von mir zu erbringende Prüfungsleistung nur dann bewertet wird, wenn die Nachprüfung durch das Zentrale Prüfungsamt der TUHH meine offizielle Zulassung vor Beginn der Prüfung ergibt.

Unterschrift:

Aufg.	Punkte	Korrekteur
1		
2		
3		
4		

$\sum =$

Exercise 1: [7 points]

Given the following initial value problem for $u(x,t)$:

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} \frac{1}{2} & x \leq 0, \\ 0 & 0 < x \leq 1, \\ -2 & 1 < x\,. \end{cases}$$

a) Compute the weak solution for $t \in [0, \tilde{t}\,]$ with a sufficiently small $\tilde{t}$.

b) To what maximum t^* can the solution from part a) be continued?

c) Determine the weak solution for $t > t^*$.

Solution:

a) At the two jump points of the initial data, we introduce two shock waves.
The jump condition requires:
$\dot{s}_1(t) = \dfrac{\frac{1}{2}+0}{2} = \frac{1}{4}$ and $\dot{s}_2(t) = \dfrac{0-2}{2} = -1\,.$
We obtain the shock fronts
$s_1(t) = \frac{1}{4}\,t$ and $s_2(t) = 1 - t\,.$
For sufficiently small t we have

$$u(x,t) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{4}\,t, \\ 0 & \frac{1}{4}\,t < x \leq 1 - t, \\ -2 & 1 - t < x\,. \end{cases} \qquad \textbf{(3 points)}$$

as a weak solution.

b) At t^* with
$\frac{1}{4}\,t^* = 1 - t^* \iff \frac{5}{4}\,t^* = 1 \iff t^* = \frac{4}{5}$ **(1 point)**
the shock fronts meet and the solution from part a) becomes ambiguous.

c) For $t^* = \frac{4}{5}$ it holds $s_1(t) = s_2(t) = \frac{1}{5}$ and

$$u(x, \frac{4}{5}) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{5}\,, \\ -2 & x > \frac{1}{5}\,. \end{cases}$$

We add the new shock front

$$s_3(t) = \frac{1}{5} + \dot{s}_3(t - \frac{4}{5}) = \frac{1}{5} + \frac{\frac{1}{2}-2}{2}\,(t - \frac{4}{5})$$

and obtain for $t > \frac{4}{5}$

$$u(x,t) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{5} - \frac{3}{4}\,(t - \frac{4}{5}) = \frac{4}{5} - \frac{3}{4}\,t. \\ -2 & x > \frac{4}{5} - \frac{3}{4}\,t. \end{cases} \qquad \textbf{(3 points)}$$

Exercise 2: [3 points]

Given is the following differential equation for $u(x,y)$:

$$u_{xx} + 6u_{xy} + (x-y)\,u_{yy} + x^3\,u_x + 5u = 6.$$

Determine the order and the type of the differential equation.

Solution:

The differential equation is of order two. **(1 point)**

Since $(x-y) - 3^2 = 0 \iff y = x - 9$ holds.

The differential equation is $\begin{cases} \text{elliptic} & y < x-9, \\ \text{parabolic} & y = x-9, \\ \text{hyperbolic} & y > x-9. \end{cases}$ **(2 points)**

Exercise 3: [7 points]

a) Given the initial boundary value problem

$$\begin{aligned} u_t - 5u_{xx} &= \frac{\pi x}{4}\sin(\pi t) && \text{for } x \in (0,4), t > 0, \\ u(x,0) &= 2\sin(\pi x) + 3\sin(2\pi x) && \text{for } x \in [0,4], \\ u(0,t) = 0, \quad u(4,t) &= 1 - \cos(\pi t) && \text{for } t > 0. \end{aligned}$$

Transform the problem into an initial boundary value problem with homogeneous boundary data using a suitable homogenization of the boundary conditions.

b) Solve the following initial boundary value problem:

$$\begin{aligned} v_t - 5v_{xx} &= 0 && \text{for } x \in (0,4), t > 0, \\ v(x,0) &= 2\sin(\pi x) + 3\sin(2\pi x) && \text{for } x \in [0,4], \\ v(0,t) = 0, \quad v(4,t) &= 0 && \text{for } t > 0. \end{aligned}$$

c) Provide the solution to the initial boundary value problem from part a).

Solution:

a) Homogenization:

$$v(x,t) = u(x,t) - 0 - \frac{x}{4}(1 - \cos(\pi t) - 0) = u(x,t) - \frac{x}{4}(1 - \cos(\pi t))$$

or

$$u(x,t) = v(x,t) + \frac{x}{4}(1 - \cos(\pi t)). \qquad \textbf{(1 point)}$$

Then it holds:

$$u_t = v_t + \frac{\pi x}{4}\sin(\pi t), \qquad v_{xx} = u_{xx}$$

New differential equation: $v_t + \frac{\pi x}{4}\sin(\pi t) - 5v_{xx} = \frac{\pi x}{4}\sin(\pi t) \iff$

$$\boxed{v_t - 5v_{xx} = 0} \qquad \textbf{(1 point)}$$

Initial data:

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{x}{4}(1 - \cos(0)) \\ &= 2\sin(\pi x) + 3\sin(2\pi x) - 0 \iff \end{aligned}$$

$$\boxed{v(x,0) = 2\sin(\pi x) + 3\sin(2\pi x)}$$

Boundary data: $\boxed{v(0,t) = v(4,t) = 0}$ **(1 point)**

b) With $\omega = \frac{\pi}{4}$ and $c = 5$ it holds:

$$v(x,t) \;=\; \sum_{k=1}^{\infty} a_k e^{-c\omega^2 k^2 t}\,\sin(k\omega x) \;=\; \sum_{k=1}^{\infty} a_k e^{-\frac{5k^2\pi^2}{16}t}\,\sin(\frac{k\pi}{4}x) \qquad \textbf{(1 point)}$$

Inserting the initial values gives:

$v(x,0) \;=\; \sum_{k=1}^{\infty} a_k \sin(\frac{k\pi}{4}x) \;\overset{!}{=}\; 2\sin(\pi x) \;+\; 3\sin(2\pi x)$

$\Longrightarrow a_4 = 2,\; a_8 = 3,\; a_k = 0 \qquad \forall k \notin \{4,8\}.$

$$v(x,t) \;=\; 2\,e^{-\frac{5\cdot 4^2\pi^2}{16}t}\,\sin(\frac{4\pi}{4}x) \;+\; 3\,e^{-\frac{5\cdot 8^2\pi^2}{16}t}\,\sin(\frac{8\pi}{4}x) \qquad \textbf{(2 points)}$$

c) For the solution of a) we thus get

$$u(x,t) = \; v(x,t) + \;\frac{x}{4}\,(1-\cos(\pi t))$$

$$= \; 2\,e^{-5\pi^2 t}\,\sin(\pi x) \;+\; 3\,e^{-20\pi^2 t}\,\sin(2\pi x) + \;\frac{x}{4}(1 - \;\cos(\pi t)). \qquad \textbf{(1 point)}$$

Exercise 4: [1+2 points]

Given the initial value problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \mathbb{R}^2 \\ u = f & \text{on } \mathbb{R} \times \{y = 0\} \\ u_y = g & \text{on } \mathbb{R} \times \{y = 0\} \end{cases}$$

a) When is this initial value problem called well-posed?

b) Using the following ansatz, show that this initial value problem is ill-posed (not well-posed). Choose

$$f \equiv 0, \quad g \equiv 0$$
$$f_n \equiv 0, \quad g_n = \frac{1}{n}\sin(nx)$$

Hint: The solution to the problem with initial values f_n and g_n is

$$u_n(x, y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

.

Solution:

a) The following properties have to hold:

(i) Existence of the solution,

(ii) Uniqueness of the solution,

(iii) Continuous dependence on the data, i.e. for $c_1, c_2 \in \mathbb{R}$

$$\|u - u_n\| \leq c_1\|f - f_n\| + c_2\|g - g_n\|.$$

b)
- If $f \equiv g \equiv 0$, also $u \equiv 0$.
- If $f_n \equiv 0$ and $g_n = \frac{1}{n}\sin(nx)$, we have

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n = 0.$$

But on the other hand

$$\lim_{n\to\infty} \|u_n\|_\infty = \infty.$$

This contradicts the requirement for continuous dependence on the data,

$$\|u - u_n\|_\infty = \infty \geq 0 = \|f - f_n\|_\infty + \|g - g_n\|_\infty.$$