Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 1

Exercise 1: (Repetition Analysis II)

For the derivation of parameter-dependent integrals for sufficiently smooth f holds the **Leibniz–Rule** :

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))$$

Find the derivative of the function $F(x)$ defined as

$$F(x) := \int_{-x}^{x^2} e^{xt}\,dt$$

and compute $\lim\limits_{x\to 0} F'(x)$.

Exercise 2:

A simple traffic flow model:

We consider a one-dimensional flow of vehicles along an infinitely long, single-lane road. In a so-called macroscopic model, one does not consider individual vehicles, but the total flow of vehicles. For this purpose, we introduce the following quantities:

$$\begin{aligned} u(x,t) &= \text{(length-)density of the vehicles at the point } x \text{ at the time } t \\ &= \text{vehicles/unit length at point } x \text{ at the time } t \\ v(x,t) &= \text{ speed at the point } x \text{ at the time } t, \\ q(x,t) &= u(x,t)\cdot v(x,t) = \text{ flow} \\ &= \text{ amount of vehicles passing the point } x \text{ at the time } t \text{ per unit time} \end{aligned}$$

a) Assume that there are no entrances or exits, no vehicles are disappearing, and no new vehicles are appearing. Let $N(t,a,\Delta a) :=$ number of vehicles on a space interval $[a, a+\Delta a]$ at the time t.

   Then on the one hand it holds that

$$N(t,a,\Delta a) = \int_a^{a+\Delta a} u(x,t)\,dx$$

and on the other hand it also holds

$$N(t, a, \Delta a) - N(t_0, a, \Delta a) \;=\; \int_{t_0}^{t} q(a, \tau) - q(a + \Delta a, \tau) d\tau \,.$$

Derive from this the so-called conservation equation for the mass (number of vehicles)

$$u_t + q_x = 0.$$

Hints on how to proceed:

- Derive both formulas for N with respect to t. Please note that for the derivation of parameter-dependent integrals with sufficiently smooth f holds the **Leibniz rule**:

$$\boxed{\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t)\, dt \;=\; \int_{a(x)}^{b(x)} \frac{d}{dx} f(x, t) dt \;+\; b'(x)\, f(x, b(x)) \;-\; a'(x)\, f(x, a(x))}$$

- Divide by Δa.
- Consider the limit $\Delta a \;\to\; 0$.

b) Additionally assume that the velocity depends only on the density: $v = v(u)$. Show that in this case the equation

$$\frac{\partial u}{\partial t} + \frac{dq}{du} \cdot \frac{\partial u}{\partial x} \;=\; 0$$

describes the conservation of mass.

c) We now assume in a first simple model that the speed increases in inverse proportion to the density and that the density is positive.

$$v(x, t) \;=\; c \;+\; \frac{k}{u(x, t)}$$

What is the continuity equation (=conservation equation for the mass)?

d) Solve the continuity equation derived in part c) for $c = 3$ and the initial condition $u(x, 0) = e^{-x^2}$.

Show that every sufficiently smooth function $u(x, t) = f(x - ct)$ solves the differential equation. Define f such that the initial condition is satisfied.

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